Property (T) and aTmenability from a geometrical viewpoint.

Plan for this series of talks

(1) Property (T)
(2) aTmenability
(3) CAT(0) cubical complexes

<u>Property (T)</u> What / Why / Who? $\to$ $SL_n\mathbb{Z}$, not $\mathbb{Z}^n$, not $\mathbb{F}_n$

↳ lattices are f.g.

What?

For G locally compact, second countable group TFAE

(1) The trivial representation is isolated in the unitary dual $\hat{G}$ for the Fell topology

(2) Any unitary representation with almost invariant vectors, has invariant vectors

(3) Any isometric affine action on a Hilbert space has a fix point (also $H^1(G,\pi)=0\ \forall\pi$)

(4) Any isometric action on a median space has a bounded orbit.

If any of those conditions hold, then G has property (T)

Isometric affine action on a Hilbert space

$$\alpha : G \longrightarrow \mathrm{Isom}(\mathcal{H}) = \mathcal{H} \rtimes \mathcal{U}(\mathcal{H})$$

group ~~homomorphism~~

$$\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[drr, "\pi \text{ unitary representation}"] \arrow[dr, "b"'] & \mathcal{H} \rtimes \mathcal{U}(\mathcal{H}) \arrow[d] \arrow[dr] & \\
& \mathcal{H} & \mathcal{U}(\mathcal{H})
\end{tikzcd}$$

b is not a homomorphism but since α is an action, b will satisfy

$$b(gh) = b(g) + \pi(g)\, b(h) \quad \text{(cocycle relation)}$$

if the action has fix point : $\xi \in \mathcal{H}$ s.t $\alpha(g)\xi = \xi$, then $\alpha(g)(\xi) = \pi(g)\xi + b(g)$
$$= \xi$$

so $b(g) = \xi - \pi(g)\xi$ – this is a coboundary

$$H^1(G,\pi) = Z^1(G,\pi) \,/\, B^1(G,\pi)$$
$$\parallel$$
$$\{0\}$$

$Z^1(G,\pi)$ ← cocycles

$B^1(G,\pi)$ ← coboundaries

Equivalence between (2) and (3)

$H^1(G,\pi) = \{0\} \Rightarrow$ property (T) Guichardet

$\Uparrow$ if this Hausdorff & π has almost inv vect.

$\Rightarrow$ find inv. vectors.

property (T) $\Rightarrow H^1(G,\pi) = 0$ Delorme

$\mathcal{H}$, $\forall t > 0$ $\mathcal{H}_t \rightsquigarrow \bigoplus \mathcal{H}_t \leftarrow$

<u>Remark</u>: λ reg. repr. never has invariant vectors

$\lambda : G \to \mathcal{U}(\ell^2 G)$ (unless G finite)

but $H^1(G,\lambda) = \{0\}$ if G has (T)

but fix points for any affine action.

<u>Proposition</u>

G locally compact, 2nd countable group TFAE

(1) G admits a proper action on a Hilbert space

$\left(\|\alpha(g)\xi - \xi\| \xrightarrow[g\to\infty]{} \infty \right)$

(2) $\exists$ a unitary representation π of G with almost invariant vectors ($\pi \in \tilde{G}$ & $T \in \overline{\pi}$) and C_0 (matrix coefficients $\langle \pi(g)\xi, \eta\rangle \xrightarrow[g\to\infty]{} 0$ $\forall \xi,\eta$)

(3) $\exists \Psi : G \to \mathbb{R}^+$ conditionally negative

definite and proper $\left(\lim_{g\to\infty} \psi(g) = \infty\right)$

$\hookrightarrow \psi(g) = \psi(g^{-1})$ and

$\sum_{i,j} a_i a_j \psi(g_i^{-1} g_j) \leq 0$

$\forall a_1 \dots a_n \in \mathbb{C}$ s.t $\sum a_i = 0$

$g \mapsto \|b(g)\|^2$

(4) $\exists$ sequence $(\phi_n)_{n\in\mathbb{N}}$ of continuous, normalized ($\phi_n(1) = 1$) positive definite functions on G vanishing at ∞ on G and $\longrightarrow 1$ uniformly on compact sets.

(5) G admits a proper action on a median space

If G satisfies one of those equivalent conditions, G is called a-T-menable (G has the Haagerup property)

Who?

all amenable groups: λ the reg representation has almost inv. vectors and is C_0

Valette & co: explicit proper action on a Hilbert space using Følner sequences

$SO(n,1)$ $SU(n,1)$ wr.prod , $\mathbb{F}_n$

<u>open</u>: $\mathrm{Mod}(S_g)$ $g \geq 3$ Braid groups

NOT groups with property (T)

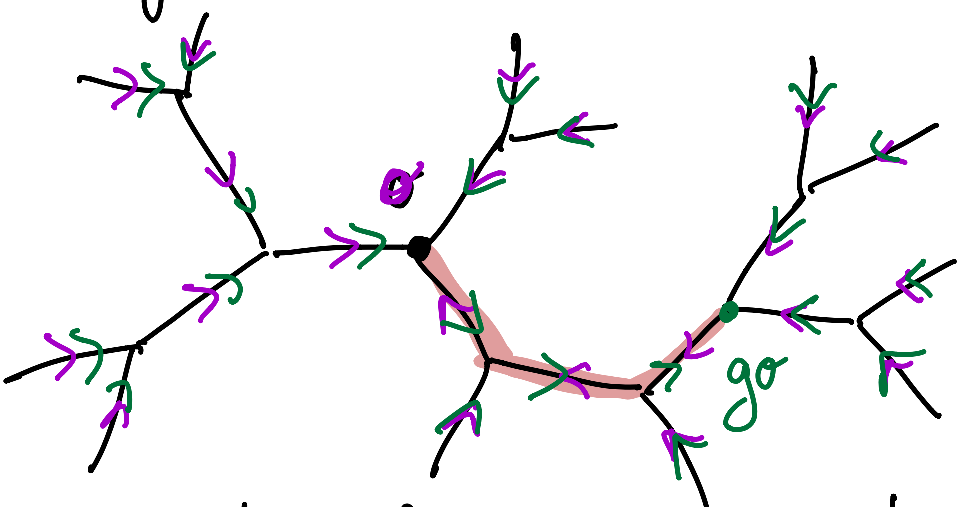
NOT $SL_2\mathbb{Z} \ltimes \mathbb{Z}^2$

<u>$\mathbb{F}_n$ is aTmenable</u> (any G discrete acting properly on a tree $T = (T^0, T^1)$

$G \curvearrowright T^0$

$T^0 \times T^0$

$\rightsquigarrow \pi$



we shall define a proper action of G on $\ell^2(T^0 \times T^0) = \mathcal{H}$

we need $b : G \longrightarrow \mathcal{H}$ proper

$$\underline{\xi_o} : T^0 \times T^0 \longrightarrow \mathbb{R}, \quad (x,y) \longmapsto \begin{cases} 1 & \text{is } \{x,y\} \text{ edge} \to o \\ 0 & \text{otherwise} \end{cases}$$

obviously $\xi_o \notin \ell^2(T^0 \times T^0)$

$b(g) = \xi_o - \underbrace{\pi(g)\xi_o}_{\xi_{g.o}}$ has finite support on $T^0 \times T^0$

$b(g)$ is a cocycle

$\|b(g)\| \sim d(o, go) \xrightarrow[g \to \infty]{} \infty$

because we assumed that G acts properly on T.

Let (X,d) be a metric space

$\forall x, y \in X$ denote by $[x,y]$ the interval between x and y, namely

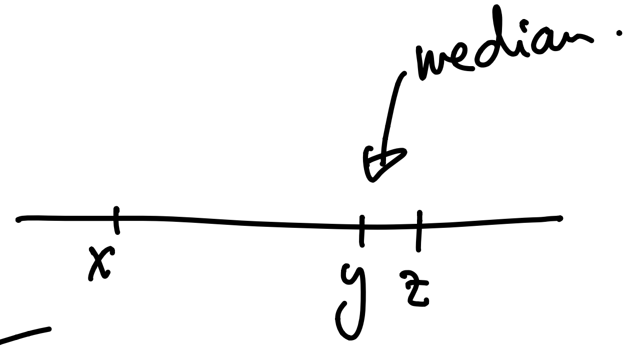
$[x,y] = \{ t \in X \text{ s.t. } d(x,t) + d(t,y) = d(x,y) \}$

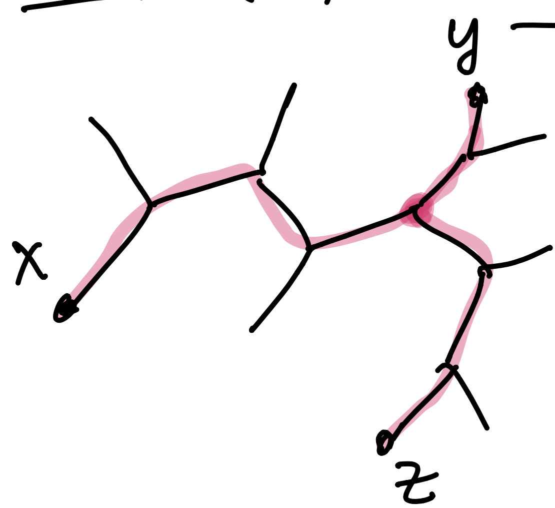
<u>Def</u>: (X,d) is a median space if $\forall x, y, z \in X$ then

$$[x,y] \cap [y,z] \cap [z,x] = \{m\}$$

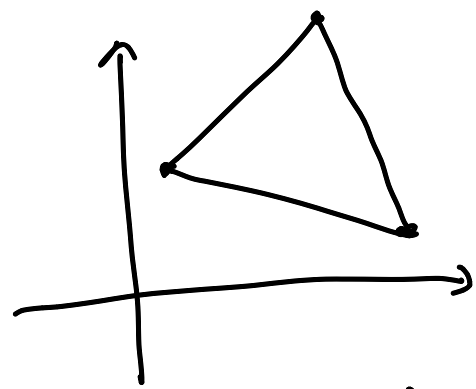
$m = m(x,y,z)$ is called the median (of x, y, z).

Ex: $(\mathbb{R}, |\,|)$

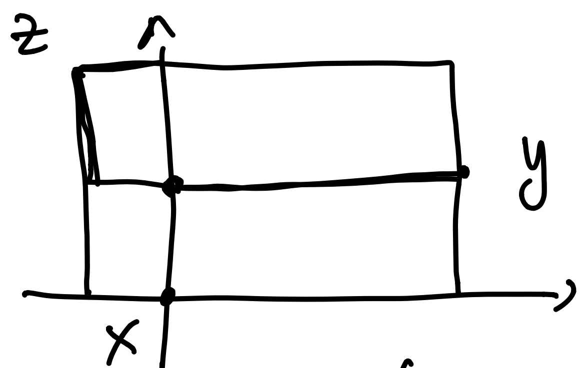




not $(\mathbb{R}^2, \text{eucl})$



but $(\mathbb{R}^2, \ell^1)$

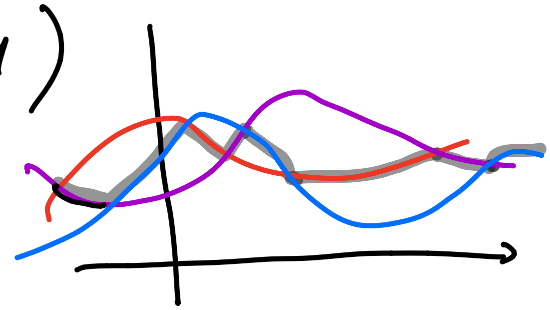


if (X_1, d_1)
(X_2, d_2)
are median spaces then
$(X_1 \times X_2, d_1 + d_2)$
is also a median space.

• If (X, μ) is a measure space

then $L^1(X,\mu)$ is a median space

(for $\|f\|_1 = \int_X |f| \, d\mu$)



- (X,μ) measure space

$\mathcal{M}(X) = \{$measurable subsets of X, finite measure$\}$

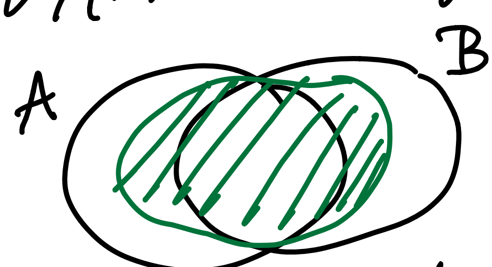
$pd(A,B) = \mu(A \Delta B)$

$(\mathcal{M}(X), pd)$ pseudo-metric space

$\overline{\mathcal{M}}(X)$ metric space

$C \in [A,B] \iff$

$A \cap B \subseteq C \subseteq A \cup B$

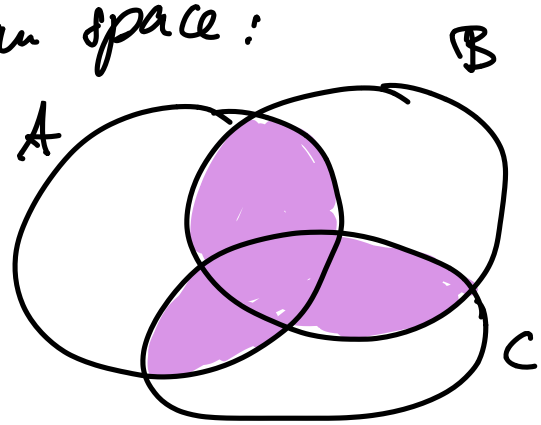


this is a median space:

$m(A,B,C) =$

$(A \cap B) \cup (B \cap C) \cup (C \cap A)$

$A \Delta B \Delta C$



tomorrow : CAT(0) cubical complexes

↳ median spaces