MSRI virtual semester:
"Random and Arithmetic Structures in Topology".
Introductory workshop,
8 September 2020.
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Anosov representations 1.

<u>Motivation:</u> G noncompact real semisimple Lie group,

e.g. $G = \begin{cases} SL(d, \mathbb{K}), & \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C} \\ PSL \\ PGL \end{cases}$

$\begin{cases} SO(p,q), & p,q \geq 1 \\ SU \end{cases}$

$Sp(2n, \mathbb{K})$ ... etc.

<u>Pb:</u> Find "nice" classes of discrete subgroups Γ of G.

① interesting examples, geometric interpretations?

② control behavior under deformation?

③ control dynamics of Γ on relevant homogeneous spaces $G/\bullet$?

...

<u>Important class:</u>

Γ lattice in G $\overset{\text{def.}}{\Longleftrightarrow}$ Γ discrete $< G$

and $\mathrm{Haar}(\underbrace{\Gamma \backslash G}_{\text{compact or not.}}) < +\infty$

Other classes? In this minicourse:

$\boxed{\text{Anosov subgroups}}$ = images of $\boxed{\text{Anosov representations}}$ $\rho : \Gamma_0 \to G$

where Γ_0 word hyperbolic group

$\left(\begin{array}{ll} \text{e.g. } \pi_1(S), & S \text{ closed hyp. surf.} \\ \pi_1(M), & M \text{ closed neg.-curved manifold} \\ F_r & \end{array} \right)$

↳ introduced by Labourie (2006) for $\Gamma_0 = \pi_1(M)$, generalised by Guichard–Wienhard (2012).

NB: two points of view:

discrete subgroups Γ of G $\longleftrightarrow$ images of inj. & discrete repr. $\Gamma_0 \to G$, Γ_0 abstract group.

I. Case $\mathrm{rank}_{\mathbb{R}} G = 1$.

e.g. $G = PSL(2,\mathbb{R}) \simeq SO(2,1)_0$

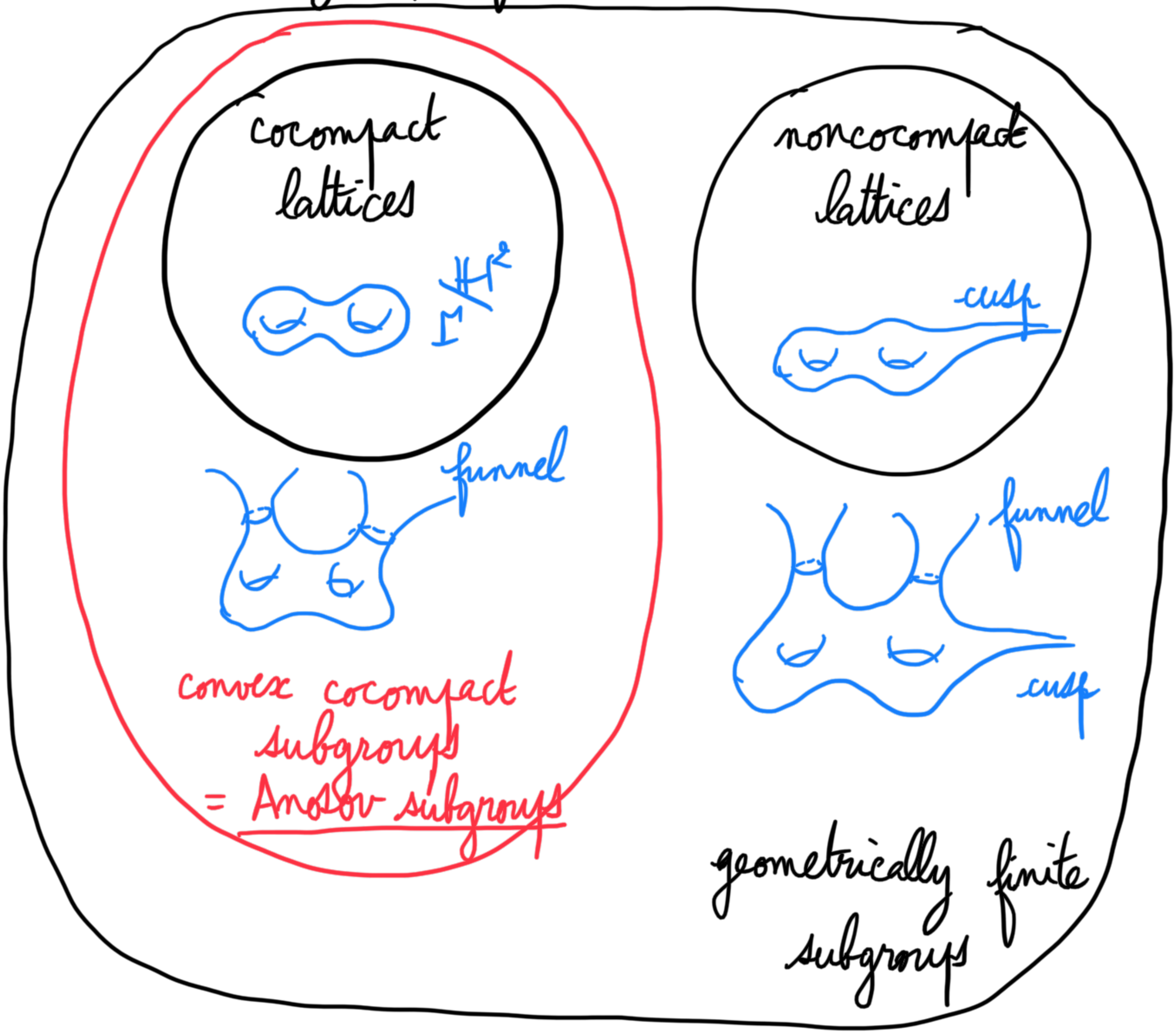
$PSL(2,\mathbb{C}) \simeq SO(3,1)_0$

$SO(n,1)$

$SU(n,1)$

$Sp(n,1)$

Finitely generated discrete subgroups of G



E.g. $G = PSL(2, \mathbb{R})$:

Γ discrete $< G \rightsquigarrow \Gamma \backslash \mathbb{H}^2$ hyperbolic surface (orbifold)

General rank-1 G :

K max. compact subgroup of G

$\rightsquigarrow G/K$ Riemannian symmetric space of G

Γ discrete $< G \rightsquigarrow \Gamma \backslash G/K$ neg.-curved locally symmetric manifold (orbifold)

G	K	G/K
$PSL(2,\mathbb{R}) \simeq SO(2,1)_0$	$PSO(2)$	$\mathbb{H}^2$
$PSL(2,\mathbb{C}) \simeq SO(3,1)_0$	$PSU(2)$	$\mathbb{H}^3$
$SO(n,1)$	$S(O(n)\times O(1))$	$\mathbb{H}^n$
$SU(n,1)$	$S(U(n)\times U(1))$	$\mathbb{H}^n_{\mathbb{C}}$
$Sp(n,1)$	$Sp(n)\times Sp(1)$	$\mathbb{H}^n_{\text{quaternions}}$

See Bowditch (1993, 1995) for geometric finiteness.

Today: $\boxed{\text{convex cocompact (CC) subgroups}}$

= Anosov subgroups.

Def.: A discrete subgroup Γ of G is CC if $\exists\ \underset{\neq \emptyset}{\mathcal{C}}$ convex $\subset G/K$, Γ-inv. with $\Gamma\backslash\mathcal{C}$ compact

$\iff \Gamma\backslash\mathcal{C}^{\text{core}}_\Gamma$ is compact, where $\mathcal{C}^{\text{core}}_\Gamma := \mathrm{Conv}_{G/K}(\Lambda_\Gamma)$.

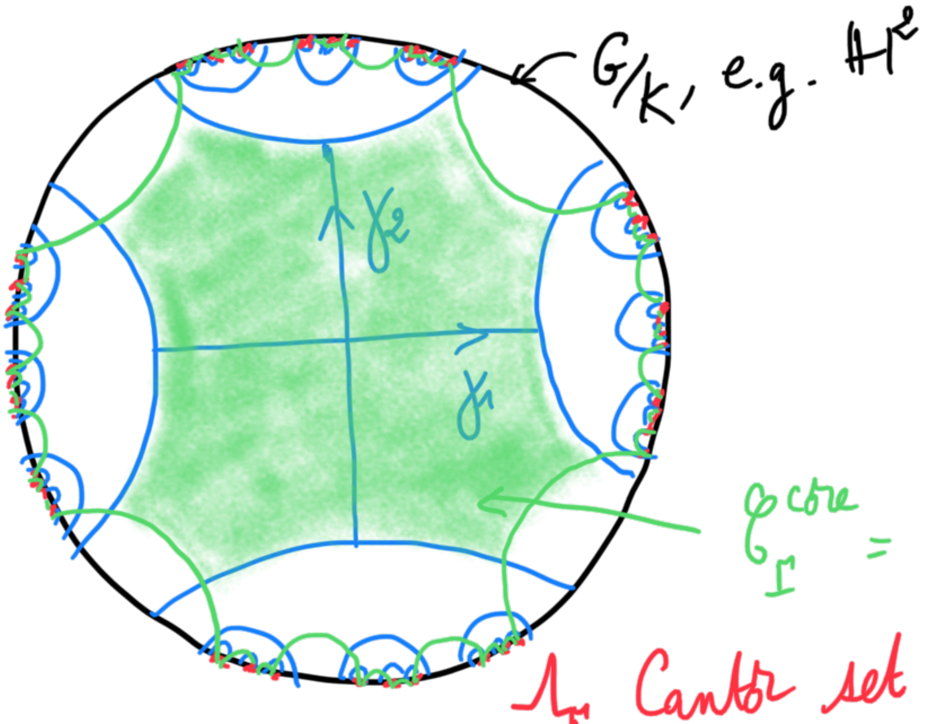
A representation $\rho : \Gamma_0 \to G$ is CC if it has finite kernel and discrete, CC image.

Here $\Lambda_\Gamma \subset \partial_\infty(G/K)$ is the limit set of Γ:

$\Lambda_\Gamma = \{$accumulation points of $\Gamma \cdot x\}$
for some $x \in G/K$ (indep. of x).

① Examples.

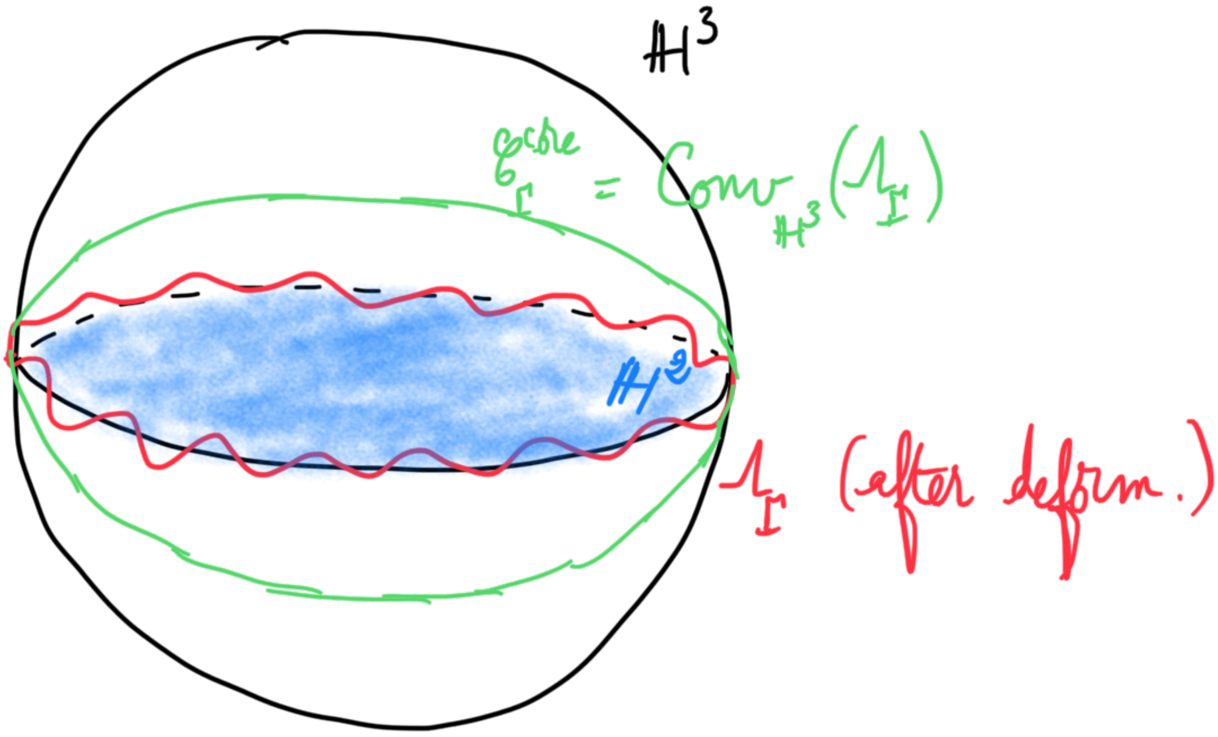
* cocompact lattices ($\mathcal{C}^{core}_\Gamma = G/K$).
* Schottky groups:
$\Gamma = \langle \gamma_1, \dots, \gamma_r \rangle$ playing ping pong on G/K (or $\partial_\infty(G/K)$)
$\rightsquigarrow$ nonabelian free group, discrete in G



* quasi-Fuchsian groups:

$\underset{\text{cocompact lattice}}{\Gamma <} PSL(2,\mathbb{R}) \hookrightarrow PSL(2,\mathbb{C})$ is CC in $PSL(2,\mathbb{C})$.

Deform a bit in $PSL(2,\mathbb{C})$ $\rightsquigarrow$ still CC.



② More on deformations.

* <u>Facts</u>: Γ_0 f.g. group.

Then
- $\mathrm{Hom}_{cc}(\Gamma_0, G)$ is open in $\mathrm{Hom}(\Gamma_0, G)$ (compact-open topology)
- $\Gamma_0 \xrightarrow{cc} G \hookrightarrow G'$, $\mathrm{rk}_{\mathbb{R}} G = 1$, $\mathrm{rk}_{\mathbb{R}} G' = 1$: $\Gamma_0 \to G'$ is still cc

$\rightsquigarrow \mathrm{Hom}_{cc}(\Gamma_0, G) \hookrightarrow \mathrm{Hom}_{cc}(\Gamma_0, G')$.

* E.g. (as above) $\Gamma_0 = \pi_1(S)$, S closed orientable surface of genus $g \geq 2$

$G = \mathrm{PSL}(2, \mathbb{R})$

$G' = \mathrm{PSL}(2, \mathbb{C})$

- $\mathrm{Hom}_{cc}(\Gamma_0, G) = \mathrm{Hom}_{\substack{inj.\\ discr.}}(\Gamma_0, G)$: two connected components of $\mathrm{Hom}(\Gamma_0, G)$

$\longrightarrow$ $\mathrm{Hom}_{cc}(\Gamma_0, G)/G$: two copies of $\underbrace{\mathrm{Teich}(S)}_{\{\text{marked hyperb. struct. on } S\}/\sim} \simeq \mathbb{R}^{6g-6}$

(action by conjugation)

- $\mathrm{Hom}_{cc}(\Gamma_0, G')$: open but not closed in $\mathrm{Hom}(\Gamma_0, G')$

$\longrightarrow$ $\mathrm{Hom}_{cc}(\Gamma_0, G')/G' \simeq \mathrm{Teich}(S) \times \mathrm{Teich}(S) \simeq \mathbb{R}^{12g-12}$
(Bers 1960)

* E.g. $\Gamma_0 = \pi_1(M)$, M closed hyperb. n-manifold $n \geq 3$

$G = SO(n,1)$

$G' = SO(n',1)$, $n' > n$

- $\mathrm{Hom}_{cc}(\Gamma_0, G)/G$ is trivial (Mostow rigidity)
- $\mathrm{Hom}_{cc}(\Gamma_0, G')/G'$ may not be

(bending deformations — Johnson-Millson).

③ Some classical characterisations.

(see survey of Kapovich-Leeb for more details, references, and more characterisations).

- **NB:** $\boxed{\partial_\infty(G/K) \simeq G/P}$, P proper parabolic subgroup of G.
- Fix basepoint $x_0 \in G/K$.

G	K	G/K	P
$PSL(2,\mathbb{R}) \simeq SO(2,1)_0$	$PSO(2)$	$\mathbb{H}^2$	$\left.\right\}$ $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$
$PSL(2,\mathbb{C}) \simeq SO(3,1)_0$	$PSU(2)$	$\mathbb{H}^3$	
$SO(n,1)$	$S(O(n)\times O(1))$	$\mathbb{H}^n$	$\left.\right\}$ $\mathrm{stab}_G\left(\text{isotropic line of } \mathbb{K}^{n,1}\right)$
$SU(n,1)$	$S(U(n)\times U(1))$	$\mathbb{H}^n_{\mathbb{C}}$	
$Sp(n,1)$	$Sp(n)\times Sp(1)$	$\mathbb{H}^n_{\text{quaternions}}$	

- **Fact:** Γ_0 f.g. group, $\rho : \Gamma_0 \to G$ representation.

TFAE: (i) ρ is CC

i.e. $\exists\ \mathcal{C}$ convex $\subset G/K$ with $\mathcal{C}/\rho(\Gamma_0)$ compact
$\quad \neq \emptyset$ $\rho(\Gamma_0)$-inv.

$$\left(\begin{array}{c} \Gamma_0 \overset{\rho}{\curvearrowright} \mathcal{C} \text{ is geometric} \\ \Longrightarrow \text{the orbital map} \\ \Gamma_0 \longrightarrow \mathcal{C} \\ \gamma \longmapsto \rho(\gamma)\cdot x_0 \\ \text{is a quasi-isometry} \end{array} \right)$$

(ii) ρ is a QI embedding

i.e. $\exists\, c, c' > 0$ s.t. $\forall \gamma \in \Gamma_0$,

$$d_{G/K}(x_0, \rho(\gamma)\cdot x_0) \geq c\, d_{Cay(\Gamma_0)}(e,\gamma) - c'$$

(iii) Γ_0 is word hyperbolic

<u>and</u> $\exists$ ρ-equiv. map $\xi : \partial_\infty \Gamma_0 \longrightarrow \partial_\infty(G/K) = G/P$

- continuous
- injective
- dynamics-preserving

i.e. $\forall \gamma \in \Gamma_0$ ∞-order, $\xi\begin{pmatrix}\text{attracting fixed point} \\ \text{of } \gamma \text{ in } \partial_\infty \Gamma_0\end{pmatrix} = \begin{pmatrix}\text{attracting f.-p.} \\ \text{of } \rho(\gamma) \text{ in } G/P\end{pmatrix}$

($\partial_\infty \Gamma_0$: Gromov boundary of Γ_0

$\longrightarrow$ circle if $\Gamma_0 = \pi_1(S)$

$\longrightarrow$ $\partial_\infty \tilde{M}$ if $\Gamma_0 = \pi_1(M)$, M closed neg.-curved

$\longrightarrow$ Cantor set if $\Gamma_0 = \mathbb{F}_r$)

(iv) Γ_0 is word hyperbolic

<u>and</u> $\Gamma_0 \underset{\rho}{\curvearrowright} G/K$ is well-displacing

i.e. $\exists c, c' > 0$ s.t. $\forall \gamma \in \Gamma_0$,

$$\underbrace{\begin{pmatrix}\text{transl. length} \\ \text{in } G/K\end{pmatrix}(\rho(\gamma))}_{= \lim\limits_{m \to +\infty} \frac{1}{m} d_{G/K}(x_0, \rho(\gamma)^m \cdot x_0)} \geq c \underbrace{\begin{pmatrix}\text{transl. length} \\ \text{in } Cay(\Gamma_0)\end{pmatrix}(\gamma)}_{\simeq \lim\limits_{m \to +\infty} \frac{1}{m} d_{Cay(\Gamma_0)}(e, \gamma^m)} - c'$$

$\simeq$ $\leftarrow$ up to bounded additive error

(v) ρ is finite-kernel and discrete

<u>and</u> any $z \in \Lambda_{\rho(\Gamma_0)}$ is a conical limit point

i.e. $\exists (\gamma_m) \in \Gamma_0^{\mathbb{N}}$ s.t. $\rho(\gamma_m)\cdot x_0 \longrightarrow z$

and $\sup_m d_{G/K}\left(\rho(\gamma_m)\cdot x_0, \mathrm{ray}(x_0,z)\right) < +\infty$

$\mathbb{H}^2$
(upper half-plane model)

(vi) ρ is finite-kernel and discrete

<u>and</u> $\Gamma_0 \overset{\rho}{\curvearrowright} G/P$ is expanding at $\Lambda_{\rho(\Gamma_0)}$

i.e. $\forall z \in \Lambda_{\rho(\Gamma_0)}$, $\exists\, \mathcal{U}$ neighborhood of z in G/P

$\exists\, \gamma \in \Gamma_0$

(see Sullivan 1985)

s.t. $\displaystyle \inf_{\substack{z_1 \neq z_2 \\ \text{in } \mathcal{U}}} \frac{d_{G/P}\left(\rho(\gamma)\cdot z_1, \rho(\gamma)\cdot z_2\right)}{d_{G/P}(z_1,z_2)} > 1.$

II. Case $\mathrm{rank}_{\mathbb{R}} G \geq 2$.

e.g. $G = \begin{cases} SL(d, \mathbb{K}), \\ PSL \\ PGL \end{cases}$ $\quad d \geq 3$

$\begin{cases} SO(p,q), \\ SU \end{cases}$ $\quad p,q \geq 2$

$Sp(2n, \mathbb{K}),$ $\quad n \geq 2$

Discrete subgroups of G ?

* <u>Lattices :</u>

Margulis : G has no compact factor

$\Longrightarrow$ all irreducible lattices of G are <u>superrigid</u> and <u>arithmetic</u>.

These groups are <u>not</u> word hyperbolic.

* <u>Another class:</u>

$\boxed{\text{Anosov subgroups}}$ = images of $\boxed{\text{Anosov representations}}$ $\rho : \Gamma_0 \to G$

where Γ_0 word hyperbolic group

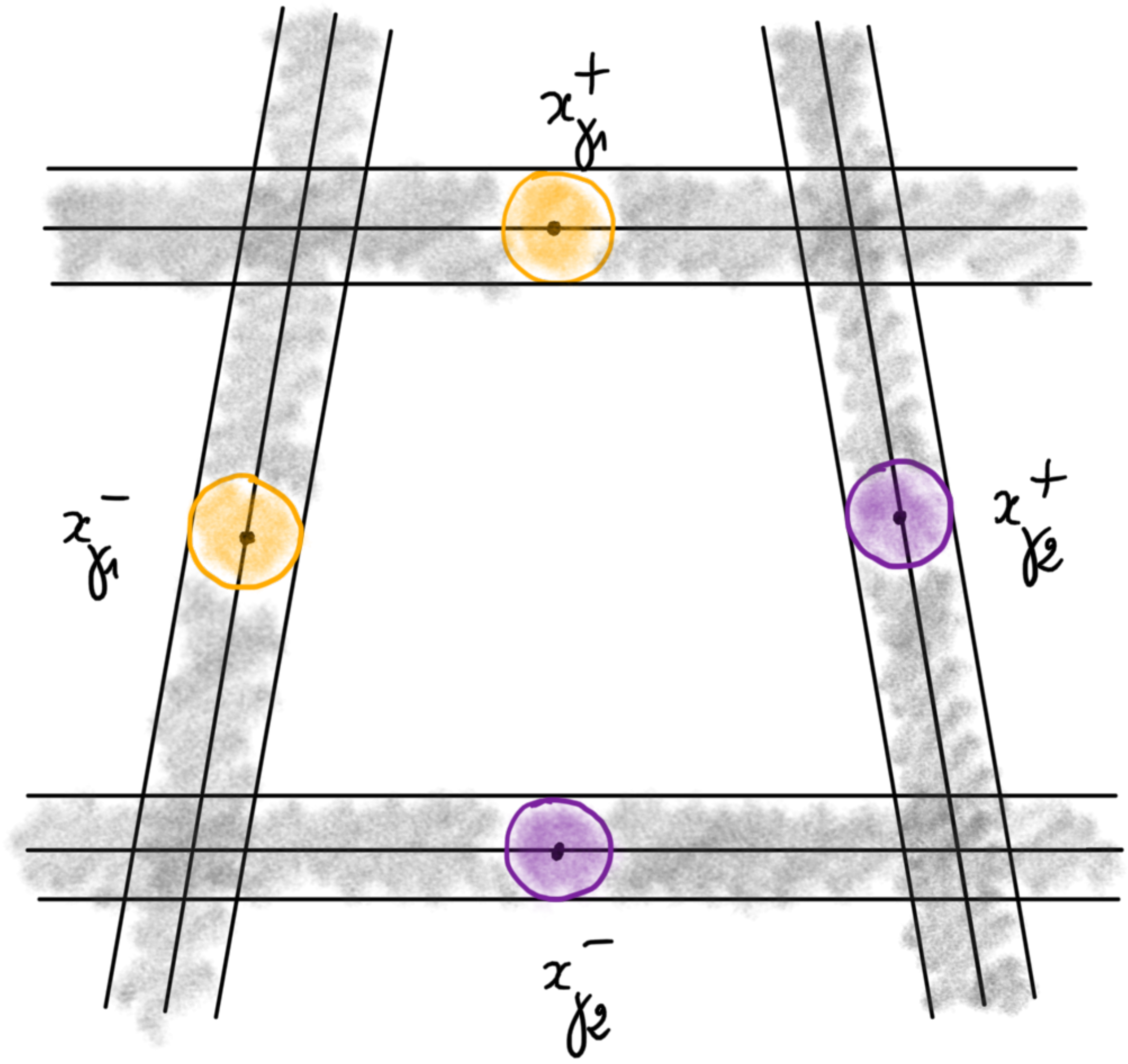
(e.g. $\pi_1(S)$, S closed hyp. surf.
$\pi_1(M)$, M closed neg.-curved mfd
$\mathbb{F}_r$)

<u>Property:</u> $\mathrm{Hom}_{\mathrm{Anosov}}(\Gamma_0, G)$ is open in $\mathrm{Hom}(\Gamma_0, G)$.

· <u>Ex.:</u> free groups playing ping pong in $\mathbb{P}(\mathbb{R}^d)$
(Tits, Benoist, Canary-Lee-Sambarino-Stover,
Kapovich-Leeb-Porti, ...)

e.g. $d=3$:
$$\gamma_1 = \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix} \in SL(3,\mathbb{R}),\ t \gg 1$$

γ_2 conjugate of γ_1
"transverse"



Observation: Any reduced word in $\gamma_1^{\pm 1}, \gamma_2^{\pm 1}$ sends the white region into the union of the four colored disks.

Consequence: $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ nonabelian free group, discrete in $SL(3,\mathbb{R})$.

↳ still true after small deformation.