Invariant currents for surface maps with transcendental dynamical degrees

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Potential theory and the Julia Set

Recall the following result of Lyubich and Friere/Lopes/Manẽ.

Theorem

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be rational with $\deg f \geq 2$. Then $\exists$ an f-invariant ergodic probability measure μ with $\operatorname{supp} \mu = \mathcal{J}(f)$, obtained as the weak limit of preimages of any non-exceptional point $z \in \mathbb{P}^1$:

$$\lim \frac{1}{(\deg f)^n} \sum_{f^n(w)=z} \delta_w = \mu.$$

In particular, we have *backward* invariance $f^*\mu = (\deg f) \cdot \mu$

Two complex variables

Examples of rational maps of two complex variables:

- Monomial map: $h_A(x_1, x_2) := (x_1^{A_{11}} x_2^{A_{12}}, x_1^{A_{21}} x_2^{A_{22}})$ for some non-singular 2×2 integer matrix A.
- Birational involution: $g(x_1, x_2) = \left(x_1 \frac{1-x_1+x_2}{x_1+x_2-1}, x_2 \frac{1+x_1-x_2}{x_1+x_2-1}\right)$.
- Main character: $f := g \circ h_A$.

Any smooth compact rational surface X can be the domain of f. Choose $X = \mathbb{P}^2$ for now. In general $\exists$ (domain dependent) sets:

- $\mathrm{Ind}(f_X) =$ finite set of 'indeterminacy' points where $f : X \dashrightarrow X$ is undefined.
- $\mathrm{Exc}(f_X) =$ finite union of 'exceptional' (algebraic) curves f_X contracts to points.

Degrees in $\dim_{\mathbb{C}} = 2$

For rational $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $\exists$ several relevant 'degrees'.

- Topological degree $d_{top}(f) := \#f^{-1}(x)$ for general $x \in \mathbb{P}^2$;
- Algebraic degree $d_{alg}(f) := \deg f^{-1}(H)$, for general line $H \subset \mathbb{P}^2$.
- Dynamical degree $\lambda(f) := \lim_{n\to\infty} d_{alg}(f^n)^{1/n} \leq d_{alg}(f)$.

Cases:

- Involution eg. $d_{alg}(g) = 2$, $d_{top}(g) = \lambda(g) = 1$;
- Monomial egs. $d_{top}(h_A) = |\det A|$, $\lambda(h_A) = \max |\text{eigenvalue of } A|$, $d_{alg}(h_A) =$ exercise.

Algebraic stability

Definition

A rational map f is *algebraically stable* on a rational surface X if $f_X^n(\mathrm{Exc}(f_X)) \cap I(f_X) = \emptyset$ for all $n > 0$.

Proposition

If $f_X : X \dashrightarrow X$ is an algebraically stable rational map on a rational surface X, then for all $n \neq 0$,

$$(f_X^n)^* = (f_X^*)^n : H^2(X, \mathbb{R}) \to H^2(X, \mathbb{R}).$$

It follows that $\lambda(f)$ is the the spectral radius of f_X^*, in particular an algebraic integer.

Invariant currents for algebraically stable maps

Theorem (D-Dujardin-Guedj)

Let $f_X : X \dashrightarrow X$ be an algebraically stable rational map on a smooth rational surface X. If $\lambda(f)^2 > d_{top}(f)$, then $\exists$ a positive closed $(1,1)$ current T^* on X such that $f_X^* T^* = \lambda T^*$.

If in fact $\lambda(f) > d_{top}(f)$, then

- *T^* does not charge curves in X;*
- *For almost all irreducible curves $H \subset X$ there exists $c(H) > 0$ such that*

$$\frac{f_X^{-n}(H)}{\lambda(f)^n} = cT^*;$$

- *T^* is strongly laminar;*

Monomial maps and toric surfaces

A *toric surface* is, for present purposes, a smooth projective surface X with an embedding $(\mathbb{C}^*)^2 \hookrightarrow X$ of the algebraic 2-torus such that

- The natural action of $(\mathbb{C}^*)^2$ on itself extends holomorphically to X;
- (alternatively) X can be obtained[1] from $\mathbb{P}^2$ by repeated 'satellite' blowups along the coordinate axes.
- (alternatively) the irreducible components of $X \setminus (\mathbb{C}^*)^2$ are the simple poles of the meromorphic 2-form $\eta := \frac{dx_1 \wedge dx_2}{x_1 x_2}$.

Irreducible components of $X \setminus (\mathbb{C}^*)^2$ are indexed by a finite set $\Sigma_1(X)$ of rational rays $\sigma \subset \mathbb{R}^2$ $(= N_{\mathbb{R}})$.

[1] convenient fib alert

Toric surfaces and monomial maps (cont)

Any monomial map h_A is a self-cover of $(\mathbb{C}^*)^2$, semiconjugated to $A : \mathbb{R}^2 \to \mathbb{R}^2$ via the 'logarithm' map

$$L(x) := (-\log|x_1|, -\log|x_2|).$$

This makes it easy to understand the extension of h_A to a toric surface.

Theorem (Favre)

Suppose $\zeta \in \mathbb{Z}[i]$ *satisfies* $\frac{1}{2\pi}\arg\zeta \notin \mathbb{Q}$. *If* $A = \begin{pmatrix} \operatorname{Re}\zeta & -\operatorname{Im}\zeta \\ \operatorname{Im}\zeta & \operatorname{Re}\zeta \end{pmatrix}$, *then there exists no surface* X *such that* $h_A : X \dashrightarrow X$ *is algebraically stable.*

Toric maps

Jan-Li Lin and I considered the following class of rational maps at length.

Definition

A rational map $f : (\mathbb{C}^*)^2 \dashrightarrow (\mathbb{C}^*)^2$ is *toric* if $f^*\eta = \delta\eta$ for some $\delta \in \mathbb{C}^*$.

- A monomial map h_A is toric with $\delta = \det A$;
- The above involution g is toric with $\delta = 1$;
- Compositions of toric maps are toric (e.g. $f := g \circ h_A$);
- (Blanc) The full group of toric **bi**rational surface maps is known;
- Question: are all toric maps finite compositions of monomial and birational toric maps?

Toric maps: good news/bad news

Toric maps f do *not* usually respect the group structure of $(\mathbb{C}^*)^2$, but they act fairly well on toric surfaces. If X and Y are toric surfaces, then

- there exists a toric blowup $\hat{X} \to X$ such that $\mathrm{Ind}(f) := \mathrm{Ind}(f_{\hat{X}Y})$ is a finite subset of $\hat{X} \setminus (\mathbb{C}^*)^2$ with image $f_{\hat{X}Y}(\mathrm{Ind}(f))$ equal to finitely many 'internal' curves in Y;
- ...analogous statement for the exceptional set $\mathrm{Exc}(f) := \mathrm{Exc}(f_{X\hat{Y}})$ under suitable toric blowup $\hat{Y} \to Y$ of the range.
- These 'persistent' indeterminacy and exceptional sets and their images by f are independent of the toric surfaces.
- f maps 'poles to poles' according to some induced 1-homogeneous PL map $T_f : \mathbb{R}^2 \to \mathbb{R}^2$.

The bad map

Let A be the matrix associated to multiplication by $\zeta \in \mathbb{C}[i]$ with $\zeta^n \notin \mathbb{C}$ for any $n > 0$. From now on $f = g \circ h_A$. Note f is a toric map with $T_f = T_{h_A} = A$.

Theorem (Bell-D-Jonsson)

The dynamical degree $\lambda(f)$ of such a map is transcendental.

Remarks: $\lambda(f)^2 > d_{top}$ always, but one can choose A to get either $\lambda(f) > d_{top}(f)$ or $\lambda(f) < d_{top}(f)$.

Proposition (It's not *that* bad)

f is 'internally stable'. That is, $f^n(\mathrm{Exc}(f)) \cap \mathrm{Ind}(f) = \emptyset$ for any $n \in \mathbb{N}$.

Warning: to make sense of the last sentence, the domain and range toric surfaces need to be different and depend on n.

The main theorem:

Theorem (D-Roeder)

There exists a positive closed current T^* (independent of the toric surface) such that

- T^* *does not charge curves;*
- $f^* T^*|_{(\mathbb{C}^*)^2} = \lambda T^*$;
- *for any internal curve $C \subset X$, there exists $c > 0$ such that*

$$\lim_{n \to \infty} \frac{1}{\lambda(f)^n} f^{n*} C|_{(\mathbb{C}^*)^2} = cT^*;$$

- *hence T^* is strongly laminar.*

The bad map is good to volume

Key (new) ingredients in the proof. First,

Theorem

Given a toric surface X with volume form dV and $\rho > d_{top}$, there exist constants $a, b > 0$ such that for any open $U \subset X$ and any $n \geq 0$,

$$\mathrm{Vol}(f^n(U)) \geq (a\,\mathrm{Vol}(U))^{b\rho^{n/2}}.$$

The invariant two form η and the fact that A acts by irrational rotation on $\mathbb{R}^2$ are important in the proof of this fact.

Every class on every toric surface, all at once

An idea from Boucksom/Favre/Jonsson and Cantat.

- If X and Y are toric surfaces, then we say $X \succ Y$ if the birational map $\mu_{XY} : X \to Y$ is a morphism.
- Identify a class $\alpha_X \in H^2(X, \mathbb{R})$ with its pushforward $\alpha_Y = \mu_{XY*}\alpha_X$.
- A *toric* class $\alpha \in H^2$ is a choice of a class α_X on every toric surface X compatible under pushforwards by birational morphisms.

Corollary

There exists a nef class $\theta^* \in H^2$ such that for any other nef class $\theta \in H^2$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda(f)^n} f^{n*}\theta = c(\theta)\theta^*,$$

where $c(\theta) > 0$ depends linearly on θ. In particular $f^*\theta^* = \lambda\theta^*$.

Currents and classes

One can define (positive) toric *currents* exactly the same way as toric classes.

- Toric currents S, T are *completely cohomologous* if for each toric surface X the representatives T_X and S_X are cohomologous.
- Then $S_X - T_X = dd^c\varphi$ for φ integrable and independent of X.
- T and its complete cohomologues define a toric class.
- Conversely each (nef) toric class has a canonical (positive) 'homogeneous' representative.
- Positive toric currents in a compact set of nef classes form a compact set in the weak topology...
- ...which leads to uniform volume estimates for sublevel sets of potentials.

...yada, yada, yada.

I'm done now

Thanks to the organizers and to MSRI, and thanks for your attention!