Spiraling domains in dimension 2

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Work in progress with Jasmin Raissy

Main result

Theorem

For $a \in \mathbb{R} \setminus \{\mathbf{0}\}$, the polynomial endormorphism $F_a : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$F_a\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ x^2 \end{pmatrix} + a\begin{pmatrix} x(x-y) \\ y(x-y) \end{pmatrix}$$

has infinitely many spiraling domains contained in distinct Fatou components.

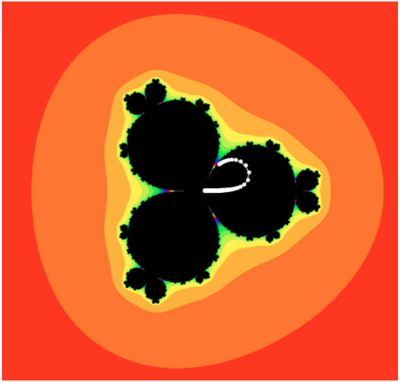
Tools

- homogeneous vector fields;
- affine surfaces;
- triangular billiards.

Maps tangent to the identity in dimension 1

- $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is tangent to the identity and $f \neq \mathrm{id}$:

$$f(z) = z + az^{k+1} + \mathrm{O}(z^{k+2}) \quad \text{with} \quad a \in \mathbb{C} \setminus \{0\}.$$



There are $k = 3$ parabolic petals/domains

Maps tangent to the identity in dimension 2

Assumptions:

- $\vec{\boldsymbol{v}}$ is a homogeneous vector field of degree k on $\mathbb{C}^2$:

$$\vec{\boldsymbol{v}} := U\partial_x + V\partial_y$$

  with U and V homogeneous polynomials of degree $k+1$;

- $$\Phi := xV - yU$$

  vanishes on $k+2$ *characteristic directions*, counting multiplicities;

- $$F(\boldsymbol{x}) = \boldsymbol{x} + \vec{\boldsymbol{v}}(\boldsymbol{x}) + \mathrm{O}\big(\|\boldsymbol{x}\|^{k+2}\big).$$

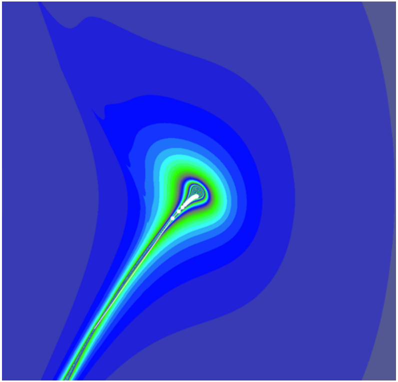
Observation:

- Near $\mathbf{0}$, orbits of F shadow real-time trajectories of $\vec{\boldsymbol{v}}$.

Known results

Proposition (Écalle, Hakim, Abate, . . . , López-Hernanz, Rosas)

For any F, tangentially to each characteristic direction, there is either a curve of fixed points, or at least one *parabolic petal*.

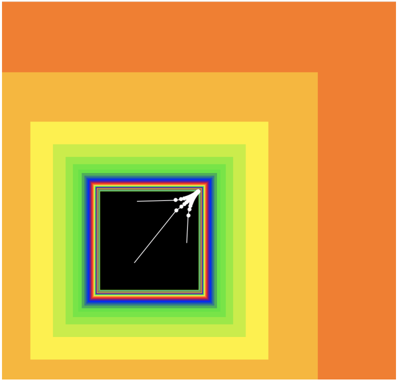


$$F(x, y) = (x + y^2 + x^3, y + x^2)$$

Known results

Proposition (Écalle, Hakim)

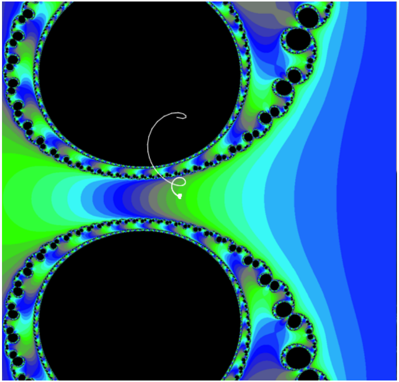
Existence of F which have *parabolic domains* on which orbits converge to **0** tangentially to a characteristic direction.



$$F(x, y) = (x + x^2, y + y^2)$$

Proposition (Rivi, Rong)

Existence of F which have *parabolic domains* on which orbits converge to **0** *spiraling* around a characteristic direction.



$$F(x,y) = (x - x^2, y + y^2 + 4x^2)$$

Known results

Proposition (Rivi, Rong)

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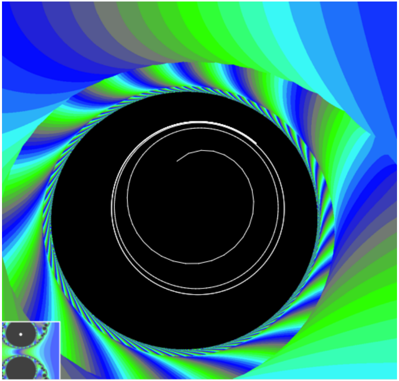

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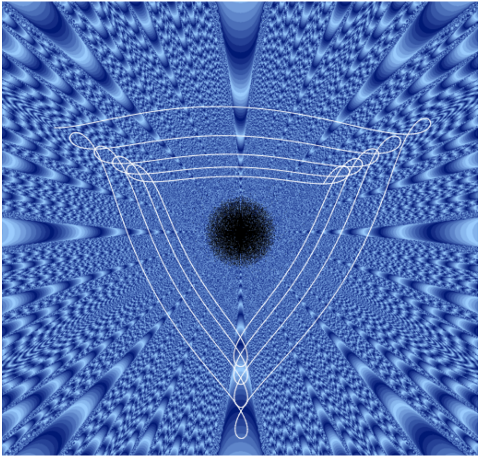
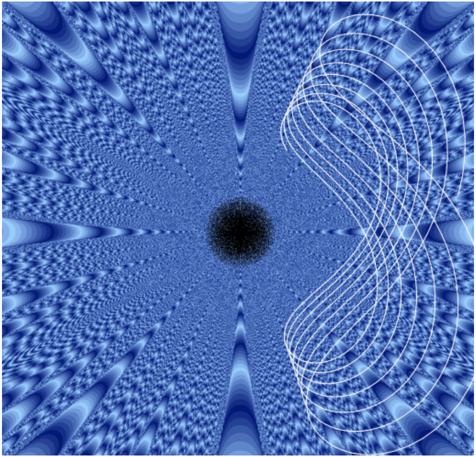
The family F_a

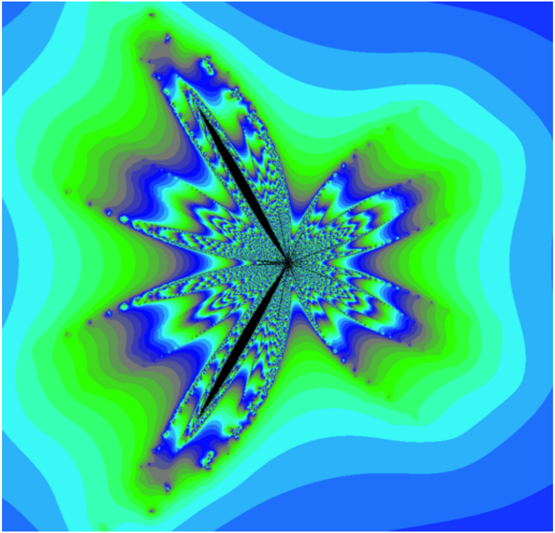
$$F_a\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ x^2 \end{pmatrix} + a\begin{pmatrix} x(x-y) \\ y(x-y) \end{pmatrix}$$

The dynamics of F_a for $a = 0$





The dynamics of F_a for $a = 0.1$



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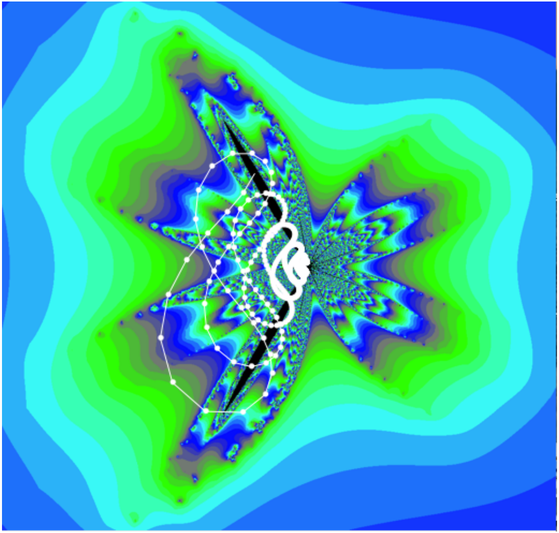

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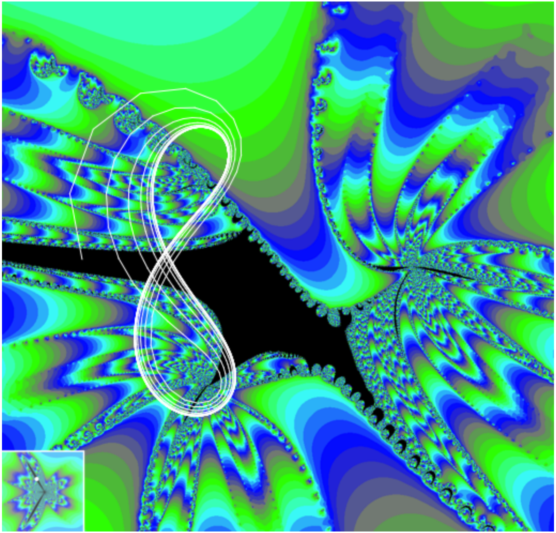

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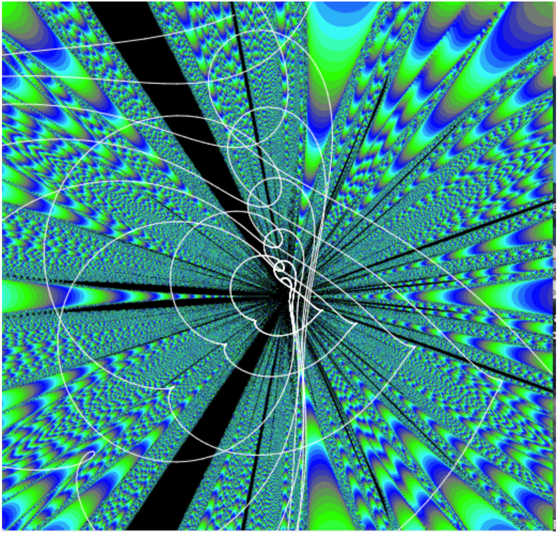

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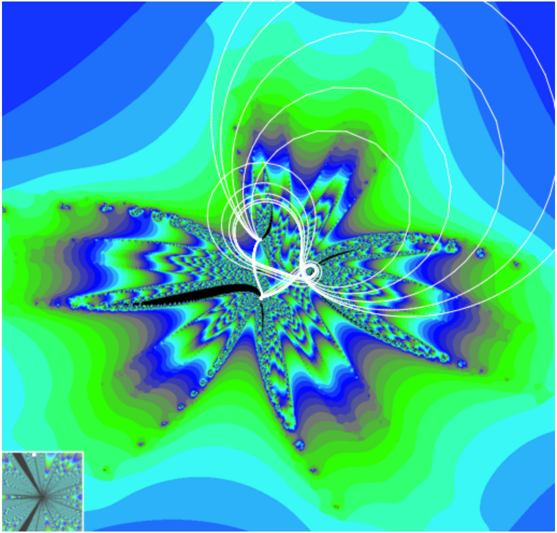

Dynamics of homogeneous vector fields

- A trajectory for $\vec{\boldsymbol{v}}$ is a solution of the differential equation

$$\dot{\gamma} = \vec{\boldsymbol{v}} \circ \gamma.$$

- Complex-time trajectories are Riemann surfaces which cover $\mathbb{CP}^1$ minus the characteristic directions.
- What does the projection to $\mathbb{CP}^1$ of a real-time trajectory look-like?

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Proposition (Abate)

We may equip $\mathbb{CP}^1$ with the structure of an affine surface $\mathbf{S}_{\vec{\boldsymbol{v}}}$ so that the projection to $\mathbf{S}_{\vec{\boldsymbol{v}}}$ of real-time trajectories of $\vec{\boldsymbol{v}}$ are geodesics.

Affine surfaces and geodesics

Definition (Affine surface)

An *affine surface* **S** is a Riemann surface whose change of charts are affine maps $z \mapsto \lambda z + \mu$ with $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$.

Example : **C** is the complex plane with its canonical affine structure.

Definition (Affine map)

A map between affine surfaces is an *affine map* if its expression in affine charts is of the form $z \mapsto \lambda z + \mu$.

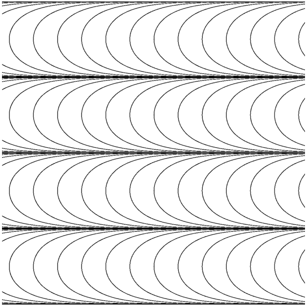
Definition (Geodesic)

A curve $\delta : I \to$ **S** defined on an interval $I \subseteq \mathbb{R}$ is a *geodesic* if δ is the restriction of an affine map $\varphi : U \to$ **S** defined on an open subset $U \subseteq$ **C**.

An example

- The dilation plane $\widetilde{\mathbf{C}}$ with underlying Riemann surface $\mathbb{C}$, whose affine charts are the restrictions of

$$\exp(z) : \widetilde{\mathbf{C}} \to \mathbf{C} \setminus \{0\}.$$



A family of parallel geodesics in $\widetilde{\mathbf{C}}$.

Non linearity

- The non linearity of a holomorphic map $\varphi : \mathbf{S} \to \mathbf{T}$ with non vanishing derivative is the 1-form $\mathcal{N}_\varphi$ defined on **S** by

$$\mathcal{N}_\varphi := \mathrm{d}(\log \varphi') = \frac{\mathrm{d}\varphi'}{\varphi'}.$$

- $\mathcal{N}_\varphi = 0$ if and only if φ is an affine map.
- If $\varphi : \mathbf{S} \to \mathbf{T}$ and $\psi : \mathbf{T} \to \mathbf{U}$ are holomorphic maps, then

$$\mathcal{N}_{\psi\circ\varphi} = \mathcal{N}_\varphi + \varphi^*(\mathcal{N}_\psi).$$

Affine surface of a homogeneous vector field

- $\vec{\boldsymbol{v}} = U\partial_x + V\partial_y$ is homogeneous of degree k.
- $z : \mathbb{CP}^1 \ni [x : y] \mapsto \dfrac{x}{y} \in \widehat{\mathbb{C}}$.
- $f\left(\dfrac{x}{y}\right) = \dfrac{U(x,y)}{V(x,y)}$.
- $p\left(\dfrac{x}{y}\right) = \dfrac{xU(x,y) - yV(x,y)}{y^{k+2}}$.

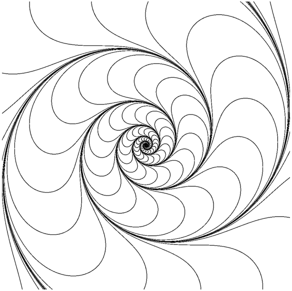
Proposition

The non linearity of $z : \mathbf{S}_{\vec{\boldsymbol{v}}} \to \mathbf{C}$ is

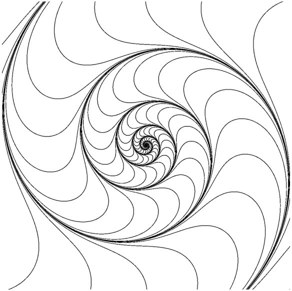
$$\nu := \left(\frac{p'(z)}{p(z)} - \frac{k}{z - f(z)}\right) \mathrm{d}z.$$

Affine surface of a homogeneous vector field

- Singularities of ν are characteristic directions.
- Assume there is a simple pole and let ρ be the residue.



$\mathrm{Re}(\rho) > 1$



$\mathrm{Re}(\rho) < 1$

Proposition (Écalle, Hakim)

If ν has a simple pole and $\mathrm{Re}(\rho) > 1$, there is a parabolic domain on which orbits converge to **0** tangentially to the characteristic direction.

Affine surface of a homogeneous vector field

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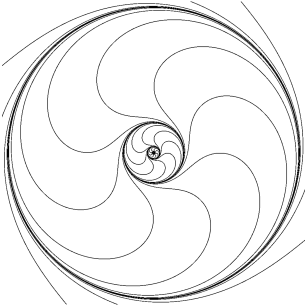
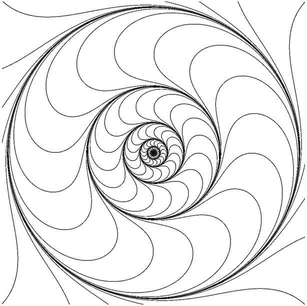


$\rho = 1 - 2\mathrm{i}$



$\rho = 1 - 4\mathrm{i}$

Proposition (Rivi,Rong)

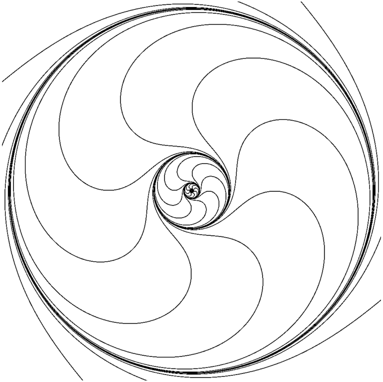
If ν has a simple pole and $\mathrm{Re}(\rho) = 1$, there is a parabolic domain on which orbits converge to **0** spiraling around the characteristic direction.

Closed geodesics

- A geodesic $\delta : I \to \mathbf{S}$ is *closed* if there exists $\lambda \in (0, +\infty)$ and $t_0 < t_1$ in I such that

$$\delta(t_1) = \delta(t_0) \quad \text{and} \quad \dot{\delta}(t_1) = \lambda \dot{\delta}(t_0).$$

- Such a geodesic is *attracting* if $\lambda \in (0, 1)$.



Spiraling domains

- If an affine surface contains an attracting closed geodesic, it contains an *attracting dilation cylinder* foliated by attracting closed geodesic.

Proposition (In progress)

Assume $F(\boldsymbol{x}) = \boldsymbol{x} + \vec{\boldsymbol{v}}(\boldsymbol{x})$ with $\vec{\boldsymbol{v}}$ homogeneous. If $\mathbf{S}_{\vec{\boldsymbol{v}}}$ contains an attracting dilation cylinder $\mathcal{C}$, then F has a spiraling domain in which orbits converge to $\mathbf{0}$, spiraling towards an attracting closed geodesic of $\mathcal{C}$.

Proposition

Assume $a \in \mathbb{R} \setminus \{0\}$ and

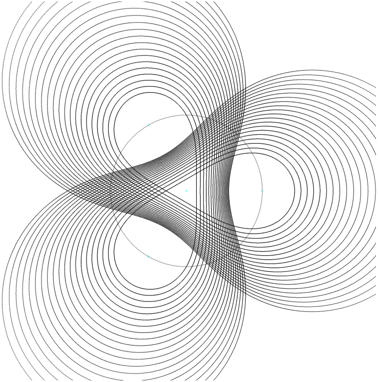
$$\vec{\boldsymbol{v}} := \big(y^2 + ax(x-y)\big)\partial_x + \big(x^2 + ay(x-y)\big)\partial_y.$$

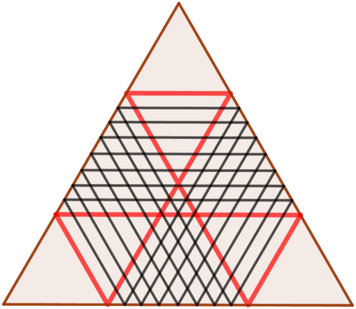
Then, $\mathbf{S}_{\vec{\boldsymbol{v}}}$ contains infinitely many non homotopic attracting dilation cylinders.

Polygonal billiards

Proposition (Valdez)

If the residues of the 1-form $\frac{\mathrm{d}z}{z-f(z)}$ are real and positive, the real-time dynamics of $\vec{v}$ is controlled by the dynamics in a polygonal billiard.





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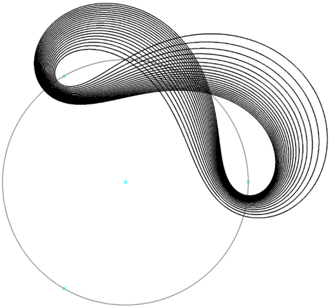
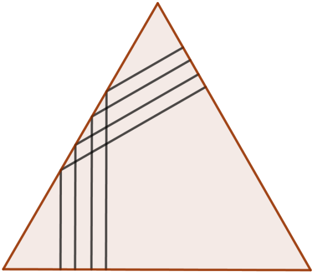
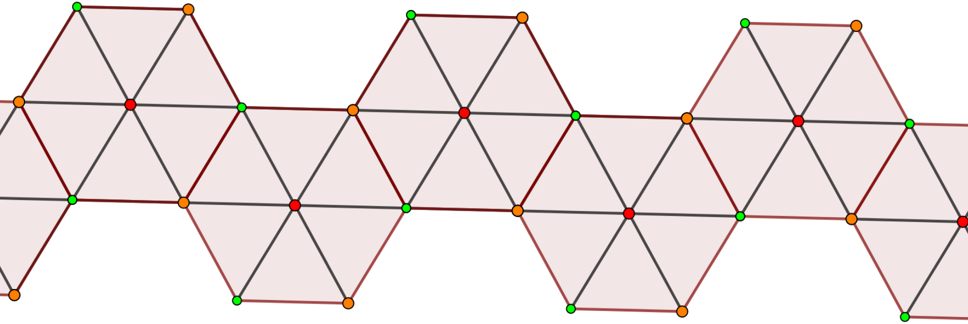




Polygonal models

- If
$$\vec{v} = y^2\partial_x + x^2\partial_y,$$
the affine surface $\mathbf{S}_{\vec{v}}$ may be obtained by gluing equilateral triangles.



- If

$$\vec{v} := \left(y^2 + ax(x-y)\right)\partial_x + \left(x^2 + ay(x-y)\right)\partial_y.$$

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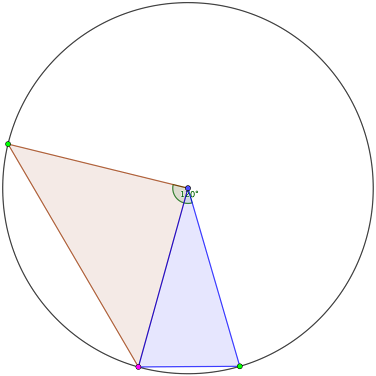

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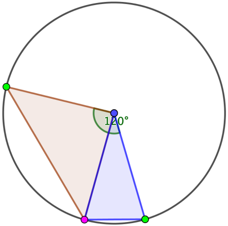

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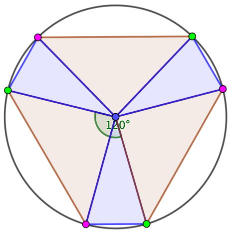

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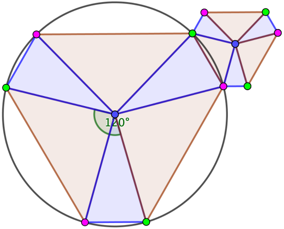

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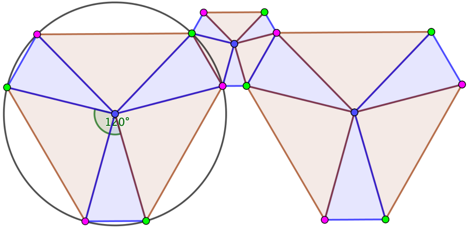

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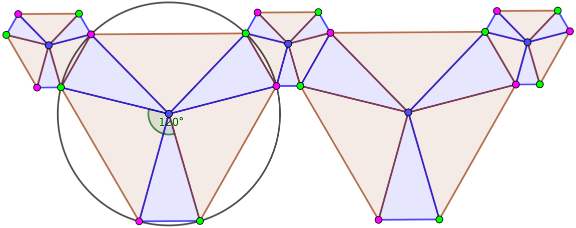

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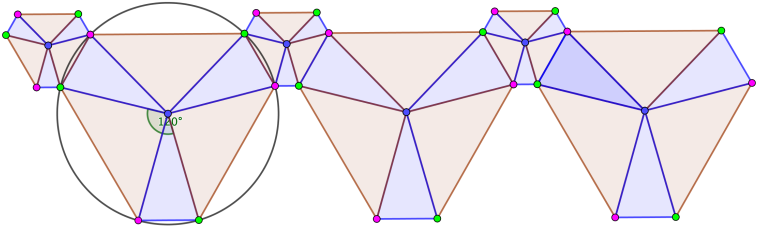
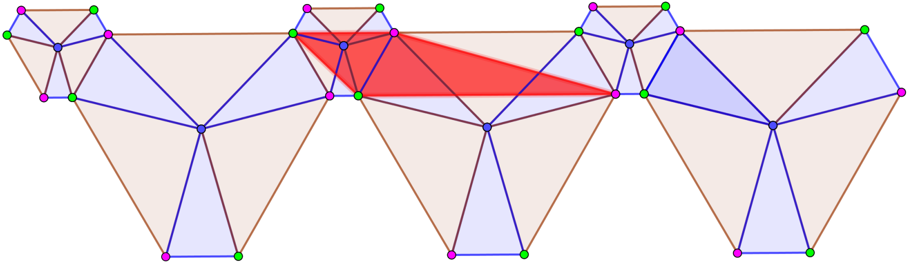
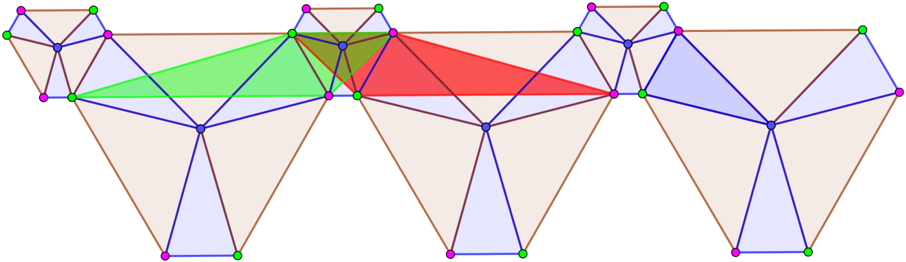


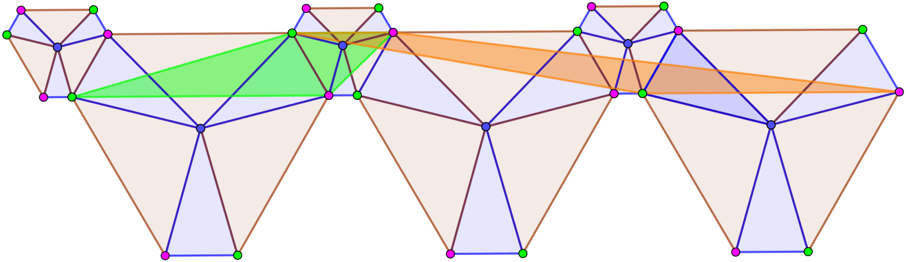
One attracting cylinder



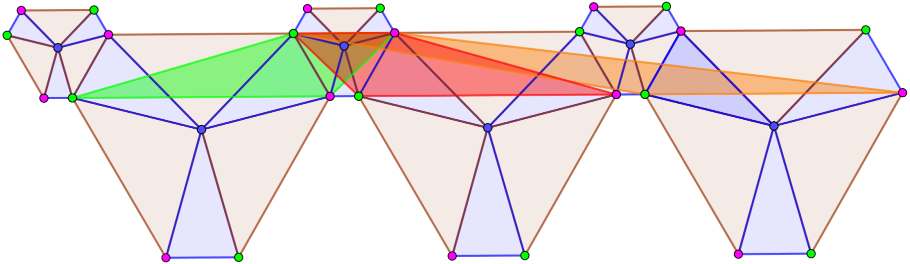
A second attracting cylinder



A third attracting cylinder



Three attracting cylinders



Thank you for your attention

Special thanks to
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