Bounded gaps between volumes of Manifolds

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Recall (from Tuesday's lecture):

Let $\pi(V,S)$ denote the maximum cardinality of a collection of pairwise non-commensurable arithmetic hyperbolic 2–manifolds derived from quaternion algebras, each of which has volume less than V and geodesic length spectrum containing S.

Theorem (Linowitz, McReynolds, Pollack, T., 2018)

If $\pi(V,S) \to \infty$ as $V \to \infty$, then there are integers $1 \leq r, s \leq |S|$ and constants $c_1, c_2 > 0$ such that

$$\frac{c_1 V}{\log(V)^{1-\frac{1}{2^r}}} \leq \pi(V,S) \leq \frac{c_2 V}{\log(V)^{1-\frac{1}{2^s}}}$$

for all sufficiently large V.

Conclusions:

1. There are lots of pairwise non-commensurable 3-manifolds w/ a great deal of overlap in their geodesic lengths.

Do the volumes just get farther & farther apart?

2. The Counting function looks a bit like the count of prime numbers...

Other Similarities between $\pi(V, S)$ & $\pi(x)$

Theorem (Linowitz, 2018)

Fix a finite set S of nonnegative real numbers for which $\pi(V,S) \to \infty$ as $V \to \infty$. Let r be the cardinality of S and define $\theta = \frac{8}{3}$ if $r = 1$ and $\theta = \frac{1}{2^r}$ otherwise. If $\epsilon > 0$ and $V^{1-\theta+\epsilon} < W < V$ then as $V \to \infty$ we have

$$\pi(V+W, S) - \pi(V,S) \geq \frac{1}{2^r} \cdot \frac{W}{\log V}.$$

analog of "primes in short intervals" results

Iwaniec + Jutila '79 → E.g. For $\theta \geq 13/23$ and x suff. large,
$$\pi(x) - \pi(x - x^\theta) > \frac{1}{177} \frac{x^\theta}{\log x}$$

Today's goal:

Let $\pi(V, S)$ denote the maximum cardinality of a collection of pairwise non-commensurable arithmetic hyperbolic 2–orbifolds derived from quaternion algebras, each of which has volume less than V and geodesic length spectrum containing S.

Theorem (Linowitz, McReynolds, Pollack, T., 2017)

Suppose that $\pi(V, S) \to \infty$ as $V \to \infty$. Then, for every $k \geq 2$, there is a constant $C > 0$ such that there are infinitely many k–tuples $M_1, \ldots, M_k$ of arithmetic hyperbolic 2–orbifolds which are pairwise non-commensurable, have length spectra containing S, and volumes satisfying $|\mathrm{vol}(M_i) - \mathrm{vol}(M_j)| < C$ for all $1 \leq i, j \leq k$.

bounded gaps between volumes of manifolds!

Bounded gaps between primes: a brief history.

Conjecture (de Polignac, 1849)

For all positive even integers h, there are infinitely many pairs of primes $p, p + h$.

obviously false when h is odd!

A Probabilistic interpretation of Chebyshev's Inequality:

Theorem (Chebyshev, 1852)

Approximately $\frac{1}{\log x}$ *of the integers in* $[1, x]$ *are prime.*

(Chebyshev's Ineq: $\exists\, C_1, C_2 > 0$ st.

$$C_1 \frac{x}{\log x} \leq \pi(x) \leq C_2 \frac{x}{\log x}$$
)

Sketch of Heuristic Argument

For integers in $[1, x]$:

$$\mathbb{P}(p \text{ is prime}) = \frac{1}{\log x}$$

$$\mathbb{P}(p+2 \text{ is prime}) = \frac{1}{\log x}$$

If these two events are <u>independent</u>,

$$\mathbb{P}(p \,\&\, p+2 \text{ prime}) = \mathbb{P}(p \text{ prime}) \cdot \mathbb{P}(p+2 \text{ prime})$$

$$= \frac{1}{\log x} \cdot \frac{1}{\log x}$$

$$= \left(\frac{1}{\log x}\right)^2.$$

So, we'd expect:

$$\#\{p \leq x : p \ \& \ p+2 \text{ prime}\} \approx \frac{x}{(\log x)^2}.$$

Since $\lim\limits_{x\to\infty} \dfrac{x}{(\log x)^2} \to \infty$, then

perhaps there are ∞ly many pairs of twin primes!

Problem: Events "p prime" & "p+2 prime" are NOT indepent. The same argument shows

False!

$$\#\{p \leq x : p \ \& \ p+1 \text{ prime}\} \approx \frac{x}{(\log x)^2}.$$

An improved heuristic

To correct for non-independence:

Let p, p' be independently chosen random integers. Look at:

$$\boxed{\frac{\mathbb{P}(p,\, p+2 \text{ not both divisible by } q)}{\mathbb{P}(p,\, p' \text{ not both divisible by } q)}}$$

for each small prime q.

Since $\mathbb{P}(q \mid p) = \frac{1}{q}$ then

$$\mathbb{P}(q \nmid p \;\&\; q \nmid p') = \left(1 - \frac{1}{q}\right)^2$$

OTOH:

$$\mathbb{P}(q \nmid p \ \& \ q \nmid (p+2)) = \mathbb{P}(p \not\equiv 0 \text{ or } -2 \pmod{q})$$

$$= \begin{cases} 1 - \frac{2}{q} & \text{if } q > 2 \\ 1 - \frac{1}{2} & \text{if } q = 2 \end{cases}$$

Hence, if $q > 2$, the correction factor for divisibility by q is: $\boxed{\dfrac{1 - \frac{2}{q}}{\left(1 - \frac{1}{q}\right)^2}}$

If $q = 2$, it is $\dfrac{1 - \frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = \boxed{2}$

Thus, define

$$C := 2 \cdot \prod_{\substack{q \text{ prime} \\ q \geq 3}} \frac{1 - 2/q}{(1 - 1/q)^2}$$

$$\approx 1.3203236\ldots$$

This suggests:

$$\#\{p \leq x : p,\ p+2 \text{ prime}\} \approx \boxed{C \cdot \frac{x}{(\log x)^2}}$$

Still not a proof that $\exists$ ∞ly many twin primes, but the problem is a bit more subtle...

Where the heuristics fail

* The heuristic argument relies heavily on the assumption that the primes are uniformly distributed among the residue classes (mod q).

Let $\pi(x;q,a) := \#\{p \le x : p \equiv a \pmod{q}\}$

If the primes were uniformly distributed (mod q), we'd expect

$$\boxed{\pi(x;q,a) \approx \frac{x}{\varphi(q)\log x}}$$

when $\gcd(a,q)=1$.

Equidistribution for "small q"

Theorem (Bombieri-Vinogradov)

For any constant $A > 0$, *there exists* $B = B(A)$ *such that*

$$\sum_{q \le Q} \max_{\substack{a \pmod q \\ (a,q)=1}} \left| \pi(x;q,a) - \frac{x}{\varphi(q)\log x} \right| \ll_A \frac{x}{(\log x)^A},$$

where $Q = \frac{x^{1/2}}{(\log x)^B}$.

This really just says: the primes are essentially uniformly distributed among the residue classes $a \pmod q$ when $q \le x^{1/2}$.

Conjecture (Elliott-Halberstam)

The Bombieri-Vinogradov theorem still holds if we take $Q = x^\theta$, for any $\theta < 1$.

We call θ the *level of distribution* of the set of primes.

↳ BV says: can take θ up to about $1/2$.

Tuples of primes

Q: Are the ∞ly many tuples of primes $(p+h_1, \ldots, p+h_k)$?

* Some tuples clearly fail:

Ex $p, p+2, p+4$ can't simultaneously be prime ∞ly often.

One of these must be $\equiv 0 \pmod 3$ (& thus it can't be prime ∞ly often)

Which k-tuples might work?

Definition

We say that a k-tuple $(h_1, ..., h_k)$ of nonnegative integers is *admissible* if it doesn't cover all of the possible remainders (mod p) for any prime p.

Ex $(0, 2, 6, 8, 12)$

Residue classes not covered:

$1 \pmod 2$

$1 \pmod 3$

$4 \pmod 5$

$3 \pmod 7$

$3 \pmod{11}$

A Conditional proof of bounded gaps between primes

G P Y

Theorem (Goldston, Pintz and Yıldırım, 2009)

If $(h_1, \ldots, h_k)$ is admissible and the Elliot-Halberstam Conjecture holds with $Q = x^{1/2+\eta}$, then there are infinitely many n such that at least 2 of $n + h_1, \ldots, n + h_k$ are prime.

Just need to get θ a bit above $\frac{1}{2}$ to get bdd gaps

Zhang's "relaxation" of Bombieri-Vinogradov

Recall

Theorem (Bombieri-Vinogradov)

For any constant $A > 0$*, there exists* $B = B(A)$ *such that*

$$\sum_{q \le Q} \max_{\substack{a \pmod q \\ (a,q)=1}} \left| \pi(x;q,a) - \frac{x}{\varphi(q)\log x} \right| \ll_A \frac{x}{(\log x)^A},$$

where $Q = \frac{x^{1/2}}{(\log x)^B}$.

Theorem (Zhang, 2013)

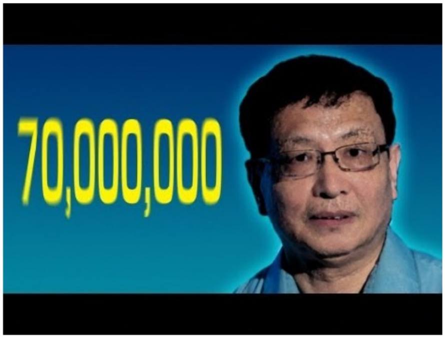
There exist $\eta, \delta > 0$ *such that for any given* a,

$$\sum_{\substack{q \le Q \\ (q,a)=1}} \left| \pi(x;q,a) - \frac{x}{\varphi(q)\log x} \right| \ll_A \frac{x}{(\log x)^A},$$

q squarefree & y -smooth

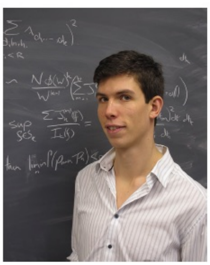
where $Q = x^{1/2+\eta}$ *and* $y = x^{\delta}$.

Bounded gaps between primes (at last!)



Corollary (Zhang, 2013)

There are infinitely many pairs of primes that differ by at most $70,000,000$.



Theorem (Maynard-Tao, November 2013)

Let $m \geq 2$. *There for any admissible* k*-tuple* $\mathcal{H} = (h_1, ..., h_k)$ *with "large enough"* k *(relative to* m*), there are infinitely many* n *such that at least* m *of* $n + h_1, ..., n + h_k$ *are prime.*

How large are the gaps?

Theorem (D. H. J. Polymath, 2014)

There are infinitely many pairs of primes that are at most 246 *apart.*

Can get this down to 6 by assuming generalized Eliot-Halberstam Conjecture

Sketch of Maynard-Tao

Goal: Find values of n for which the tuple $n+h_1, \dots, n+h_k$ contains several primes.

Setup: For large N, look for n in $[N, 2N)$

Let $W := \prod_{p \leq \log\log\log N} p$.

$\mathcal{H}$ admissible $\Rightarrow$ we can choose $v \in \mathbb{Z}$ s.t. $\gcd(v+h_i, W) = 1 \quad \forall\ 1 \leq i \leq k$.

W-trick: pre-sieve the set to just those n satisfying $n \equiv v \pmod{W}$.

Our sample space becomes

$$\Omega := \{ N \leq n < 2N : n \equiv v \pmod{W} \}$$

Let $w(n)$ denote nonnegative weights, $\chi_{\mathcal{P}}$ the char. function of the set of primes.

$$S_1 := \sum_{\substack{N \le n < 2N \\ n \equiv v \pmod W}} w(n)$$

$$S_2 := \sum_{\substack{N \le n < 2N \\ n \equiv v \pmod W}} \left(\sum_{i=1}^{k} \chi_{\mathcal{P}}(n+h_i) \right) w(n)$$

S_2/S_1 is a weighted avg. of the # of primes among $n+h_1, \ldots, n+h_k$ over Ω.

Key Idea: If $S_2/S_1 > m-1$ for some $m \in \mathbb{Z}^+$ then at least m of the $n+h_1, \dots, n+h_k$ are prime, for some $n \in \Omega$.

For this method to work, we need to choose weights $w(n)$ st:

* S_2 & S_1 can be estimated using tools of asymptotic analysis

* S_2/S_1 is large

Bounded gaps between primes in Chebotarev Sets

Theorem (Thorner, 2014)

Let $K/\mathbb{Q}$ be a Galois extension of number fields with Galois group G and discriminant Δ, and let $\mathcal{C}$ be a conjugacy class of G. Let $\mathcal{P}$ be the set of primes $p \nmid \Delta$ for which $\left(\frac{K/\mathbb{Q}}{p}\right) = \mathcal{C}$. Then there are infinitely many pairs of distinct primes $p_1, p_2 \in \mathcal{P}$ such that $|p_1 - p_2| \leq c$, where c is a constant depending on $G, \mathcal{C}, \Delta$.

Some examples of Chebotarev sets:

- The set of primes $p \equiv 1 \pmod 3$ for which 2 is a cube $\pmod p$.
- Fix $n \in \mathbb{Z}^+$. The set of primes expressible in the form $\mathsf{x}^2 + ny^2$.
- Let τ be the Ramanujan tau function. The set of primes p for which $\tau(p) \equiv 0 \pmod d$ for any positive integer d.
- The set of primes p for which $\#E(\mathbb{F}_p) \equiv p + 1 \pmod d$ for any positive integer d.

Generalizing Thorner's work

Theorem (Linowitz, McReynolds, Pollack, T., 2017)

Let L/K be a Galois extension of number fields, let $\mathcal{C}$ be a conjugacy class of $\mathrm{Gal}(L/K)$, and let k be a positive integer. Then, for a certain constant $c = c_{L/K,\mathcal{C},k}$, there are infinitely many k–tuples $P_1, \ldots, P_k$ of prime ideals of K for which the following hold:

1. $\left(\frac{L/K}{P_1}\right) = \cdots = \left(\frac{L/K}{P_k}\right) = \mathcal{C}$,
2. *$P_1, \ldots, P_k$ lie above distinct rational primes,*
3. *each of $P_1, \ldots, P_k$ has degree 1,*
4. *$|N(P_i) - N(P_j)| \le c$, for each pair of $i, j \in \{1, 2, \ldots, k\}$.*

Thorner's theorem is the case where $K = \mathbb{Q}$. ← This was the hard part

Thorner had pairs of primes instead of tuples of prime ideals ← This part was an easy generalization

Two proofs:

1. Use a different version of Bombieri–Vinogradov & check that the Maynard–Tao machinery still works.
   ↳ lots of messy analytic NT

due to Murty + Petersen, 2013

2. Give an algebraic argument to show that our problem can be reduced to Thorner's
   ↳ No analytic NT

Back to geometry...

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Also holds when $M_1, \ldots, M_k$ are arithmetic hyperbolic 3-manifolds

Pf Sketch

Borel's Covolume Formula:

$$\mathrm{vol}(\mathbf{H}^2/\Gamma^1_{\mathcal{O}}) = \frac{|\Delta_K|^{3/2}\zeta_K(2)}{(4\pi^2)^{n_K-1}} \prod_{P\in \mathrm{Ram}_f(B)} (N(P)-1).$$

Only depends on K

This depends on B

BCF $\Rightarrow$ If two orbifolds have the same field of def. K but their associated B's ramify at different primes P then their volumes will differ by some function of the N(P)'s.

Moral: primes w/ gaps between them produce orbifolds w/ volumes lying in bad length intervals.

What's missing?

Need the orbifolds to have length spectra containing S.

$\Updownarrow$

quadratic extensions k_γ embed into the B's

Can arrange this by choosing primes (ramifying in the B's) to lie in certain Chebotarev sets

That's why we need bad gaps between primes in Chebotarev sets

Thank you!