MSRI

talk Thomas GAUTHIER

joint with G. Vigny

Motivation:

- $(f_\lambda)_{\lambda\in\Lambda}$ family of rational maps.

Λ $\mathbb{C}$-manifold connected,

$$f : \Lambda \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \text{ holomorphic}$$
$$(\lambda, z) \longmapsto f_\lambda(z)$$

f_λ rational of degree $\underline{\underline{d}}$.

- $(f_\lambda)_{\lambda\in\Lambda}$ is algebraic if Λ quasi-proj. variety and f is a morphism

- <u>Theorem</u> (McMullen 87) If $(f_\lambda)_{\lambda\in\Lambda}$ algebraic family and stable, then $(f_\lambda)_{\lambda\in\Lambda}$ is either isotrivial or Lattès.

* <u>isotrivial</u>: $\forall \lambda, \lambda' \in \Lambda$, $\exists\, \phi \in \mathrm{Aut}(\mathbb{P}^1)$ s.t.

$$\phi \circ f_\lambda = f_{\lambda'} \circ \phi \text{ on } \mathbb{P}^1.$$

* <u>Lattès</u>: $\forall \lambda \in \Lambda$, $\exists\, E_\lambda$ elliptic curve, $\exists N \in \mathbb{Z}$ & and $\psi : E_\lambda \xrightarrow[\text{finite}]{} \mathbb{P}^1$

s.t.

$$\begin{tikzcd}
E_\lambda \arrow[r, "\times N"] \arrow[d, "\psi"'] & E_\lambda \arrow[d, "\psi"] \\
\mathbb{P}^1 \arrow[r, "f_\lambda"'] & \mathbb{P}^1
\end{tikzcd}$$

$(f_\lambda)_{\lambda \in \Lambda}$ is stable if $\forall \lambda_0 \in \Lambda$, $\exists\, U \subset \Lambda$ neigh. of λ_0

s.t. $\forall \lambda \in U$, $\exists\, h : J_{\lambda_0} \xrightarrow{\simeq} J_\lambda$

s.t. $h \circ f_{\lambda_0} = f_\lambda \circ h$.

$\bigcup_{n \geq 1} f_\lambda^n(C_{f_\lambda})$ finite

Combination of 2 results:

① $(f_\lambda)_{\lambda \in \Lambda}$ is algebraic and stable
$\Longrightarrow$ $(f_\lambda)_{\lambda \in \Lambda}$ is isotrivial or PCF.

② $(f_\lambda)_{\lambda\in\Lambda}$ s.t. $\forall \lambda$ f_λ is PCF, then

$(f_\lambda)_{\lambda\in\Lambda}$ is either Lattès or isotrivial.

① relies on Montel thm + Riemann-Hurwitz theorem

+ Mañé-Sad-Sullivan + Lyubich.

② relies on Thurston theory.

Focus on ① and its generalizations

Hypothesis: $\exists c_1, \dots, c_{2d-2} : \Lambda \longrightarrow \mathbb{P}^1$ holomorphic

and s.t. $C_{f_\lambda} = \{c_1(\lambda), \dots, c_{2d-2}(\lambda)\}$
with multiplicities.

Theorem (Mañé-Sad-Sullivan, Lyubich)

$(f_\lambda)_{\lambda\in\Lambda}$ holomorphic family. $(f_\lambda)_{\lambda\in\Lambda}$ stable

$\iff$ [$\forall j$, $\{\lambda \longmapsto f_\lambda^n(c_j(\lambda))\}_{n\geq 0}$ is a normal family on Λ.]

The set Stab $\subset \Lambda$ of stable parameters is open and dense in any family.

Theorem ①: If $\{\lambda \mapsto f_\lambda^n(c_j(\lambda))\}_{\substack{n\geq 0\\ 1\leq j\leq 2d-2}}$ is normal and if $(f_\lambda)_{\lambda\in\Lambda}$ is stable, then

either $(f_\lambda)_{\lambda\in\Lambda}$ is isotrivial or each c_j is persistently preperiodic: $\exists\, n > m \geq 0$ s.t.

$$f_\lambda^n(c_j(\lambda)) = f_\lambda^m(c_j(\lambda)) \quad \forall \lambda.$$

* Dujardin–Favre (08)

If $\exists j$ s.t. $\{\lambda \mapsto f_\lambda^n(c_j(\lambda))\}_{n\geq 1}$ is equicont.

then c_j is persistently preperiodic or $(f_\lambda)_\lambda$ isotrivial.

* DeMarco (16)

If $\exists\, a : \Lambda \longrightarrow \mathbb{P}^1$ rational functn s.t.

$\{\lambda \mapsto f_\lambda^n(a(\lambda))\}_{n\geq 1}$ is equicontinuous,

then either a is persistently preperiodic, or

(f, a) is isotrivial:

Given $\lambda_0 \in \Lambda$

$\exists\, p : \Lambda' \longrightarrow \Lambda$ finite, $\exists\, \phi : \Lambda' \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ s.t.

$\forall t \in \Lambda'$, $\phi_t \in \mathrm{Aut}(\mathbb{P}^1)$ s.t. $\phi_t^{-1}(a(p(t))) \equiv \text{cte}$.

$$\phi_t \circ f_{\lambda_0} = f_{p(t)} \circ \phi_t \quad \text{on } \mathbb{P}^1$$

$\dim \Lambda = 1$:

$\Big[$ * (f, a) stable ($\{\lambda \mapsto f_\lambda^n(a(\lambda))\}_n$ is normal)
$\Rightarrow \deg(\lambda \mapsto f_\lambda^n(a(\lambda))) \leq C$ indep of n.

↳ # Riemann-Hurwitz.

Higher dimensional situation

$(f_\lambda)_{\lambda \in \Lambda}$ family of endomorphisms of $\mathbb{P}^k$, $k \geq 2$.

$$f : \Lambda \times \mathbb{P}^k \longrightarrow \Lambda \times \mathbb{P}^k$$
$$(\lambda, z) \longmapsto (\lambda, f_\lambda(z)).$$

Λ = quasi-proj. curve, $\deg(f_\lambda) = d$, $\forall \lambda$

ω = Fubini-Study form of $\mathbb{P}^k$,

$\hat{\omega} = \pi_{\mathbb{P}^k}^* \omega$, $\quad \pi_{\mathbb{P}^k} : \Lambda \times \mathbb{P}^k \to \mathbb{P}^k$

$\pi_\Lambda : \Lambda \times \mathbb{P}^k \to \Lambda$.

$$\frac{1}{d^n}(f^n)^* \hat{\omega} \longrightarrow \hat{T} = \hat{\omega} + dd^c g$$

fibered Green current.

$a : \Lambda \to \mathbb{P}^k$ morphism.

. bifurcation measure of (f, a):

$$\mu_{f,a} := (\pi_\Lambda)_* \left(\underbrace{\widehat{T}_f \wedge [\widehat{\Gamma}_a]}_{} \right) = (id, a)^* (\widehat{T})$$

Theorem (DeMarco, Dujardin-Favre)

$\mu_{f,a} = 0 \iff (f, a)$ is stable. on $\mathbb{P}^1$

Theorem (G-Vigny) if $a_n : \lambda \longmapsto f_\lambda^n(a(\lambda))$,

$$\deg(a_n) = d^n \underbrace{\int_\Lambda \mu_{f,a}}_{\text{if } =0} + O(1)$$

$O(1)$ depends on f and $\deg(a)$.

$\Rightarrow (\deg(a_n))_n$ bounded.

What is the correct statement?

⚠ there tricky examples!

Ex: $f_\lambda(z, w) = (z^2, w^2 + \lambda z)$, $\lambda \in \mathbb{C}$. $(z, w) \in \mathbb{C}^2$

f_λ extends holomorphically to $\mathbb{P}^2 \xrightarrow[f_\lambda]{} \mathbb{P}^2$.

$f_\lambda([z : w : 0]) = [z^2 : w^2 : 0]$ indep. of λ.

• $(f_\lambda)_{\lambda\in\mathbb{C}}$ is not "trivial".

• the action on $L_\infty = $ (line at ∞) is indep of λ.

take $a = [a_0 : b_0 : 0]$ $\quad (a_0, b_0) \neq (0,0)$

then a_n is constant $\forall n$.

$\{a \,/\, (f, a) \text{ stable}\}$ not countable!

Here we have $\mathcal{Y} \subsetneq \Lambda \times \mathbb{P}^k$ subvariety s.t.

$\pi_\Lambda|_{\mathcal{Y}} : \mathcal{Y} \longrightarrow \Lambda$ is surjective proper flat (over an open set $= \Lambda$)

$\forall \lambda \in \Lambda$ $Y_\lambda = (\pi_\Lambda|_{\mathcal{Y}})^{-1}\{\lambda\}$, $\exists N \geq 1$, s.t.

$f_\lambda^N(Y_\lambda) \subset Y_\lambda$ and $(f_\lambda^N)_{\lambda\in\Lambda}$ is isotrivial on Y_λ:

$\forall \lambda, \lambda' \in \Lambda$, $\exists\, \phi : Y_\lambda \longrightarrow Y_{\lambda'}$ isomorphism s.t.

$$\phi \circ f_\lambda^N|_{Y_\lambda} = f_{\lambda'}^N|_{Y_{\lambda'}} \circ \phi .$$

$(\mathcal{Y}, f^N)$ is isotrivial.

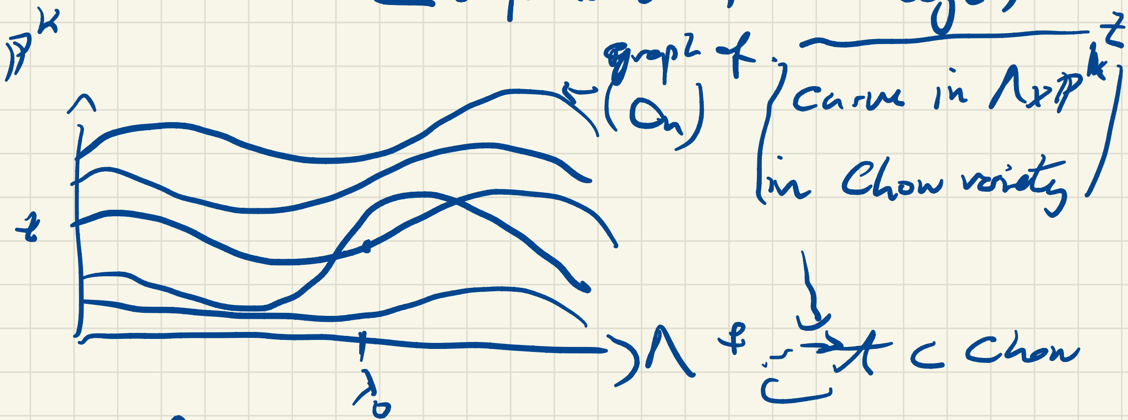
Theorem (G-Vigny) If (f, a) is stable and $(f_\lambda)_{\lambda\in\Lambda}$ not isotrivial, $\exists\, \mathcal{Y} \subsetneq \Lambda \times \mathbb{P}^k$, N st. $(\mathcal{Y}, f^N)$ isotrivial, $\exists q \geq 1$ st. $\forall \lambda \in \Lambda$, $\forall n \geq q$, $a_n(\lambda) \in Y_\lambda$

• If Y_λ finite $\forall \lambda$, then a is persistently preperiodic.

Step 1: $\deg(a_n) = d^n \underbrace{\int_\Lambda \mu_{f,a}}_{=0} + O(1)$

$\Rightarrow \deg(a_n) \leq C \quad \forall n$

C depends on f and $\deg(a)$.



$\Lambda \ni \lambda \mapsto \mathcal{A} \subset$ Chow

$$\hat{\mathcal{A}} = \{(C, \lambda, z) \;/\; (\lambda, z) \in C\}$$

$C \in \mathcal{A}$

$\pi : \hat{\mathcal{A}} \longrightarrow \Lambda \times \mathbb{P}^k$ isomorphism.

$(C, \lambda, z) \longmapsto (\lambda, z)$