Property (T) and a-T-menability from a geometrical viewpoint.

Plan for this series of talks

(1) Property (T)

(2) a-T-menability

(3) CAT(0) cubical complexes

Property (T) What / Why / Who?

What?

For G locally compact, second countable group TFAE

(1) The trivial representation is isolated in the unitary dual $\hat{G}$ for the Fell topology

(2) Any unitary representation with almost invariant vectors, has invariant vectors

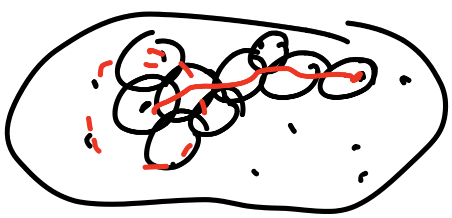
(3) Any isometric affine action on a Hilbert space has a fix point (also $H^1(G,\pi)=0$ $\forall \pi$)

(4) Any isometric action on a median space has a bounded orbit.

If any of those conditions hold, then G has property (T)

Why? Kazhdan 67 uses property (T) to show that all lattices in $SL_n\mathbb{R}$ are finitely generated.

That $SL_n\mathbb{Z} = \langle \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} = e_{ij},\ \sigma_{ij} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$ is f.g. is an easy computation and if $\Gamma < SL_n\mathbb{R}$ s.t. $SL_n\mathbb{R}/_{\Gamma}$ is compact, is



but for general lattices it was open

Thm: Let G be a countable discrete group with property (T), then G is finitely generated.

We'll see a proof using the first def of property (T). Recall that the unitary dual $\hat{G}$ = all unitary & irreducible repr.

$\tilde{G}$ = all unitary representations

$= \{\pi : G \longrightarrow \mathcal{U}(\mathcal{H}) = \{P : \mathcal{H} \to \mathcal{H}\ \|Pu\| = \|u\|\}$

continuous group homomorphism

On this set, the Fell topology is described as follows: given $\pi \in \tilde{G}$ and a triple $(\xi \in \mathcal{H}, K \subseteq G, \varepsilon > 0)$ a representation $\sigma \in \tilde{G}$ is in the (ξ, K, ε)-nbhd of π if $\exists \eta \in \mathcal{H}$ s.t.

$$|\langle \pi(g)\xi, \xi\rangle - \langle \sigma(g)\eta, \eta\rangle| < \varepsilon \quad \forall g \in K$$

The trivial representation is

$$T : G \longrightarrow \mathcal{U}(\mathcal{H}) \quad g \mapsto Id$$

<u>Proof</u>: Enumerate $G = \{g_1, g_2, \dots\}$ and set $\Gamma_n = \langle g_1, \dots, g_n\rangle$ want n s.t. $\Gamma_n = G$. If not, then $\Gamma_n < \Gamma_{n+1} < \dots$

$\lambda_n : G \longrightarrow \mathcal{U}(\ell^2(G/\Gamma_n))$ (G acts on G/Γ_n) semi-regular representation. so that, for any $\gamma \in \Gamma_n$ $\lambda_n(\gamma)$ will act trivially on $\ell^2(G/\Gamma_n)$. Now, take any $\xi \in \mathcal{H}$

any $\varepsilon > 0$ and

$\langle T(g)\xi, \xi \rangle$, so for any $\varepsilon > 0$ and

K compact in G — so finite if $n \geq |K|$

$K \subseteq \Gamma_n$ and $\lambda_n(\gamma)\eta = \eta \quad \forall \gamma \in K$

$\Rightarrow \lambda_n \in (\xi, K, \varepsilon)$-nbhd of T

↯ since T is isolated, hence at some point $\lambda_n = G$

<u>Back to some of the what</u> & almost invariant vectors

Given a representation $\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$

$Q \subseteq G$, $\varepsilon > 0$ a vector $\xi \in \mathcal{H}$ is called (Q, ε)-invariant if

$$\sup_{g \in Q} \| \pi(g)\xi - \xi \| < \varepsilon \|\xi\|$$

say that the representation $(\pi, \mathcal{H})$ has almost invariant vectors if it has (Q, ε)-invariant vectors $\forall\, Q \subseteq G$ compact and $\forall\, \varepsilon > 0$.

Say that $(\pi, \mathcal{H})$ has invariant vectors if $\exists\, \xi \neq 0$ s.t. $\pi(g)\xi = \xi \quad \forall g \in G$.

<u>Remark</u>: $\|\pi(g)\xi - \xi\| = \langle \pi(g)\xi - \xi, \pi(g)\xi - \xi\rangle$

$$= \|\pi(g)\xi\| + \|\xi\| - 2\langle \pi(g)\xi, \xi\rangle =$$

$$= 2\,(1 - \langle \pi(g)\xi, \xi\rangle|$$

$\langle T(g)\xi, \xi\rangle$

Who? Who has (T) and who doesn't?

- non-finitely generated discrete groups.
- finite groups
- $SL_n\mathbb{R}$, $SL_n\mathbb{Z}$: property (T) goes from lattices to the ambient group and vice-versa

sketch of why $SL_n\mathbb{R}$ has property (T)

<u>key proposition</u>: G locally compact, $N \lhd G$ abelian (here it will be $G = SL_2\mathbb{R} \ltimes \mathbb{R}^2$, $N = \mathbb{R}^2$)

If the only G-invariant mean on the Borel subsets of $\hat{N}$ is δ_1, ($G \curvearrowright \hat{N}$

... $\{1_N\}$ (...
dual to the conj. action). Then any π unit rep. of G with almost invariant vectors, has N-invariant vectors.

$\to$ this is called relative property (T) or also (T) for the pair (G, N)

For $G = SL_2\mathbb{R} \ltimes \mathbb{R}^2$ $N = \mathbb{R}^2$ this proposition follows from the Dirac mass at 0 being the only $SL_2\mathbb{R}$-invariant mean on the Borel subsets of $\mathbb{R}^2$

$$\begin{pmatrix} \boxed{SL_2\mathbb{R}} & \begin{matrix} * \\ * \end{matrix} \\ 0 \quad 0 & 1 \end{pmatrix} < SL_3\mathbb{R}$$

then get those $\mathbb{R}^2$-inv. vectors move them around to get $SL_3\mathbb{R}$-invariant vectors

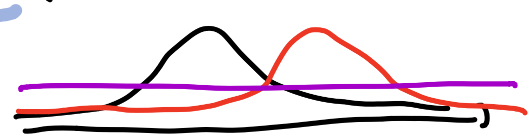
More groups with property (T)

Aut $\mathbb{F}_n$ $n \geq 5$ (recent results) Out$(\mathbb{F}_n)$

<u>Open</u> Mod(S_g) $g \geq 3$

Who doesn't have (T)? $\mathbb{Z}^n$, $\mathbb{Z}$

Let us see why the regular representation $\lambda : \mathbb{Z} \to \mathcal{U}(\ell^2(\mathbb{Z}))$ has almost invariant vectors: it won't have any invariant vector



want (Q,ε)-invariant vectors $\forall Q \subseteq \mathbb{Z}$ compact – I will assume $Q = [-k,k]$

$\varepsilon > 0$. Define $\xi = \frac{1}{\sqrt{n}} \mathbb{1}_{[0,n]}$ for n large enough so that $\frac{2k}{n} < \varepsilon^2$

for $g \in Q$

$$\|\lambda(g)\xi - \xi\|^2 = \sum_{z \in \mathbb{Z}} |\lambda(g)\xi(z) - \xi(z)|^2$$

$$\leq \frac{2k}{n} < \varepsilon^2$$

$\Rightarrow \xi$ is (Q,ε)-invariant

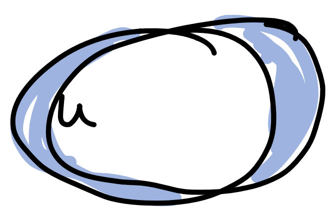
$\Rightarrow \mathbb{Z}$ does not have property (T)

The same proof works for the regular representation on any amenable group: here pick the Følner def:

(discrete) G amenable $\iff$ $\forall F \subseteq G$ finite $\forall \varepsilon > 0$

$$\exists\, U = U(F, \varepsilon) \text{ s.t. } \frac{|FU \Delta U|}{|U|} < \varepsilon$$

so then the exact same proof as for the case $\mathbb{Z}$ works



$$\xi = \mathbb{1}_U$$

Property (T) inherited by quotients gives many non property (T) groups

$\mathbb{F}_n \twoheadrightarrow \mathbb{Z}$.

Property (T) by Bekka, de la Harpe Valette criterion for (T) | Yann Ollivier

$\boxed{G \text{ has } (T) \iff \text{any } G\text{-inv. sym rw has a spectral gap}}$

G semisimple with (T) – higher rk

– $Sp(n,1)$, $F_{4(-20)}$

not $SO(n,1) \leftarrow K=-1$

$SU(n,1) \leftarrow -1 \le K \le -\frac{1}{2}$

$\mathbb{H}^2 \times \mathbb{R}$ $\quad$ $\mathbb{H}^2 \times \mathbb{H}^2$

$SU(n,1)$

Sphere $\mathbb{R}$

$\mathbb{H}^n$

$\mathbb{C}\mathbb{H}$

$Sl_n \mathbb{R}$

$\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$

$Mf(x) = \sum_{y \in X} \mu(x \to y) f(y)$ averaging operator

M will act on $\mathcal{H}$ so spectrum if $\operatorname{spec} M \subseteq \{-1\} \cup [-\sigma, \sigma] \cup \{1\}$

$\sigma < 1$

Gromov hyperbolic — $T < Sp(n,1)$ hyp. with (T)

— F_n hyp not (T)

Y_n :