Cubical dimension of groups

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Goal:

compare various notions of dimension of a group.

+ some examples of groups with dimension gaps

Dimensions of G :

1) Geometric dimension of G:

$\dim G := \inf\{k \mid G$ has a k-dimensional $K(G,1)$ complex$\}$

2) CAT(0) dimension of G:

by semi-simple isometries

$\mathrm{CAT(0)}\dim_{ss} G := \inf\{k \mid G$ acts properly on a k-dim CAT(0) complex$\}$

3) Cubical dimension of G:

$\mathrm{cubdim}\, G := \inf\{k \mid G$ acts properly on a k-dim CAT(0) cube complex$\}$

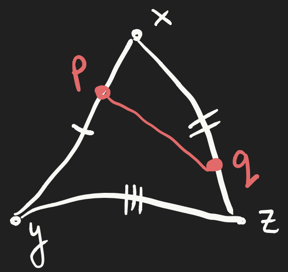
4) Cohomological dimension of G:

$\mathrm{cd}\, G := \inf\{k \mid$ there exists a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ of length $k\}$

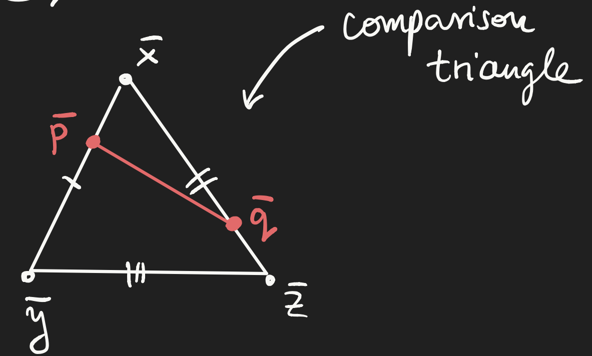
CAT(0) SPACES

X - geodesic metric space, $x, y, z \in X$

X



$\mathbb{E}^2$



We say $\Delta(x,y,z)$ satisfies <u>CAT(0) inequality</u> if for all $p, q \in \Delta(x,y,z)$

$$d_X(p,q) \leqslant d_{\mathbb{E}^2}(\bar{p},\bar{q})$$

X is <u>CAT(0)</u> if every geodesic triangle satisfies the CAT(0) inequality.

<u>Example</u>: CAT(0) cube complexes with the metric induced by the Euclidean metric on each cube

Isometries of CAT(0) spaces:

An isometry $g : X \to X$ is

semi-simple

- *elliptic* if g has a fixed point in X
- *hyperbolic* if g preserves a geodesic line in X and acts on it by translation

- *parabolic* otherwise

For the rest of the talk: G is finitely generated and torsion-free

$\operatorname{cd} G = \inf\{k \mid \mathbb{Z} \text{ admits a projective resolution over } \mathbb{Z}G \text{ of length } k\}$

$\dim G = \inf\{k \mid G \text{ has a } k\text{-dimensional } K(G,1)\text{-complex}\}$

$\mathrm{CAT}(0)\dim_{ss} G = \inf\{k \mid G \text{ acts properly by semi-simple isometries on a CAT(0) complex of dim } k\}$

$\operatorname{cubdim} G = \inf\{k \mid G \text{ acts properly on a CAT(0) cube complex of dim } k\}$

$$\operatorname{cd} G \leq \dim G \leq \mathrm{CAT}(0)\dim_{ss} G \leq \operatorname{cubdim} G$$

cellular structure gives a free resolution of length k.

CAT(0) spaces are contractible
G torsion-free ⇒ proper action is free

every isometry of a CAT(0) cube complex is semi-simple. (Haglund, 2007)

$\mathrm{cd}\, G \leq \dim G \leq \mathrm{CAT}(0) \dim_{ss} G \leq \mathrm{cubdim}\, G$

__Question__: Can these inequalities be strict?

1) (Stallings '57): If $\mathrm{cd}\, G = 1$, then G is free.

↳ all dimensions are equal 1.

2) There are many groups that don't act properly on CAT(0) cube complexes / CAT(0) spaces so $\mathrm{cubdim}\, G = \infty$ or $\mathrm{CAT}(0) \dim G = \infty$.

↳ infinite gaps e.g. $BS(1,2)$

$\mathrm{cd}\, G \leq \dim G \leq \mathrm{CAT(0)} \dim_{ss} G \leq \mathrm{cubdim}\, G$

Question: Can there be a finite gap?

3) Theorem (Eilenberg-Ganea, '57)

If $\mathrm{cd}\, G = n \geq 3$, then $\dim G = n$.

If $\mathrm{cd}\, G = 2$, then $\dim G \leq 3$.

Eilenberg-Ganea conjecture: $\mathrm{cd}\, G = \dim G$

(Bestvina-Brady '97): Only one of the conjectures can be true: Eilenberg-Ganea conjecture, or Whitehead conjecture ('41): connected subcomplex of an aspherical 2-dim. CW-complex is aspherical.

$$\mathrm{cd}\,G \le \underbrace{\dim G \le \mathrm{CAT}(0)\dim_{ss} G}_{} \le \mathrm{cubdim}\,G$$

4) (Brady–Crisp, 2002): certain 3-generator Artin groups

$$\dim G < \mathrm{CAT}(0)\dim_{ss} G < \infty$$

$$A_{mnp} = \langle a,b,c \mid \underbrace{aba\ldots}_{m} = \underbrace{bab\ldots}_{m},\ \underbrace{bcb\ldots}_{n} = \underbrace{cbc\ldots}_{n},\ \underbrace{cac\ldots}_{p} = \underbrace{aca\ldots}_{p} \rangle$$

(Charney–Davis, '95): $\dim A_{mnp} = 2 \iff \frac{1}{m} + \frac{1}{n} + \frac{1}{p} \le 1$

(Brady–Crisp, 2002): 1) $\mathrm{CAT}(0)\dim_{ss} A_{2np} > 2$

2) All but finitely many A_{2np} are π_1 (3-dim NPC piecewise Euclidean complex)

$\hookrightarrow$ $\dim A_{2np} = 2 < \mathrm{CAT}(0)\dim_{ss} A_{2np} = 3$

$\mathrm{cd}\, G \leq \dim G \leq \mathrm{CAT}(0)\dim_{ss} G \leq \mathrm{cubdim}\, G$

E.g. $B_4 = A_{233} = \langle a, b, c \mid aba = bab,\ bcb = cbc,\ ca = ac \rangle$

braid group on 4 strands

$\dim B_4 = 2 < \mathrm{CAT}(0)\dim_{ss} B_4 = 3$

Question: What about cubdim?

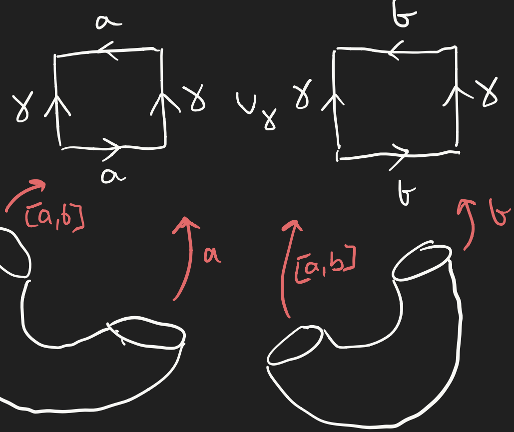
Thm (Huang-J.-Przytycki, Haettel, 2016)

A_{mnp} does not act properly and cocompactly on a CAT(0) cube complex, unless $(m,n,p)=(2,2,p)$

Without cocompactness: open.

$$\mathrm{cd}\,G \le \underline{\dim G \le \mathrm{CAT}(0)\dim_{ss} G \le \mathrm{cubdim}\, G}$$

5) (Bridson, 2001)

$X =$ 

$G = \pi_1 X$

Bridson:

1) X is aspherical

$\Rightarrow \dim G = 2$

2) $\pi_1 X$ does not act properly by semi-simple isometries on any 2-dim CAT(0) complex

$X \simeq$ 3-dim CAT(0) cube complex

$\Rightarrow \mathrm{CAT}(0)\dim G = \mathrm{cubdim}\, G$

$= 3$

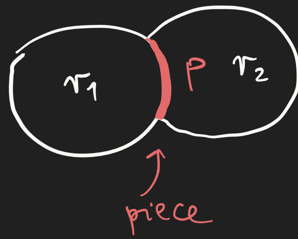
3) $\hat{X} \xrightarrow[\text{double cover}]{} X$ $\quad$ $\hat{X} \simeq$ 2-dim NPC piecewise Euclidean complex

$$\mathrm{cd}\,G \le \dim G \le \underbrace{\mathrm{CAT}(0)\dim_{ss} G \le \mathrm{cubdim}\, G}$$

6) <u>Thm</u> (J.)

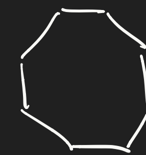
For every n, there exists a finitely presented $\underbrace{C'(1/6)\text{-small cancellation group}}_{\text{always hyperbolic, } \dim G = 2}$ G with $\mathrm{cubdim}\, G > n$.

$G = \langle S \mid R \rangle$ is $C'(\frac{1}{6})$ if in $Cay(G, S)$:



$$|p| < \frac{1}{6} |r|$$

Example:
$\pi_1 \Sigma_2 = \langle a, b, c, d \mid [a,b][c,d] \rangle$

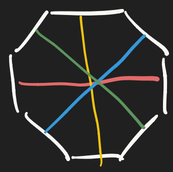


pieces = single edges
$1 < \frac{1}{6} \cdot 8$

- $C'(\frac{1}{6})$ groups are hyperbolic, dim 2.
- (Gromov, S. Brown 2016): For uniformly $C'(\frac{1}{6})$ group Cayley complex can be folded into a CAT(−1) complex. $\Rightarrow$ CAT(0) dim $G = 2$
- (Wise, 2004): $C'(\frac{1}{6})$ groups act properly and cocompactly on CAT(0) cube complexes $\Rightarrow$ cubdim $G < \infty$

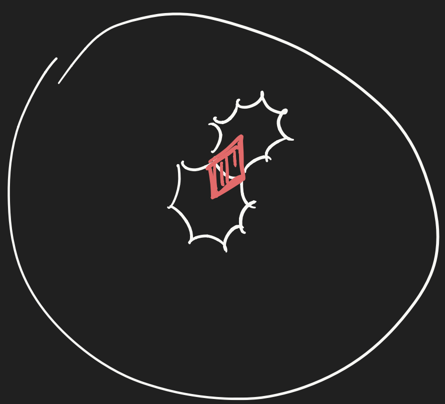
Wise's construction:

$\pi_1 \Sigma_2$



$\rightsquigarrow$ action of $\pi_1 \Sigma_2$ on a 4-dimensional CAT(0) cube complex

but $\pi_1 \Sigma_2 \curvearrowright \mathbb{H}^2$



dual CAT(0) square structure

$\Rightarrow$ cubdim $\pi_1 \Sigma_2 = 2$

One consequence:

<u>Example</u> (J.-Wise, 2020)

There exists finitely generated G such that

- $\dim G = 2$
- $\operatorname{cubdim} G = \infty$
- G acts properly on a locally finite CAT(0) cube complex

Thank you.