MSRI virtual semester :
"Random and Arithmetic Structures in Topology".
Introductory workshop, 9 September 2020.
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$\boxed{\text{Anosov representations 2.}}$

Recap from yesterday:

G noncompact real semisimple Lie group

(e.g. $\begin{smallmatrix}SL\\PGL\end{smallmatrix}(d, \mathbb{K})$, $\begin{smallmatrix}SO\\SU\end{smallmatrix}(p,q)$, $Sp(2n, \mathbb{K})$, ...)

$\rightsquigarrow$ class of discrete subgroups of G :

$\boxed{\text{Anosov subgroups}}$ = images of $\boxed{\text{Anosov representations}}$

$\rho : \Gamma_0 \to G$

Γ_0 word hyperbolic group

(e.g. $\pi_1(S)$, S closed hyperb. surf.
$\pi_1(M)$, M closed, neg.-curved manifold
$\mathbb{F}_r$)

$\hookrightarrow$ introduced by Labourie (2006) for $\Gamma_0 = \pi_1(M)$,
generalized by Guichard–Wienhard (2012).

* When $\mathrm{rank}_{\mathbb{R}} G = 1$, Anosov subgroups
= convex cocompact subgroups
(CC)

$$G \rightsquigarrow \text{Riem. symmetric space } G/K \rightsquigarrow \partial_\infty G/K = G/P$$

(P proper parabolic subgroup of G)

$PSL(2,\mathbb{R})$	$\mathbb{H}^2$	$\mathbb{S}^1$
$PSL(2,\mathbb{C})$	$\mathbb{H}^3$	$\mathbb{S}^2$
$SO(n,1)$	$\mathbb{H}^n$	$\mathbb{S}^{n-1}$

- Def.: A discrete subgroup Γ of G is CC if $\exists\, \mathcal{C}$ convex $\subset G/K$ s.t. $\Gamma\backslash\mathcal{C}$ compact
 ($\mathcal{C} \neq \emptyset$, Γ-inv.)

$$\Longleftrightarrow \Gamma\backslash\mathcal{C}^{core}_\Gamma \text{ is compact, where } \mathcal{C}^{core}_\Gamma := \mathrm{Conv}_{G/K}(\Lambda_\Gamma)$$

limit set $\Lambda_\Gamma \subset G/P$
$= \{\text{accumulation points of } \Gamma\cdot x\}$
for some (any) $x \in G/K$.

A representation $\rho : \Gamma_0 \to G$ is CC if it has finite kernel and discrete, CC image.

- Ex.: cocompact lattices, Schottky groups, quasi-Fuchsian groups ...

- <u>Characterization:</u> $\rho : \Gamma_0 \to G$ is CC

$\iff$ Γ_0 is word hyperbolic

<u>and</u> $\exists$ ρ-equivariant map $\xi : \partial_\infty \Gamma_0 \to G/P$

- continuous
- injective
- dynamics-preserving

i.e. $\forall \gamma \in \Gamma_0$ ∞-order, $\xi\begin{pmatrix}\text{attracting fixed point} \\ \text{of } \gamma \text{ in } \partial_\infty\Gamma_0\end{pmatrix} = \begin{pmatrix}\text{attr. f.p.} \\ \text{of } \rho(\gamma) \text{ in } G/P\end{pmatrix}$

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* <u>today:</u> $\mathrm{rank}_{\mathbb{R}} G$ arbitrary.

I. Definition and basic properties.

<u>General setting:</u>

G semisimple Lie group

$\cup$

P parabolic subgroup

$\rightsquigarrow G/P$ flag variety.

<u>Today:</u>

$G = \overset{SL}{PGL}(d, \mathbb{K})$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

$\cup$

$P_i = \begin{array}{cc} & \overbrace{\quad}^{i} \\ i\{ & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array} \rightsquigarrow G/P_i = \mathrm{Gr}_i(\mathbb{K}^d)$

$1 \leq i \leq d-1$.

- <u>Def.:</u> Γ_0 word hyperbolic group.

A representation $\rho : \Gamma_0 \to G$ is <u>P_i-Anosov</u>

if $\exists$ ρ-equivariant maps $\xi_{i}, \xi_{d-i} : \partial_\infty \Gamma_0 \longrightarrow G/P_{i}, G/P_{d-i}$

- continuous
- transverse : $\forall \eta \neq \eta'$ in $\partial_\infty \Gamma_0$, $\xi_i(\eta) \oplus \xi_{d-i}(\eta') = \mathbb{K}^d$
- uniform contraction condition which strengthens dynamics-preserving.

- Basic properties : If ρ is P_i-Anosov, then

ⓐ ξ_i and ξ_{d-i} are unique

ⓑ ——————— injective ($\Rightarrow$ homeos onto image)

ⓒ ——————— compatible : $\xi_{\min(i,d-i)}(\eta) \subset \xi_{\max(i,d-i)}(\eta)$ $\forall \eta \in \partial_\infty \Gamma_0$

ⓓ ρ has finite kernel and discrete image.

Proof : * ⓐ + ⓒ : dynamics-preserving $\Rightarrow$ ξ_i and ξ_{d-i} are determined and compatible on {attracting f. p. of ρ}, which is a dense subset of $\partial_\infty \Gamma_0$. Use continuity.

* ⓑ : compatibility + transversality.

* ⓓ : action of Γ_0 on $\partial_\infty \Gamma_0$ $\left\{ \begin{array}{l} \text{has finite kernel } (\star) \\ \text{is a convergence action:} \end{array} \right.$

$\forall (\gamma_m) \in \Gamma_0^{\mathbb{N}}$ unbounded, up to subsequence, $\exists \eta^\pm \in \partial_\infty \Gamma_0$ s.t. $\gamma_m \cdot \eta \longrightarrow \eta^+$ $\forall \eta \in \partial_\infty \Gamma_0 \smallsetminus \{\eta^-\}$ unif. on compact sets

Consider $\begin{cases} \gamma \in \Gamma_0 . \\ (\gamma_m) \in \Gamma_0^{\mathbb{N}} . \end{cases}$

$$\begin{cases} \rho(\gamma) = \mathrm{id} \\ (\rho(\gamma_m))_{m\in\mathbb{N}} \text{ bounded in } G \end{cases} \qquad\qquad \begin{cases} \gamma \in \text{finite subset of } \Gamma_0 \\ (\gamma_m)_{m\in\mathbb{N}} \text{ bounded in } \Gamma_0 \end{cases}$$

$$\Downarrow \qquad\qquad\qquad\qquad\qquad\qquad \Uparrow (\star)$$

$$\begin{cases} \rho(\gamma) = \mathrm{id} \text{ as homeos} \\ (\rho(\gamma_m))_{m\in\mathbb{N}} \text{ bounded} \end{cases} \text{ of } G/P_i \text{, hence of } \xi_i(\partial_\infty \Gamma_0) \ \underset{\substack{\xi_i \text{ homeo.} \\ \text{onto its} \\ \text{image}}}{\Longrightarrow} \ \begin{cases} \gamma = \mathrm{id} \text{ as homeos} \\ (\gamma_m)_{m\in\mathbb{N}} \text{ bounded} \end{cases} \text{ of } \partial_\infty \Gamma_0 .$$

This shows that ρ has finite kernel & discrete image. □

- <u>Contraction condition</u> :

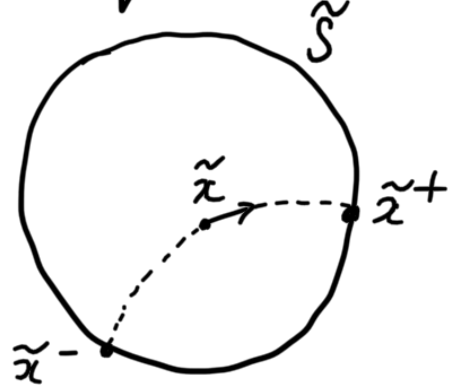
Say $\boxed{\Gamma_0 = \pi_1(S)}$, S closed hyperbolic surface.

$\rightsquigarrow$ "flow space" $\boxed{T^1 \tilde{S}}$

$$\cong$$

$$(\partial_\infty \tilde{S})^{(2)} \times \mathbb{R} .$$

(Hopf parametrization)



$*$ $\tilde{x} \in T^1 \tilde{S}$ $\rightsquigarrow$ decomp. $\mathbb{K}^d = \xi_i(\tilde{x}^+) \oplus \xi_{d-i}(\tilde{x}^-)$
(transversality)

* vector bundle $\Gamma_0 \backslash (T^1\tilde{S} \times \mathbb{K}^d) = V_i \oplus V_{d-i} \circlearrowleft$ flow (ψ_t)

$(\tilde{x}, w) \sim (\gamma \cdot \tilde{x}, \rho(\gamma) \cdot w)$

$\psi_t : [\tilde{x}, w] \mapsto [\varphi_t \cdot \tilde{x}, w]$ preserves decomposition.

$$\downarrow$$

$\Gamma_0 \backslash T^1\tilde{S} = T^1 S \circlearrowleft (\varphi_t)$ geodesic flow

where $(V_{i \atop d-i})_x = \left\{ [\tilde{x}, w] \mid w \in \xi_{i \atop d-i}(\tilde{x}^{\pm}) \right\}$

<u>Condition:</u> The flow (ψ_t) uniformly contracts V_i w.r.t. V_{d-i}, i.e. $\exists\, c, c' > 0$ s.t. $\forall t \geq 0$, $\forall x \in T^1 S$, $\forall v_{i \atop d-i} \in (V_{i \atop d-i})_x$ nonzero,

$$\frac{\|\psi_t \cdot v_i\|_{\varphi_t \cdot x}}{\|\psi_t \cdot v_{d-i}\|_{\varphi_t \cdot x}} \leq e^{-ct+c'} \frac{\|v_i\|_x}{\|v_{d-i}\|_x},$$

where $\|\cdot\|_x$ continuous family of norms on $(V_i)_x \oplus (V_{d-i})_x$.

(The cond. is independent of the choice of the continuous family of norms, by compactness of $T^1 S$.)

↳ <u>Similar condition for general word hyperbolic Γ_0:</u>
define flow space $(\partial_\infty \Gamma_0)^{(2)} \times \mathbb{R}$
with natural flow on $\mathbb{R}$-factor (translation by t)
and appropriate, geometric Γ_0-action
(Gromov, Mineyev, Champetier, ...).

• <u>NB</u>: $\ast$ P_i - Anosov $\Longleftrightarrow$ P_{d-i} - Anosov

$\ast$ contraction condition is still satisfied after small deformation (compactness of $T^1 S = \Gamma_0 \backslash (\text{flow space})$).

$\Rightarrow$ $\mathrm{Hom}_{P_i\text{-Anosov}}(\Gamma_0, G)$ is open in $\mathrm{Hom}(\Gamma_0, G)$.

II. Examples.

• free groups playing ping pong in $\mathbb{P}(\mathbb{R}^d)$ (see yesterday's lecture) are P_1 - Anosov.

Here is another way of producing examples:

• <u>Fact</u> (see Guichard - Wienhard 2012):

$$\begin{tikzcd} \Gamma_0 \arrow[r, hook, "cc"] \arrow[rr, bend right, dashed, "\rho_0"'] & \underset{\mathrm{rk}_{\mathbb{R}} G = 1}{G} \arrow[r, hook, "\tau"] & \overset{SL}{PGL}(d, \mathbb{R}) \end{tikzcd}$$

$$\rho_0 \text{ is } P_i\text{-Anosov} \Longleftrightarrow \left(\frac{i^{\text{th}} \ |\text{eigenvalue}|}{(i+1)^{\text{th}} \ |\text{eigenvalue}|}\right)\left(\tau\begin{pmatrix}\text{hyperbolic} \\ \text{element} \\ \text{of } G\end{pmatrix}\right) > 1.$$

In this case, τ induces embeddings $\partial \tau_{\substack{i \\ d-i}} : G/P \hookrightarrow Gr_{\substack{i \\ d-i}}(\mathbb{R}^d)$ and $\xi_{\substack{i \\ d-i}} = \partial\tau_{\substack{i \\ d-i}} \circ \xi$ where $\xi : \partial_\infty \Gamma_0 \to G/P$ boundary map of the CC repr.!

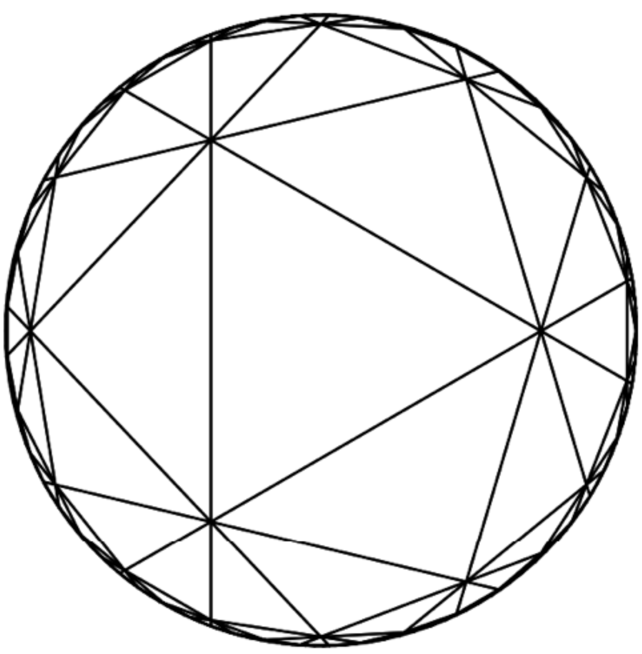
- <u>Ex. 1</u> :

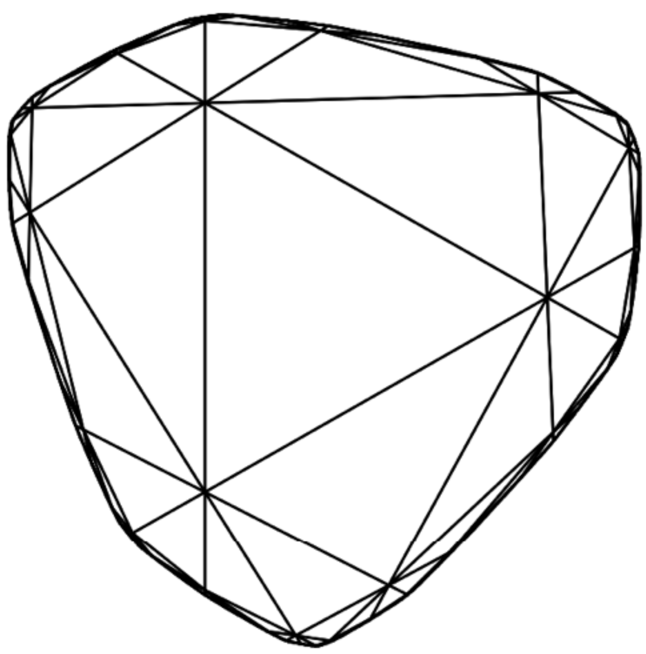
$$\begin{tikzcd} \Gamma_0 \arrow[r, hook, "cc"] \arrow[rr, bend right, dashed] & PO(n,1) \arrow[r, hook] & PGL(n+1,\mathbb{R}) \end{tikzcd}$$

ρ_0 is P_1-Anosov.

Any small enough deformation of ρ_0 in $\mathrm{Hom}(\Gamma_0, PGL(n+1,\mathbb{R}))$ is still P_1-Anosov. If the image of Γ_0 is cocompact in $PO(n,1)$, then the whole connected component of ρ_0 in $\mathrm{Hom}(\Gamma_0, PGL(n+1,\mathbb{R}))$ consists of P_1-Anosov representations ρ (Choi–Goldman 1993 for $n=2$, Benoist 2005 in general).

<u>Idea</u>: $\rho(\Gamma_0)$ acts properly discontinuously and cocompactly on ("divides") a strictly convex open subset Ω of $\mathbb{P}^n(\mathbb{R})$ (for ρ_0, Ω is the projective model of $\mathbb{H}^n$).





Fix S closed orientable surface of genus $g \geq 2$.

- <u>Ex. 2:</u>

$$\begin{tikzcd}
\Gamma_0 = \pi_1(S) \arrow[r, hook, "\text{inj.}", "\text{discr.}"'] \arrow[rr, dashed, bend right] & SL(2,\mathbb{R}) \arrow[r, hook, "\begin{pmatrix} * & 0 \\ 0 & I \end{pmatrix}"', "\text{(standard embedding)}"'] & SL(d,\mathbb{R})
\end{tikzcd}$$

ρ_0 is P_1-Anosov.

Any small enough deformation of ρ_0 in $SL(d,\mathbb{R})$ is still P_1-Anosov. (see Barbot for $d=3$)

- <u>Ex. 3:</u>

$$\begin{tikzcd}
\Gamma_0 = \pi_1(S) \arrow[r, hook, "\text{inj.}", "\text{discr.}"'] \arrow[rr, dashed, bend right] & SL(2,\mathbb{R}) \arrow[r, "\text{irred.}", "\tau_d"'] & SL(d,\mathbb{R})
\end{tikzcd}$$

ρ_0 is P_i-Anosov $\forall i$

(τ_d can be realised by: $\mathbb{R}^d \simeq \mathbb{R}[X,Y]_{\substack{\text{homog.} \\ \text{deg. } d-1}} \circlearrowleft SL(2,\mathbb{R})$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(X,Y) = P(aX+bY, cX+dY)$).

The whole connected component of ρ_0 in $\mathrm{Hom}(\Gamma_0, SL(d,\mathbb{R}))$ consists of P_i-Anosov representations $\forall i$

(Choi - Goldman 1993 for $d=3$, $\begin{cases} \text{Labourie 2006 in general} \\ \text{Fock - Goncharov 2006} \end{cases}$).

$\hookrightarrow$ "<u>Hitchin component</u>":

modulo conjugation, $\simeq \mathbb{R}^{(d^2-1)(2g-2)}$ (Hitchin 1992).

• Ex. 4 : $\Gamma_0 = \pi_1(S) \xrightarrow[\text{discr.}]{\text{inj.}} PSL(2,\mathbb{R}) \simeq SO(2,1)_0 \hookrightarrow SO(2,q)$,

ρ_0 is P_1 - Anosov

The whole connected component of ρ_0 in $\mathrm{Hom}(\Gamma_0, SO(2,q))$ consists of P_1 - Anosov representations (see, more generally, Burger - Iozzi - Wienhard 2010).

→ "maximal representations"

(maximize a topological invariant, the Toledo number, which generalizes the Euler number).

• NB : The connected components in Ex. 3 and 4, modulo conjugation, are higher Teichmüller spaces.

Analogies with Teich(S) include :

* topology (Hitchin)
* proper action of Mod(S) (Wienhard, Labourie)
* good systems of coordinates generalizing those on Teich(S) (Goldman, Fock - Goncharov, Bonahon - Dreyer, Zhang, Strubel)
* analytic Mod(S) - invariant Riemannian metric (Bridgeman - Canary - Labourie - Sambarino, Pollicott - Sharp)

* versions of the Collar Lemma for the associated locally symmetric spaces (Lee-Zhang, Burger-Pozzetti)

etc.

→ see surveys by Burger-Iozzi-Wienhard (2010), Wienhard (ICM 2018), Pozzetti (Bourbaki 2019).

III. Characterizations.

$g \in GL(d,\mathbb{R}) \rightsquigarrow \sigma_1(g) \geq \dots \geq \sigma_d(g)$: singular values of g ($= \sqrt{\text{eigenvalues of } g^t g}$)

$\varepsilon_1(g) \geq \dots \geq \varepsilon_d(g)$: |eigenvalues| of g.

- Easy fact: Anosov representations $\rho : \Gamma_0 \longrightarrow G = \begin{smallmatrix} SL \\ PGL \end{smallmatrix}(d,\mathbb{R})$ are
  * QI embeddings: $\exists c, c' > 0$ s.t. $\forall \gamma \in \Gamma_0$,
$$\underbrace{d_{G/K}(x_0, \rho(\gamma)\cdot x_0)}_{\sqrt{\sum_{i=1}^{d-1} \log \frac{\sigma_i}{\sigma_{i+1}}(\rho(\gamma))}} \geq c\, d_{Cay(\Gamma_0)}(e, \gamma) - c'$$
  * well-displacing: $\exists c, c' > 0$ s.t. $\forall \gamma \in \Gamma_0$,
$$\underbrace{\begin{pmatrix} \text{translation length} \\ \text{in } G/K \end{pmatrix}(\rho(\gamma))}_{\sqrt{\sum_{i=1}^{d-1} \log \frac{\varepsilon_i}{\varepsilon_{i+1}}(\rho(\gamma))}} \geq c \begin{pmatrix} \text{translation length} \\ \text{in } Cay(\Gamma_0) \end{pmatrix}(\gamma) - c'$$

How much better are they?

(i) ⇔ (ii) ⇔ (iv) and other characterizations (i) ⇔ (ii) ⇔ (iii)

- **Thm** (Kapovich–Leeb–Porti, Guéritaud–Guichard–Kassel–Wienhard, Bochi–Potrie–Sambarino, Kassel–Potrie)

alternative proof of (i) ⇔ (iv) (iv) ⇔ (v)

Γ_0 word hyperbolic group, $\rho : \Gamma_0 \to G = \mathrm{SL}/\mathrm{PGL}(d,\mathbb{R})$ repr., $1 \leq i \leq d-1$.

TFAE:

(i) ρ is P_i-Anosov

(ii) $\exists$ ρ-equivariant maps $\xi_{i}, \xi_{d-i} : \partial_\infty \Gamma_0 \to G/P_{i}, G/P_{d-i}$ continuous, transverse, dynamics-preserving and $\dfrac{\sigma_i}{\sigma_{i+1}}(\rho(\gamma)) \xrightarrow[d_{\mathrm{Cay}(\Gamma_0)}(e,\gamma)\to+\infty]{} +\infty$

(iii) $\exists$ ρ-equivariant maps $\xi_{i}, \xi_{d-i} : \partial_\infty \Gamma_0 \to G/P_{i}, G/P_{d-i}$ continuous, transverse, dynamics-preserving and $\dfrac{\varepsilon_i}{\varepsilon_{i+1}}(\rho(\gamma)) \xrightarrow[\left(\substack{\text{transl.}\\ \text{length}\\ \text{in } \mathrm{Cay}(\Gamma_0)}\right)(\gamma)\to+\infty]{} +\infty$

(iv) $\exists\, c, c' > 0$ s.t. $\forall \gamma \in \Gamma_0$,
$$\log \frac{\sigma_i}{\sigma_{i+1}}(\rho(\gamma)) \geq c\, d_{\mathrm{Cay}(\Gamma_0)}(e,\gamma) - c'$$

(v) $\exists\, c, c' > 0$ s.t. $\forall \gamma \in \Gamma_0$,
$$\log \frac{\varepsilon_i}{\varepsilon_{i+1}}(\rho(\gamma)) \geq c \begin{pmatrix}\text{transl.}\\ \text{length}\\ \text{in } \mathrm{Cay}(\Gamma_0)\end{pmatrix}(\gamma) - c'$$

NB: For any finitely generated group Γ_0, condition (iv) implies that Γ_0 is word hyperbolic (Kapovich - Leeb - Porti, Bochi - Potrie - Sambarino).

Kapovich - Leeb - Porti also established other characterizations, including generalizations of classical rank-1 characterizations such as conicality of the limit set or expansion at the limit set.

- Question: Can Anosov representations also be characterized by some geometric convex cocompactness condition?

  ↪ yes: see next lecture.