Fair Division and Allocation

Michael A. Jones, *AMS* | *Mathematical Reviews*

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AMERICAN MATHEMATICAL SOCIETY

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A motivating example? (Japanese internment)

- During World War II, the federal government forcefully removed and confined ∼120,000 Japanese Americans (62% were U.S. citizens).
- President Roosevelt authorized the designation of military areas by local commanders, allowing for the removal of “any or all persons”.
- The order allowed all people of Japanese ancestry to be excluded from the entire Pacific coast of the U.S.
- In 1944, the Supreme Court upheld the constitutionality of the actions “pressing public necessity”. (Korematsu v. United States)

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- The Civil Liberties Act of 1988 sponsored by Congressman Mineta and Senators Simpson and Wilson granted reparations.
- The legislation stated that the internment had been based on "race prejudice, war hysteria, and a failure of political leadership" as opposed to legitimate security reasons.
- President Reagan signed it into law.
- The act granted $20,000 to each surviving internee. Over 82,000 people received compensation.

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Is this a type of fair division problem?

Fair division and claims problems

Question. How do you divide a resource among people with different claims to the resource?
When that resource is money, we'll refer to it as the *estate*, E.

The Bankruptcy/Claims Problem

Each claimant i is owed c_i so that $\sum_{i=1}^{n} c_i = C > E$.

A division rule solves the problem by mapping the set of possible claims into the set of nonnegative vectors $\mathbf{r}$ with $0 \leq r_i \leq c_i$ and $\sum_{i=1}^{n} r_i = E$.

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Zoom poll 1. If the claims vector is $(100, 200, 300)$ and $E = 300$, which do you think is most fair?

1) $\mathbf{r} = (100, 100, 100)$ 2) $\mathbf{r} = (50, 100, 150)$ 3) $\mathbf{r} = (0, 100, 200)$

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Zoom poll 2. Was the reparation problem for interred Japanese Americans a claims problem?

The Talmud rule

The following bankruptcy solutions appeared in a table in the 2000-year old Talmud. Were they generated by the same rule?

		Estate		
		100	**200**	**300**
	100	$\frac{100}{3}$	50	50
Claims	**200**	$\frac{100}{3}$	75	100
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This question intrigued Hebrew scholars, mathematicians and social scientists.

In 1985, Aumann and Maschler showed that each solution was the nucleolus of a game. (Aumann awarded 2005 Nobel prize.)

Aumann, Robert J.; Maschler, Michael. (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory. 35, 195-213.

The Talmud rule

There was a well-described rule, the Contested Garment Principle, in the Talmud. Let's see how it works.

250	
200	**300**
200	50
0	250
100	150

$c_1 = 200, c_2 = 300,$ and $E = 250.$

Line 1: C1 owed 200; other 50 goes to C2.

Line 2: C2 owed 300; takes all 250; 0 for C1.

Line 3: C1 gets 100 $(= (200 + 0)/2)$

and C2 gets 150 $(= (50 + 250)/2)$.

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There ends up being a relationship (called *consistency*) between the two rules. A rule is consistent If claimant i takes r_i out of the problem and $E' = E - r_i$, then $r'_j = r_j$.

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The Contested Garment Principle ends up being the 2-claimant Talmud rule.

Gura, Ein-Ya; Maschler, Michael B. Insights into game theory: An alternative mathematical experience. Cambridge University Press. 2008.

Axiomatic characterization of the Talmud rule

Like in social choice theory, much of the literature focuses on the axiomatic characterization of division rules.

Theorem

The Talmud rule is the only rule satisfying 1) equal treatment of equals, 2) claims truncation invariance, 3) minimal rights first, and 4) bilateral consistency.

Axioms. Assume there are n claimants.

1) If $c_i = c_j$, then $r_i = r_j$.
2) If $c_i' = \min\{c_i, E\}$ and c_j are fixed, then $r_i' = r_i$ and $r_j' = r_j$.
3) This notion extends the Contested idea from the Contested Garment Principle.
4) The notion of consistency holds for subsets of size $n-2$ leaving.

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The point. As a modeler, the idea is to use a division rule that satisfies properties that are desirable for the problem at hand.

Thomson, William. How to divide when there isn't enough: From Aristotle, the Talmud, and Maimonides to the axiomatics of resource allocation. Cambridge University Press. 2019.

A graphical view of the division rules

As in the Talmud problem, let $c_1 = 100$, $c_2 = 200$ and $c_3 = 300$.
In what is referred to hydraulic rationing, let the claim sizes represent containers.
If $E = 300$, then we pump in 300 units of water. The water levels out to determine the allocation.

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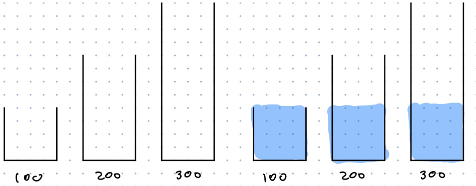
Constrained Equal Awards

Under the constrained equal awards rule, the containers start at the same level.
The solution is $\mathbf{r} = (100, 100, 100)$. (Option 1 in the Zoom poll.)



Kaminski, Marek M. "Hydraulic" rationing. Math. Social Sci. 40 (2000), no. 2, 131–155.

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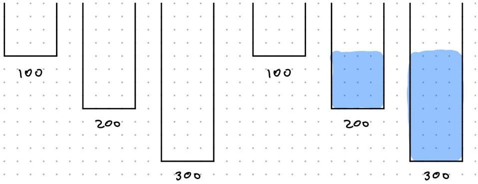
Constrained Equal Losses

Under the constrained equal losses rule, the containers hang from the same level.
The solution is $\mathbf{r} = (0, 100, 200)$. (Option 3 in the Zoom poll.)



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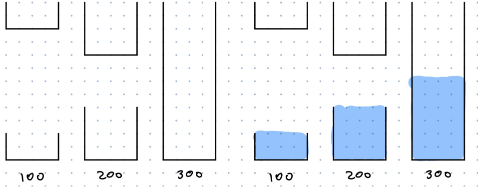
Talmud Rule

Under the Talmud rule, the containers of half size start and hang from the same level.
The solution is $\mathbf{r} = (50, 100, 150)$. (Option 2 in the Zoom poll.)



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Cake cutting

When dividing up a contiguous object like land, fair division is referred to as cake cutting.

If n claimants have equal rights to the land, then the idea is to divide the land to satisfy the following axioms:

1) proportionality (each gets at least $1/n$ of the land),

2) envyfreeness (no claimant prefers what another claimant receives),

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Question. Can cake cutting be used to divide the land between descendants?

Robertson, Jack; Webb, William Cake-cutting algorithms. Be fair if you can. A K Peters, Ltd. 1998.

Reparations as claim problems

Question. Is the Japanese American internment payout a type of fair division?

Reparations as claim problems

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The following instances of reparations all suffer from the same *one size fits all*.

- Evanston
  redlining
- Georgetown
  use of slaves to build institution
- University of South Dakota
  took land from indigenous people

A short introduction to mechanism design

- Constructing a game to achieve a particular outcome (often the truthful revelation of preferences) is called mechanism design.
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When I was at Montclair State University in NJ, the Associate Dean asked me to devise a method to award travel funds.
• Let r_i be the amount (in dollars) requested by faculty member i and $R = r_1 + \cdots + r_n$.
• Let B be the amount the Dean's office has budgeted for travel.
$\Rightarrow$ If $B < R$, then awarding travel funds is a claims problem.

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By requesting too much, a faculty member can be reimbursed for all of their travel!

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Let s_i be the amount spent by faculty member i.

Travel fund allocation–new method.
When $B < R$, faculty member i is still budgeted $b_i = (B/R)r_i$, but now the amount reimbursed depends on s_i and $\mathbf{r}$ and is given by

$$g_i(\mathbf{r}, s_i) = \begin{cases} (B/R)s_i & \text{if } s_i \leq r_i \\ (B/R)r_i & \text{if } s_i > r_i. \end{cases}$$

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To minimize out-of-pocket expenses, it is in the best interest of faculty member i to request $r_i = s_i$.

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Proposition

To minimize out-of-pocket expenses, it is in the best interest of faculty member i to spend less.

Jones, Michael A. A mechanism design approach to allocating travel funds. Submitted.

Matching

How do you pair up students to schools?
Gale and Shapley created an algorithm to solve this problem in 1962.
They simplified the problem of matching n students to n schools.

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Schools provide a linear order over all students.
Students provide a linear order over all schools.

Ouput

A stable matching of students to schools.

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Definition [Stable matching]

A matching (pairing up each school with one student) is stable if there is no school-student pair that would prefer to be matched together rather than their current pairings.

The National Resident Matching Program (the Match) has been using the deferred acceptance algorithm since the early 1950s.

Gale, D.; Shapley, L. S. College Admissions and the Stability of Marriage. Amer. Math. Monthly 69 (1962), no. 1, 9–15.

Deferred-Acceptance Algorithm

Gale-Shapley (Deferred-Acceptance) Algorithm

Round 1.

Each school offers admission to the top student on its list. A student who is admitted by more than one school turns down lower-ranked schools. Schools not turned down are in limbo, waiting for the student to accept.

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Wash. Rinse. Repeat.

The admission offers continue until no school is turned down. At this point, each school is paired to a unique student.

- As described, the schools admit students. But, it can be flipped around so that students choose.

School choice and affirmative action

The deferred-acceptance algorithm has been used in conjunction with affirmative action measures.

- California's Proposition 209 (1996) states that the government and public institutions cannot discriminate against or grant preferential treatment to persons on the basis of race, sex, color, ethnicity, or national origin in public employment, public education, and public contracting.
- California's Proposition 16 to repeal Proposition 209 was defeated at the polls in 2020.

> *Prohibitions on using race in affirmative action have spurred a number of admissions systems to adopt race-neutral alternatives that encourage diversity without appearing to explicitly advantage any particular group. ... Chicago's exam school reserves seats for students based on their neighborhood ... Tiers are based on an index of socioeconomic disadvantage. At each school, an equal fraction of seats are reserved for each tier.*

Dur, Umut; Pathak, Parag A.; Sönmez, Tayfun Explicit vs. statistical targeting in affirmative action: theory and evidence from Chicago's exam schools. J. Econom. Theory 187 (2020), 104996, 48 pp.

College admissions and private high school attendance

Supreme Court and affirmative action

The Students for Fair Admissions (SFFA) case against Harvard University contends that college admissions at Harvard discriminates against Asian Americans.

SFFA's strategy argues statistically that admission policies benefit legacies, donors, athletes and underrepresented minorities at the expense of Asian Americans.

Harvard Class of 2018

- 10% of students were recruited athletes
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Can the deferred acceptance algorithm be used to mitigate advantages from being a student at a private school?

`https://slate.com/news-and-politics/2021/06/private-schools-competitive-college-advantage-problems.html`

Take away

Mathematicians have the ability to create, to suggest, and to analyze fair division procedures to solve real-world problems.

As educators, we can introduce the mathematics of fair division and its application to solve societal problems in our classrooms.

Shameless plug, and opportunity to speak

- David McCune, Jennifer Wilson and I are organizing an AMS Special Session on the Mathematics of Decisions, Elections and Games for the 2022 Joint Math Meetings in Seattle.
- Feel free to contact me if you have a talk that would be appropriate for the session. We will apply to the Editorial Board of the AMS' Contemporary Mathematics about publishing a proceedings.
- Attending the session will be a good opportunity to meet others interested in mathematical social sciences.

The mathematics of decisions, elections, and games. Proceedings of the AMS Special Sessions held in Boston, MA, January 4, 2012, and San Diego, CA, January 11–12, 2013. Edited by Karl-Dieter Crisman and Michael A. Jones. Contemporary Mathematics, 624. American Mathematical Society, Providence, RI, 2014. x+229 pp. ISBN: 978-0-8218-9866-6

Also, if you want to learn more about the mathematics of the social sciences consider being a reviewer for Mathematical Reviews. You can ask to review published articles in narrow areas (e.g., school choice) as a way to learn about the field or to keep up on the literature.

Thank you.