MSRI virtual semester:
"Random and Arithmetic Structures in Topology".
Introductory workshop, 11 September 2020.
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Anosov representations 3.

Recall from previous lectures:

$$\begin{cases} G & \text{noncompact real semisimple Lie group} \\ P & \text{parabolic subgroup of } G \\ \Gamma_0 & \text{word hyperbolic group, with Gromov boundary } \partial_\infty \Gamma_0 \end{cases}$$

$\left(\text{e.g. } \Gamma_0 = \begin{cases} \pi_1(S), & S \text{ closed hyperbolic surface} \\ \mathbb{F}_r & \text{nonabelian free group} \end{cases} \rightsquigarrow \partial_\infty \Gamma_0 \begin{cases} \text{circle} \\ \text{Cantor set} \end{cases} \right)$

$\rightsquigarrow$ notion of <u>P-Anosov representation</u> $\Gamma_0 \to G$.

For simplicity:

$$\boxed{\begin{array}{l} G = SL/PGL\,(d, \mathbb{K}), \qquad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C} \\[1em] P = P_i = \begin{pmatrix} \overbrace{*}^{i} & * \\ 0 & * \end{pmatrix}\Big\}\, i, \qquad 1 \leq i \leq d-1. \end{array}}$$

· <u>Def.</u> (Labourie 2006, Guichard–Wienhard 2012):

A representation $\rho : \Gamma_0 \to G$ is <u>P_i-Anosov</u>

if $\exists$ ρ-equivariant maps $\xi_i / \xi_{d-i} : \partial_\infty \Gamma_0 \to G/P_i \,/\, G/P_{d-i}$ ("boundary maps")

- continuous
- transverse: $\forall \eta \neq \eta'$ in $\partial_\infty \Gamma_0$, $\xi_i(\eta) \oplus \xi_{d-i}(\eta') = \mathbb{K}^d$

– satisfying a <u>uniform contraction contraction</u>

which strengthens the fact that ξ_i and ξ_{d-i} are dynamics-preserving: $\xi_{i \,/\, d-i}\begin{pmatrix}\text{attracting fixed point}\\ \text{of } \gamma \text{ in } \partial_\infty \Gamma_0\end{pmatrix} = \begin{pmatrix}\text{attr. f.p.}\\ \text{of } \rho(\gamma) \text{ in } G/P_{i \,/\, d-i}\end{pmatrix}$

$\forall \gamma \in \Gamma_0$ ∞-order,

- <u>Properties</u>:

  * ξ_i and ξ_{d-i} are unique, injective ($\Rightarrow$ homeo onto image), compatible (i.e. $\forall \eta \in \partial_\infty \Gamma_0$, $\xi_{\min(i,d-i)}(\eta) \subset \xi_{\max(i,d-i)}(\eta)$).
  * Anosov representations have finite kernel and discrete image.
  * P_i - Anosov $\iff$ P_{d-i} - Anosov.
  * $\mathrm{Hom}_{P_i\text{-Anosov}}(\Gamma_0, G)$ is open in $\mathrm{Hom}(\Gamma_0, G)$.

- <u>Examples</u>:

  * when $\mathrm{rank}_{\mathbb{R}} G = 1$ (i.e. $d=2$ for $G = PGL(d, \mathbb{K})$), there is only one notion of Anosov, and it is equivalent to convex cocompactness in G/K.

  * $$\begin{tikzcd} \Gamma_0 \arrow[r, "\sigma_0"] \arrow[rr, bend right, dashed, "\rho_0"'] & PO(n,1) \arrow[r, hook] & PGL(n+1, \mathbb{R}) \end{tikzcd}.$$

    Compatibility: $\left[\sigma_0 \text{ is CC}\right]$ (i.e. Anosov) $\iff$ $\left[\begin{matrix}\Gamma_0 \text{ is word hyperbolic}\\ \underline{\text{and}}\ \rho_0 \text{ is } P_1\text{-Anosov}\end{matrix}\right]$.

* $$\begin{tikzcd} \Gamma_0 = \pi_1(S) \arrow[r, hook, "\text{inj.}", "\text{discr.}"'] \arrow[rr, dashed, bend right, "\rho_0"'] & PSL(2,\mathbb{R}) \arrow[r, "\text{irred.}"] & PGL(d,\mathbb{R}) \end{tikzcd}$$ is P_i-Anosov $\forall i$.

In this case the whole connected component of ρ_0 in $\mathrm{Hom}(\Gamma_0, PSL(d,\mathbb{R}))$ consists of P_i-Anosov repr. $\forall i$ ("Hitchin representations").

(Choi-Goldman for $d=3$
Labourie in general)
(see also Fock-Goncharov).

* free groups playing ping pong in $\mathbb{P}(\mathbb{R}^d)$ are P_1-Anosov.

- Anosov representations admit a number of characterisations that generalise the classical characterisations of rank-1 convex cocompact representations.

  → <u>Question:</u> Can Anosov representations also be characterised by some geometric convex cocompactness condition?

I. Anosov representations and convex cocompactness.

- <u>Def.:</u> $\rho : \Gamma_0 \to PO(n,1) = \mathrm{Isom}(\mathbb{H}^n)$.

  Say action $\Gamma_0 \curvearrowright \mathbb{H}^n$ is <u>CC</u> if it is properly discontinuous <u>and</u> $\exists \underset{\neq\emptyset \ \ \rho(\Gamma_0)\text{-inv.}}{\mathcal{C}}$ convex $\subset \mathbb{H}^n$ with $\rho(\Gamma_0)\backslash\mathcal{C}$ compact.

• Guiding fact: (see above)

$$\left[\Gamma_0 \underset{\rho}{\curvearrowright} \mathbb{H}^n \text{ is CC}\right] \iff \left[\begin{array}{l}\Gamma_0 \text{ is word hyperbolic } \underline{\text{and}} \\ \rho : \Gamma_0 \to PO(n,1) \hookrightarrow PGL(n+1,\mathbb{R}) \text{ is } P_1\text{-Anosov}\end{array}\right].$$

→ What about representations $\rho : \Gamma_0 \to PGL(n+1,\mathbb{R})$ that do not factor through $PO(n,1)$?

• 1st attempt: $\mathbb{H}^n \overset{\text{generalise}}{\rightsquigarrow} G/K$ Riem. symmetric space where K max. compact subgroup

$(G = PGL(d,\mathbb{R}) \rightsquigarrow K = PO(d))$

Thm (Kleiner–Leeb, Quint): Suppose $d \geq 3$ (more generally: G simple with $\mathrm{rank}_{\mathbb{R}} G \geq 2$).

For $\rho : \Gamma_0 \to G$,

$$\left[\Gamma_0 \underset{\rho}{\curvearrowright} G/K \text{ is CC}\right] \underset{\text{generically}}{\Longrightarrow} \left[\rho(\Gamma_0) \text{ is a cocompact lattice in } G\right].$$

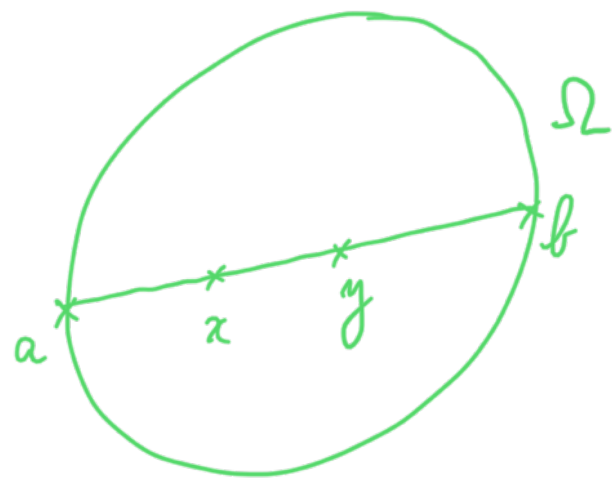
(i.e. whenever $\rho(\Gamma_0)$ is Zariski-dense in G)

In particular, such a Γ is not an Anosov subgroup.

• 2nd attempt: $\mathbb{H}^n = \{[x] \in \mathbb{P}(\mathbb{R}^{n+1}) \mid x_1^2 + \dots + x_n^2 - x_{n+1}^2 < 0\}$

$\overset{\text{generalise}}{\rightsquigarrow} \Omega \ \underbrace{\text{properly convex}}_{\text{convex and bounded in some affine chart.}} \text{ open } \subset \mathbb{P}(\mathbb{R}^d)$

* NB: On Ω there is still a natural proper metric d_Ω which is preserved by any $\gamma \in PGL(d,\mathbb{R})$ preserving Ω.



$$d_\Omega(x,y) = \frac{1}{2}\log \underbrace{[a:x:y:b]}_{\substack{\text{cross ratio,}\\ \text{normalized so that}\\ [0:1:t:\infty]=t.}}$$

(coincides with the hyperbolic metric for $\Omega = \mathbb{H}^{d-1}$, but in general only a Finsler metric, not Riemannian).

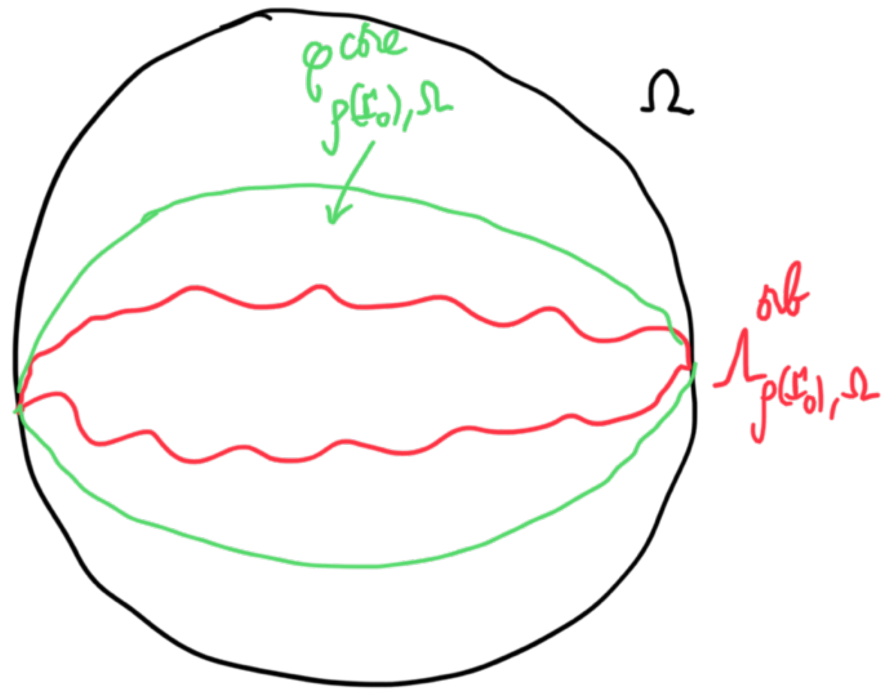
In particular, if $\Gamma \underset{\text{discrete}}{<} PGL(d,\mathbb{R})$ preserves Ω, then $\Gamma \curvearrowright \Omega$ is properly discontinuous.

* <u>Def.</u>: Suppose $\partial\Omega \not\supset$ segment ("Ω is strictly convex").
Let $\rho : \Gamma_0 \to \mathrm{Aut}(\Omega) \subset PGL(d,\mathbb{R})$.
Say action $\Gamma_0 \underset{\rho}{\curvearrowright} \Omega$ is <u>CC</u> if it is properly discontinuous <u>and</u> $\exists\, \underset{\substack{\neq\\ \emptyset}}{\mathcal{C}}$ convex $\subset \Omega$ $\rho(\Gamma_0)$-inv. with $\rho(\Gamma_0)\backslash\mathcal{C}$ compact

$\iff \Gamma \backslash \mathcal{C}^{core}_{\rho(\Gamma_0),\Omega}$ is compact, where $\mathcal{C}^{core}_{\rho(\Gamma_0),\Omega} := \mathrm{Conv}_\Omega(\Lambda^{orb}_{\rho(\Gamma_0),\Omega})$.

Here $\Lambda^{orb}_{\rho(\Gamma_0),\Omega} \subset \partial\Omega$ is the <u>orbital limit set</u>
$= \{$accumulation points of $\rho(\Gamma_0)\cdot x\}$ for some $x \in \Omega$.
(independent of x because $\partial\Omega \not\supset$ segment)

<u>Idea</u>: If $\begin{cases} \rho(\gamma_m)\cdot x \longrightarrow z \in \partial\Omega \\ \rho(\gamma_m)\cdot x' \longrightarrow z' \in \partial\Omega \end{cases}$,
then $[z,z'] \subset \partial\Omega$ by proper discontinuity, hence $z = z'$.



(first for $\rho : \Gamma_0 \to PO(p,q)$, then general case) (assuming irreducibility).

* **Thm (Danciger–Guéritaud–Kassel, A. Zimmer):**

Γ_0 finitely generated group, $\rho : \Gamma_0 \to PGL(d,\mathbb{R})$ representation.

Suppose $\rho(\Gamma_0)$ preserves a properly convex open subset of $\mathbb{P}(\mathbb{R}^d)$.

TFAE: ① Γ_0 is word hyperbolic and ρ is P_1-Anosov

② $\Gamma_0 \underset{\rho}{\curvearrowright} \Omega$ is CC for some properly convex open set $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ s.t. $\partial\Omega \not\supset$ segment.

(say ρ is "strongly CC in $\mathbb{P}(\mathbb{R}^d)$").

$\to$ extends previous work of Benoist, Crampon–Marquis, Mess, Barbot–Mérigot...

* **Remark 1:** For certain word hyperbolic groups Γ_0,

ρ P_1-Anosov $\Longrightarrow$ $\rho(\Gamma_0)$ preserves a properly convex set

(hence the assumption in the theorem may be removed).

E.g. • Γ_0 with $\partial_\infty \Gamma_0$ connected

$\Rightarrow$ true for $\rho : \Gamma_0 \to PO(p,q) < PGL(p+q,\mathbb{R})$

(Danciger - Guéritaud - Kassel)

(in this case, convex set can be taken in

$$\mathbb{H}^{p,q-1} := \{[x] \in \mathbb{P}(\mathbb{R}^{p+q}) \mid x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2 < 0\}$$

or $$\mathbb{H}^{q,p-1} := \{[x] \in \mathbb{P}(\mathbb{R}^{p+q}) \mid x_{p+1}^2 + \dots + x_{p+q}^2 - x_1^2 - \dots - x_p^2 < 0\}$$

which are pseudo-Riemannian analogues of $\mathbb{H}^n$).

• Γ_0 with $\partial_\infty \Gamma_0$ connected :

(and $\partial_\infty \Gamma_0 \smallsetminus \{\eta\}$ —— $\forall \eta \in \partial_\infty \Gamma_0$

and $\partial_\infty \Gamma_0 \smallsetminus \{\eta, \eta'\}$ —— for some $\eta \neq \eta'$ in $\partial_\infty \Gamma_0$)

true for irreducible $\rho : \Gamma_0 \to PGL(d,\mathbb{R})$ (Zimmer)

(NB: this is satisfied if Γ_0 is nonelementary, not commensurable to $\pi_1(S)$, and Γ_0 does not split over a finite group: see Stallings, Swarup, Gabai, Tukia, ...)

In general, anyway,

* Remark 2: can always reduce to preserving a convex set.

Indeed,
$$\begin{aligned} \tau : PGL(d,\mathbb{R}) &\longrightarrow PGL(\mathrm{Sym}_d(\mathbb{R})) \\ g &\longmapsto ([A] \mapsto [gAg^t]) \end{aligned}$$

preserves $\Omega_{sym} := \underbrace{\mathbb{P}(\mathrm{Sym}_d^{>0}(\mathbb{R}))}_{\text{positive definite symmetric matrices}} \underset{\substack{\text{properly convex}\\ \text{open}}}{\subset} \mathbb{P}(\mathrm{Sym}_d(\mathbb{R}))$.

<u>Fact</u> (see Guichard-Wienhard):

$$\left[\begin{array}{c}\rho : \Gamma_0 \longrightarrow PGL(d,\mathbb{R}) \\ \text{is } P_1\text{-Anosov}\end{array}\right] \iff \left[\begin{array}{c}\tau\circ\rho : \Gamma_0 \longrightarrow PGL(Sym^d(\mathbb{R})) \\ \text{is } P_1\text{-Anosov}\end{array}\right].$$

Applying the above theorem, we obtain:

<u>Corollary</u>: Γ_0 f.g. group, $\rho : \Gamma_0 \to PGL(d,\mathbb{R})$ any representation.

TFAE: ① Γ_0 is word hyperbolic and ρ is P_1-Anosov

② $\tau\circ\rho$ is strongly CC in $\mathbb{P}(Sym_d(\mathbb{R}))$.

* <u>Remark 3</u>: This yields a characterisation of P-Anosov $\forall P$.

Indeed:

$$\begin{array}{cccc} \tau_i : & PGL(d,\mathbb{R}) & \longrightarrow & PGL(\Lambda^i \mathbb{R}^d) \\ & g & \longmapsto & (v_1\wedge\ldots\wedge v_i \longmapsto (g\cdot v_1)\wedge\ldots\wedge(g\cdot v_i)) \end{array}$$

<u>Fact</u> (see Guichard-Wienhard):

$$\left[\begin{array}{c}\rho : \Gamma_0 \to PGL(d,\mathbb{R}) \\ \text{is } P_i\text{-Anosov}\end{array}\right] \iff \left[\begin{array}{c}\tau_i\circ\rho : \Gamma_0 \to PGL(\Lambda^i \mathbb{R}^d) \\ \text{is } P_1\text{-Anosov}\end{array}\right].$$

* <u>Idea of proof of Thm, ② $\Rightarrow$ ①:</u>

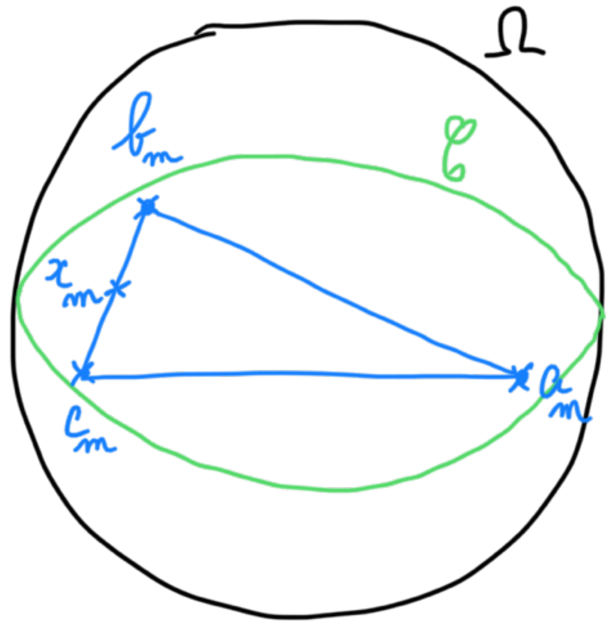
Suppose $\Gamma_0 \overset{\rho}{\curvearrowright} \Omega$ is CC, $\partial\Omega \not\supset$ segment:

$$\exists\, \underset{\substack{\neq \\ \emptyset}}{\mathcal{C}} \text{ convex} \subset \Omega \text{ with } \mathcal{C}/\rho(\Gamma_0) \text{ compact.}$$

$\rho(\Gamma_0)$-inv.

(i) $\partial\Omega \not\supset$ segment $\Longrightarrow$ $(\mathcal{C}, d_\Omega)$ is Gromov hyperbolic.

Indeed, by contradiction:



"less and less thin triangles":

$d_\Omega(x_m, [a_m, b_m] \cup [a_m, c_m]) \to +\infty$.

Up to Γ_0-action,

$x_m \in$ compact set $\subset \mathcal{C}$.

$\downarrow_{m \to +\infty}$

$x \in \mathcal{C}$ (up to subsequence).

Then $[a_m, b_m]$, $[a_m, c_m]$ $\longrightarrow$ segments in $\partial\Omega$, nontrivial since $x \in \mathcal{C}$: contradiction.

(ii) $\Gamma_0 \curvearrowright_\rho (\mathcal{C}, d_\Omega)$ geometric action (properly discontinuous, by isometries, cocompact)

$\Longrightarrow$ Γ_0 is word hyperbolic and any orbital map $\Gamma_0 \to \mathcal{C}$ extends to $\xi_1 : \partial_\infty \Gamma_0 \longrightarrow \partial_\infty \mathcal{C} \subset \mathbb{P}(\mathbb{R}^d)$ continuous, ρ-equivariant.

(iii) Dual convex set:

$$\Omega^* := \{ H \in \mathbb{P}((\mathbb{R}^d)^*) \mid H \cap \overline{\Omega} = \emptyset \}.$$

(seen as projective hyperplane in $\mathbb{P}(\mathbb{R}^d)$)

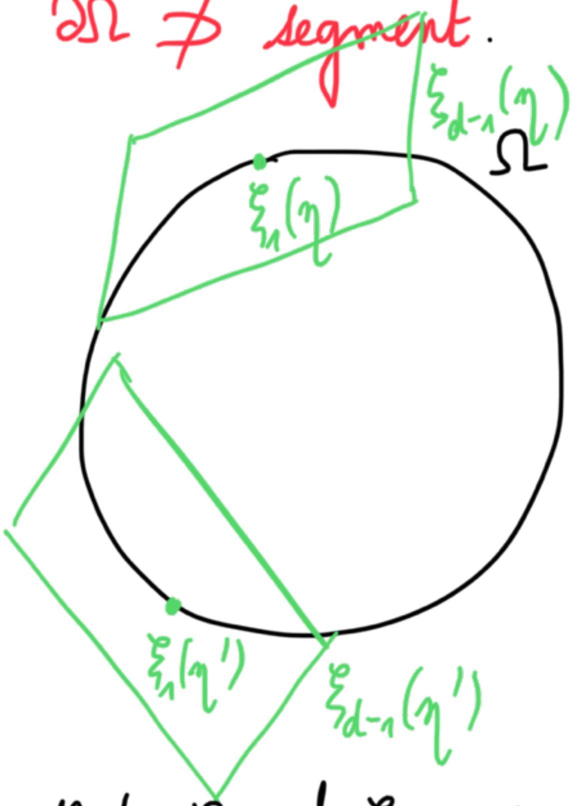
If Ω is C^1 (i.e. $\exists!$ supporting hyperplane at any point of $\partial\Omega$),

then $\partial\Omega^* \not\supset$ segment. This can always be assumed by a "smoothing" procedure (see DGK).

Show that $\underset{\text{dual action}}{\Gamma_0 \overset{\rho}{\curvearrowright} \Omega^*}$ is still CC.

$\rightsquigarrow \xi_{d-1} : \partial_\infty \Gamma_0 \longrightarrow \mathbb{P}((\mathbb{R}^d)^*)$ continuous, ρ-equivariant.

(iv) By construction, ξ_1 and ξ_{d-1} are transverse:
$\xi_{d-1}(\eta)$ is the unique supporting hyperplane to Ω at $\xi_1(\eta)$
and $\partial\Omega \not\supset$ segment.



(v) Show that ξ_1 and ξ_{d-1} are dynamics-preserving

and that $\left(\dfrac{1^{\text{st}} \text{ singular value}}{2^{\text{nd}} \text{ singular value}}\right)(\rho(\gamma)) \xrightarrow[\substack{\gamma\to\infty \\ \text{in } \Gamma_0}]{} +\infty$

Then use one of the characterizations of P_1-Anosov from last time (Kapovich - Leeb - Porti, Guéritaud - Guichard - Kassel - Wienhard). $\square$

* **Idea of proof of Thm, 1 $\Rightarrow$ 2:**

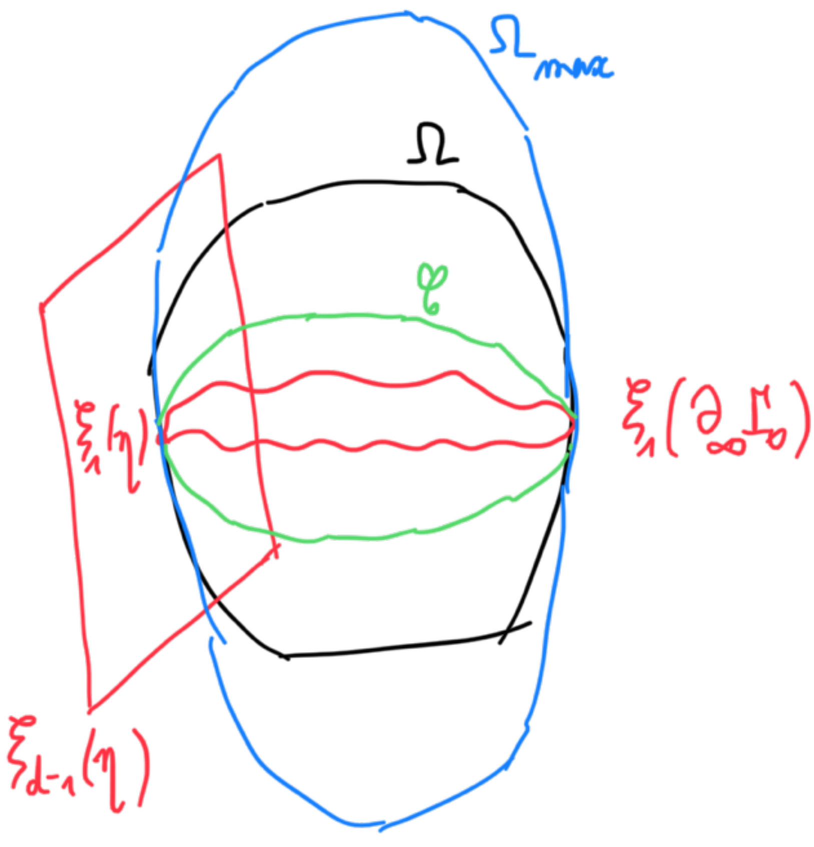
Suppose ρ is P_1-Anosov with boundary maps $\xi_1 : \partial_\infty \Gamma_0 \to \mathbb{P}(\mathbb{R}^d)$
$\xi_{d-1} : \partial_\infty \Gamma_0 \to \mathbb{P}((\mathbb{R}^d)^*)$
and $\rho(\Gamma_0)$ preserves a properly convex open $\Omega \subset \mathbb{P}(\mathbb{R}^d)$.

NB: $\partial\Omega$ may contain segments
and $\Gamma_0 \overset{\rho}{\hookrightarrow} \Omega$ may not be CC if we chose Ω badly.
However:

(i) Consider Ω_{max} := connected component of
$$\mathbb{P}(\mathbb{R}^d) \setminus \bigcup_{\eta \in \partial_\infty \Gamma_0} \xi_{d-1}(\eta)$$
containing Ω

(convex open set).

Show that $\forall x \in \Omega_{max}$, $\{\text{accum. points of } \Gamma \cdot x\} = \xi_1(\partial_\infty \Gamma_0)$.



(ii) Consider $\mathcal{C} := \mathrm{Conv}_{\Omega_{max}}(\xi_1(\partial_\infty \Gamma_0))$.

Dynamics of Anosov representations $\Rightarrow$ $\mathcal{C}/\rho(\Gamma_0)$ compact.

(expansion at $\xi_1(\partial_\infty\Gamma_0)$, see KLP and GGKW)

see KLP, generalizing a classical argument of Sullivan.

(iii) Transversality of ξ_1 and ξ_{d-1}

$\Rightarrow$ no segment in $\partial\Omega_{max}$ between points of $\xi_1(\partial_\infty\Gamma_0)$

$\Rightarrow$ may "smooth out" Ω_{max} to obtain

a $\rho(\Gamma_0)$-invariant properly convex open $\Omega' \subset \Omega_{max}$

containing $\mathcal{C}$ s.t. $\partial\Omega' \not\supset$ segment.

Then $\Gamma_0 \overset{\rho}{\hookrightarrow} \Omega'$ is CC. $\square$

II. Applications.

* <u>Geometric interpretation for Anosov representations:</u>

<u>Ex.:</u> For odd $d = 2n+1$,

any Hitchin representation $\rho : \Gamma_0 = \pi_1(S) \longrightarrow PSL(2n+1, \mathbb{R})$

is strongly CC in $\mathbb{P}(\mathbb{R}^{2n+1})$.

(Choi-Goldman 1993 for $n=2$, DGK and Z in general).

(Idea: (general case)
- injective and discrete is closed
- if limit is irreducible, then preserving a properly convex set is closed
- Hitchin representations are all P_1-Anosov (Labourie), apply theorem above and get CC.
)

* New examples of Anosov representations:

Ex. (DGK, Lee-Marquis):

For any infinite, word hyperbolic Coxeter group Γ_0 in d generators, $\exists\, \rho : \Gamma_0 \to GL(d,\mathbb{R})$ representation as a reflection group which is strongly CC in $\mathbb{P}(\mathbb{R}^d)$.

$\Downarrow$ Thm above

P_1-Anosov.

In fact,
$$\mathrm{Hom}_{CC,\,\text{reflection}}(\Gamma_0, PGL(d,\mathbb{R})) = \mathrm{Int}\big(\underbrace{\mathrm{Hom}_{\text{reflection}}(\Gamma_0, PGL(d,\mathbb{R}))}_{\substack{\text{semialgebraic set,}\\ \text{described by Vinberg,}\\ \text{in some connected component}\\ \text{of } \mathrm{Hom}(\Gamma_0,\, PGL(d,\mathbb{R})).}}\big).$$

III. Beyond Anosov representations.

We saw that Anosov representations can be characterized by a $\underbrace{\text{strong}}_{\partial\Omega \not\supset \text{segment.}}$ CC condition in real proj. space:

Generalizations:

- keep strong condition but allow for cusps
  $\rightsquigarrow$ strong geometric finiteness (Crampon-Marquis 2014).
  The corresponding groups Γ_0 are relatively hyperbolic and the corresponding representations are $\boxed{\text{relative Anosov representations}}$ as developed in recent work of Kapovich-Leeb and F. Zhu.

- relax strong condition:
  CC allowing $\partial\Omega \supset$ segments (DGK).

* $\underline{\text{Def.}}$: Ω properly convex open $\subset \mathbb{P}(\mathbb{R}^d)$ ~~with $\partial\Omega \not\supset$ segment~~
(DGK)

$\rho : \Gamma_0 \to \mathrm{Aut}(\Omega) \subset PGL(d,\mathbb{R})$.

Say $\Gamma_0 \overset{\rho}{\curvearrowright} \Omega$ is $\left\{\begin{array}{l} \underline{CC} \\ \underline{\text{naive } CC} \end{array}\right.$ if it is properly discontinuous

$\underline{\text{and}}$ $\exists\, \mathscr{C}$ convex $\subset \Omega$ with $\rho(\Gamma_0)\backslash\mathscr{C}$ compact.
$\mathscr{C} \neq \emptyset$, $\rho(\Gamma_0)$-inv.

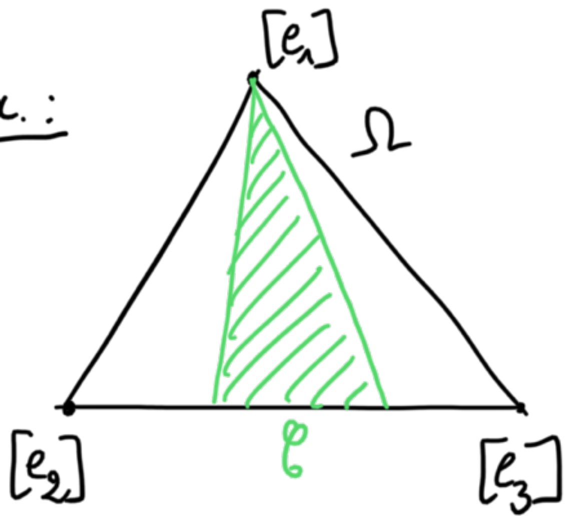
and $\mathscr{C}$ $\underbrace{\text{large enough}}_{\text{contains } \mathrm{Conv}_\Omega(\Lambda^{orb}_{\rho(\Gamma_0),\Omega})}$.

Here $\Lambda^{orb}_{\rho(\Gamma_0),\Omega}$ is the $\underline{\text{full orbital limit set}}$:

$$\Lambda^{orb}_{\rho(\Gamma_0),\Omega} := \bigcup_{x\in\Omega} \{\text{accum. points of } \rho(\Gamma_0)\cdot x\} \subset \partial\Omega .$$

* $\underline{\text{Point}}$: naive CC is $\underline{\text{not}}$ stable under small deform.

$\underline{\text{Ex.}}$:



$$\rho : \Gamma_0 = \mathbb{Z} \longrightarrow PGL(3,\mathbb{R})$$
$$1 \longmapsto \begin{pmatrix} 2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$
is naive CC (but $\underline{\text{not}}$ CC).

Deform :

$$\rho_\varepsilon : 1 \longmapsto \begin{pmatrix} 2 & & \\ & \cos\varepsilon & -\sin\varepsilon \\ & \sin\varepsilon & \cos\varepsilon \end{pmatrix}$$
does $\underline{\text{not}}$ preserve
a properly convex open set in $\mathbb{P}(\mathbb{R}^3)$.

* $\underline{\text{NB}}$: $\partial\Omega \not\supset$ segment

$$\Longrightarrow \Lambda^{orb}_{\rho(\Gamma_0),\Omega} = \{\text{accum. points of } \rho(\Gamma_0)\cdot x\} \quad \forall x \in \Omega$$

and naive CC is equivalent to CC.

<u>False in general.</u>

* <u>Def.:</u> Say $\rho : \Gamma_0 \to PGL(d,\mathbb{R})$ is <u>CC in $\mathbb{P}(\mathbb{R}^d)$</u> if $\Gamma \underset{\rho}{\curvearrowright} \Omega$ CC for some properly convex open $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ (possibly with $\partial\Omega \supset$ segments).

* <u>Thm (DGK):</u>
  - $\mathrm{Hom}_{CC \text{ in } \mathbb{P}(\mathbb{R}^d)}(\Gamma_0, PGL(d,\mathbb{R}))$ is open in $\mathrm{Hom}(\Gamma_0, PGL(d,\mathbb{R}))$.
  - $[\rho$ is strongly CC in $\mathbb{P}(\mathbb{R}^d)]$ $\iff$ $[\rho$ is CC in $\mathbb{P}(\mathbb{R}^d)$ <u>and</u> Γ_0 is word hyperbolic$]$.

* <u>Examples:</u>
  - Ω with $\partial\Omega \supset$ segment s.t. $\rho(\Gamma_0)\backslash\Omega$ compact ("nonstrictly convex, divisible convex set") (Benoist, Ballas-Danciger-Lee, Marquis, Choi-Lee-Marquis, ...)
  - Danciger-Guéritaud-Kassel-Lee-Marquis: CC representations $\rho : \Gamma_0 \to PGL(d,\mathbb{R})$ as reflection group with Γ_0 Coxeter group, <u>not</u> word hyperbolic (description of all possible examples).