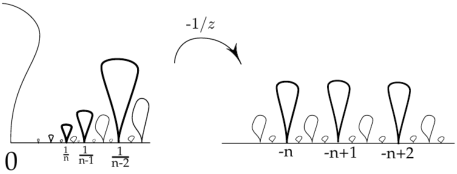
Complex rotation numbers and renormalization

Nataliya Goncharuk, University of Toronto
natalia.goncharuk@utoronto.ca

MSRI workshop "Adventurous Berkeley Complex Dynamics", May 2-6, 2022



May 2, 2022

Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\operatorname{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{x} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

f has a periodic orbit $\Leftrightarrow$ $\operatorname{rot} f$ is rational.

[Denjoy] C^2-smooth f is continuously conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is irrational.

[Arnold, Herman, Yoccoz] $f \in C^\omega$ is analytically conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is a Herman number.

Diophantine $\subset$ Herman $\subset$ Brjuno $\subset$ irrational numbers.

Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\operatorname{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{x} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

f has a periodic orbit $\Leftrightarrow$ $\operatorname{rot} f$ is rational.

[Denjoy] C^2-smooth f is continuously conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is irrational.

[Arnold, Herman, Yoccoz] $f \in C^\omega$ is analytically conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is a Herman number.

Diophantine $\subset$ Herman $\subset$ Brjuno $\subset$ irrational numbers.

Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\operatorname{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{x} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

f has a periodic orbit $\Leftrightarrow$ $\operatorname{rot} f$ is rational.

[Denjoy] C^2-smooth f is continuously conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is irrational.

[Arnold, Herman, Yoccoz] $f \in C^\omega$ is analytically conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is a Herman number.

Diophantine $\subset$ Herman $\subset$ Brjuno $\subset$ irrational numbers.

Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\operatorname{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{x} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

f has a periodic orbit $\Leftrightarrow$ $\operatorname{rot} f$ is rational.

[Denjoy] C^2-smooth f is continuously conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is irrational.

[Arnold, Herman, Yoccoz] $f \in C^\omega$ is analytically conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is a Herman number.

Diophantine $\subset$ Herman $\subset$ Brjuno $\subset$ irrational numbers.

Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

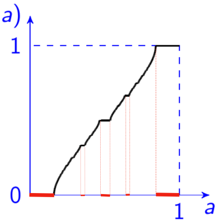
$$\operatorname{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{x} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

f has a periodic orbit $\Leftrightarrow$ $\operatorname{rot} f$ is rational.

[Denjoy] C^2-smooth f is continuously conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is irrational.

[Arnold, Herman, Yoccoz] $f \in C^\omega$ is analytically conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is a Herman number.

Diophantine $\subset$ Herman $\subset$ Brjuno $\subset$ irrational numbers.

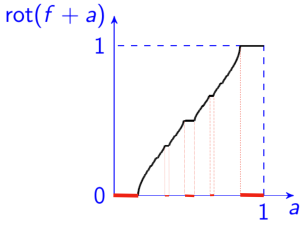


Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\mathrm{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{x} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

f has a periodic orbit $\Leftrightarrow$ rot f is rational.
[Denjoy] C^2-smooth f is continuously conjugate to the rotation by rot f if rot f is irrational.
[Arnold, Herman, Yoccoz] $f \in C^\omega$ is analytically conjugate to the rotation by rot f if rot f is a Herman number.
Diophantine $\subset$ Herman $\subset$ Brjuno $\subset$ irrational numbers.



Rotation number of a circle diffeomorphism.

Let F be a lift of $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

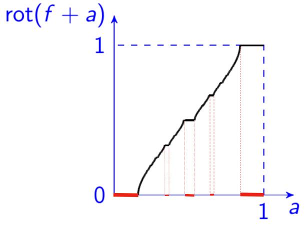
$$\mathrm{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{x} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

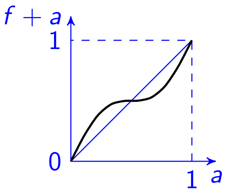
f has a periodic orbit $\Leftrightarrow$ $\mathrm{rot}\, f$ is rational.

[Denjoy] C^2-smooth f is continuously conjugate to the rotation by $\mathrm{rot}\, f$ if $\mathrm{rot}\, f$ is irrational.

[Arnold, Herman, Yoccoz] $f \in C^\omega$ is analytically conjugate to the rotation by $\mathrm{rot}\, f$ if $\mathrm{rot}\, f$ is a Herman number.

Diophantine $\subset$ Herman $\subset$ Brjuno $\subset$ irrational numbers.





Hyperbolic: $(f+a)'(\text{f.p.}) \neq 1$

Rotation number of a circle diffeomorphism.

Let F be a lift of $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

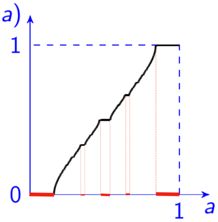
$$\operatorname{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{x} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

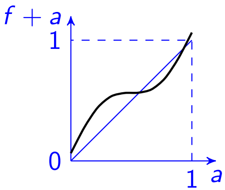
f has a periodic orbit $\Leftrightarrow$ $\operatorname{rot} f$ is rational.

[Denjoy] C^2-smooth f is continuously conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is irrational.

[Arnold, Herman, Yoccoz] $f \in C^\omega$ is analytically conjugate to the rotation by $\operatorname{rot} f$ if $\operatorname{rot} f$ is a Herman number.

Diophantine $\subset$ Herman $\subset$ Brjuno $\subset$ irrational numbers.





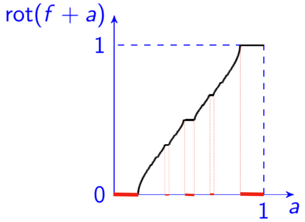
Hyperbolic: $(f+a)'(\text{f.p.}) \neq 1$

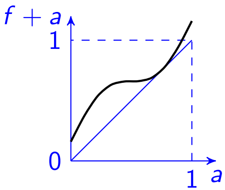
Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\mathrm{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{x} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

f has a periodic orbit $\Leftrightarrow$ $\mathrm{rot}\, f$ is rational.
[Denjoy] C^2-smooth f is continuously conjugate to the rotation by $\mathrm{rot}\, f$ if $\mathrm{rot}\, f$ is irrational.
[Arnold, Herman, Yoccoz] $f \in C^\omega$ is analytically conjugate to the rotation by $\mathrm{rot}\, f$ if $\mathrm{rot}\, f$ is a Herman number.
Diophantine $\subset$ Herman $\subset$ Brjuno $\subset$ irrational numbers.





Parabolic: $(f+a)'(\text{f.p.}) = 1$

Rotation number of a circle diffeomorphism.

Let F be a lift of $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

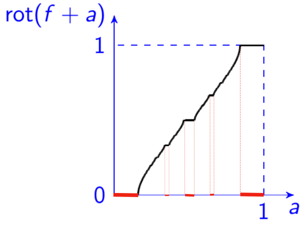
$$\mathrm{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{x} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

f has a periodic orbit $\Leftrightarrow$ $\mathrm{rot}\, f$ is rational.

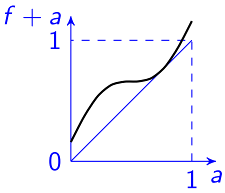
[Denjoy] C^2-smooth f is continuously conjugate to the rotation by $\mathrm{rot}\, f$ if $\mathrm{rot}\, f$ is irrational.

[Arnold, Herman, Yoccoz] $f \in C^\omega$ is analytically conjugate to the rotation by $\mathrm{rot}\, f$ if $\mathrm{rot}\, f$ is a Herman number.

Diophantine $\subset$ Herman $\subset$ Brjuno $\subset$ irrational numbers.



↑Stairs

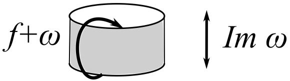


Parabolic: $(f+a)'(\text{f.p.}) = 1$

Arnold's construction (1978)

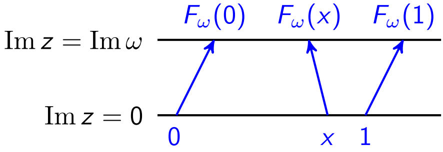
Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 < \operatorname{Im} z < \operatorname{Im} \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* $T_{F+\omega}$ with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.

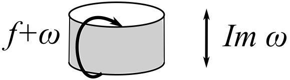


Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

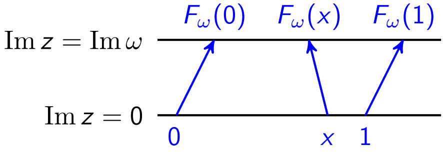


- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 < \operatorname{Im} z < \operatorname{Im} \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* $T_{F+\omega}$ with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.

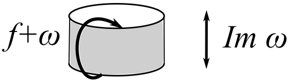


Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

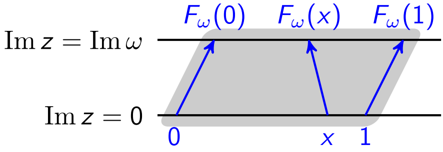


- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 < \mathrm{Im}\, z < \mathrm{Im}\, \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* $T_{F+\omega}$ with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.

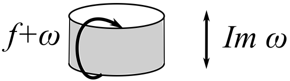


Arnold's construction (1978)

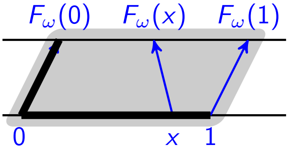
Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



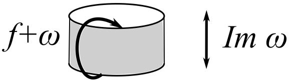
- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 < \operatorname{Im} z < \operatorname{Im} \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* $T_{F+\omega}$ with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.



Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

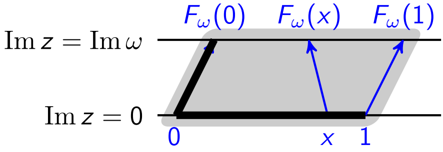


- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 < \operatorname{Im} z < \operatorname{Im} \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* $T_{F+\omega}$ with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.

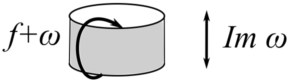


Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

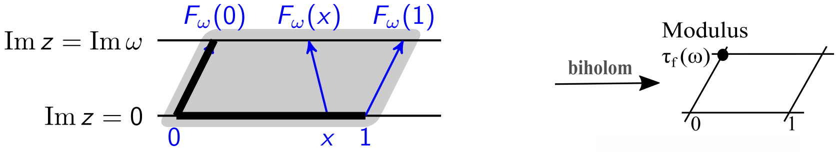


- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 < \operatorname{Im} z < \operatorname{Im} \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* $T_{F+\omega}$ with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.

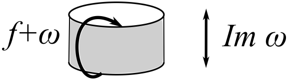


Arnold's construction (1978)

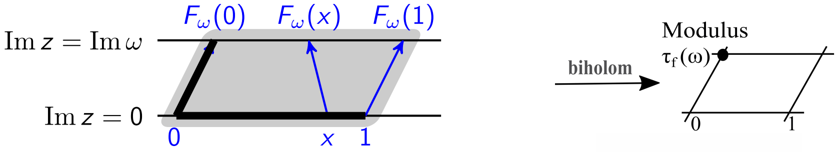
Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 < \operatorname{Im} z < \operatorname{Im} \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* $T_{F+\omega}$ with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.



Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



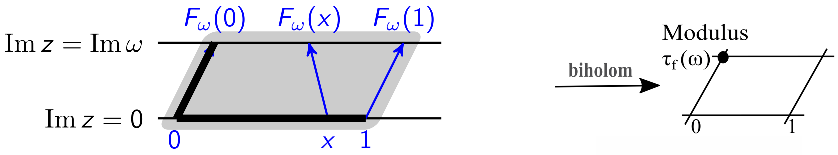
- **Example:** If $f(x)$ is a rotation by ϕ, then $\tau_f(\omega) = \omega + \phi$.
- **Remark:** τ_f is holomorphic.
- **Arnold's conjecture, 1978:**

$$\lim_{\omega \to a \in \mathbb{R}} \tau_f(\omega) = \mathrm{rot}(f + a)$$

  if $\mathrm{rot}(f + a)$ is Diophantine. [TRUE]
- **Ghys's question:** is this true for any irrational rotation number?
- What happens for the rational rotation number?

Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



- **Example:** If $f(x)$ is a rotation by ϕ, then $\tau_f(\omega) = \omega + \phi$.
- **Remark:** τ_f is holomorphic.
- **Arnold's conjecture, 1978:**

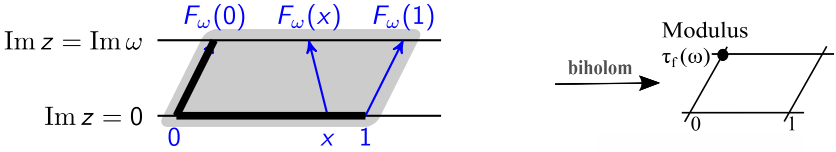
$$\lim_{\omega \to a \in \mathbb{R}} \tau_f(\omega) = \operatorname{rot}(f + a)$$

if $\operatorname{rot}(f + a)$ is Diophantine. [TRUE]

- **Ghys's question:** is this true for any irrational rotation number?
- What happens for the rational rotation number?

Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



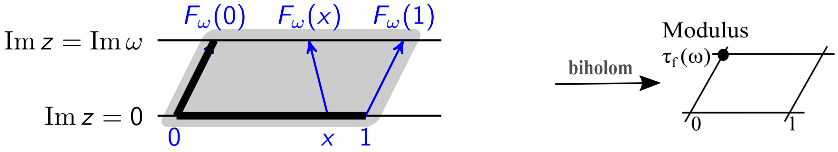
- **Example:** If $f(x)$ is a rotation by ϕ, then $\tau_f(\omega) = \omega + \phi$.
- **Remark:** τ_f is holomorphic.
- **Arnold's conjecture, 1978:**

$$\lim_{\omega \to a \in \mathbb{R}} \tau_f(\omega) = \mathrm{rot}(f + a)$$

  if $\mathrm{rot}(f + a)$ is Diophantine. [TRUE]
- **Ghys's question:** is this true for any irrational rotation number?
- What happens for the rational rotation number?

Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

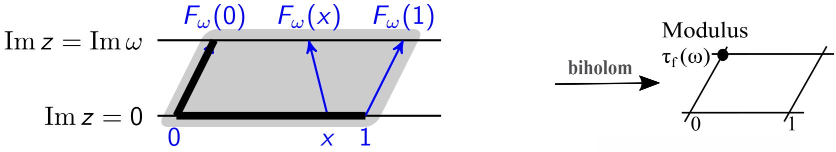


- **Example:** If $f(x)$ is a rotation by ϕ, then $\tau_f(\omega) = \omega + \phi$.
- **Remark:** τ_f is holomorphic.
- **Arnold's conjecture, 1978:**

$$\lim_{\omega \to a \in \mathbb{R}} \tau_f(\omega) = \mathrm{rot}(f + a)$$

  if $\mathrm{rot}(f + a)$ is Diophantine. [TRUE]
- **Ghys's question:** is this true for any irrational rotation number?
- What happens for the rational rotation number?

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



- **Example:** If $f(x)$ is a rotation by ϕ, then $\tau_f(\omega) = \omega + \phi$.
- **Remark:** τ_f is holomorphic.
- **Arnold's conjecture, 1978:**

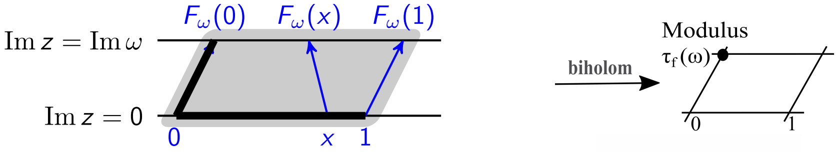
$$\lim_{\omega \to a \in \mathbb{R}} \tau_f(\omega) = \mathrm{rot}(f + a)$$

if $\mathrm{rot}(f + a)$ is Diophantine. [TRUE]

- **Ghys's question:** is this true for any irrational rotation number?
- What happens for the rational rotation number?

Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



- **Example:** If $f(x)$ is a rotation by ϕ, then $\tau_f(\omega) = \omega + \phi$.
- **Remark:** τ_f is holomorphic.
- **Arnold's conjecture, 1978:**

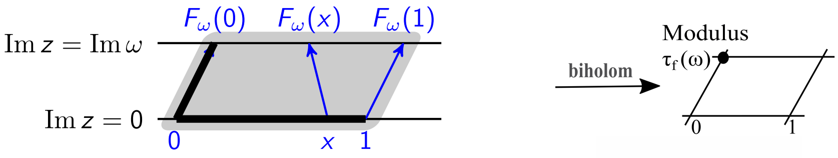
$$\lim_{\omega \to a \in \mathbb{R}} \tau_f(\omega) = \mathrm{rot}(f + a)$$

if $\mathrm{rot}(f + a)$ is Diophantine. [TRUE]

- **Ghys's question:** is this true for any irrational rotation number?
- What happens for the rational rotation number?

Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



- **Example:** If $f(x)$ is a rotation by ϕ, then $\tau_f(\omega) = \omega + \phi$.
- **Remark:** τ_f is holomorphic.
- **Arnold's conjecture, 1978:**

$$\lim_{\omega \to a \in \mathbb{R}} \tau_f(\omega) = \text{rot}(f + a)$$

if $\text{rot}(f + a)$ is Diophantine. [TRUE]

- **Ghys's question:** is this true for any irrational rotation number?
- What happens for the rational rotation number?

Bubbles: overview of results

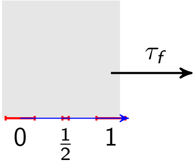
$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim\limits_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. **(Outside stairs)**
- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- $f + a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. **(Endpoints of stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.

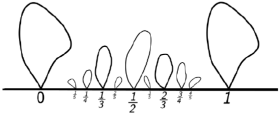
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. **(Outside stairs)**
- $f+a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- $f+a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. **(Endpoints of stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.



↑ Stairs

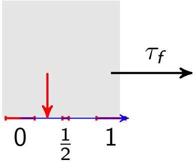


Limit values of τ_f on $\mathbb{R}$.

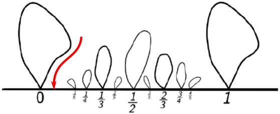
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- rot$(f + a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = $ rot$(f + a)$. **(Outside stairs)**
- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- $f + a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = $ rot$(f + a)$. **(Endpoints of stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q, \alpha))^{\xi}}{q^2}$.



↑ Stairs

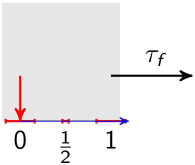


Limit values of τ_f on $\mathbb{R}$.

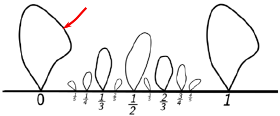
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- rot$(f + a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \text{rot}(f + a)$. **(Outside stairs)**
- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- $f + a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \text{rot}(f + a)$. **(Endpoints of stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^\xi}{q^2}$.



↑ Stairs

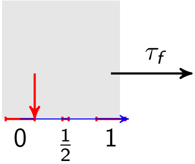


Limit values of τ_f on $\mathbb{R}$.

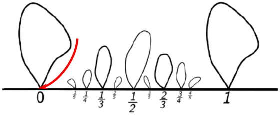
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f + a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Outside stairs)**
- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- $f + a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Endpoints of stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.



↑ Stairs

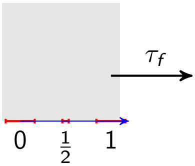


Limit values of τ_f on $\mathbb{R}$.

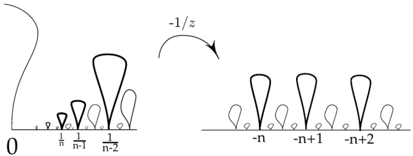
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f + a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Outside stairs)**
- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- $f + a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Endpoints of stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.



↑ Stairs

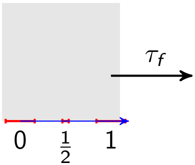


Limit values of τ_f on $\mathbb{R}$.

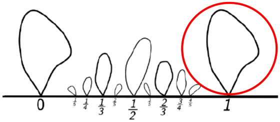
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. **(Outside stairs)**
- $f+a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- $f+a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. **(Endpoints of stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.



↑ Stairs

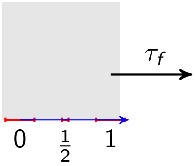


Limit values of τ_f on $\mathbb{R}$.

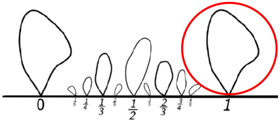
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. **(Outside stairs)**
- $f+a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- $f+a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. **(Endpoints of stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.



↑ Stairs

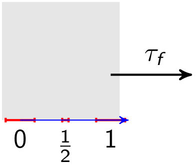


Limit values of τ_f on $\mathbb{R}$.

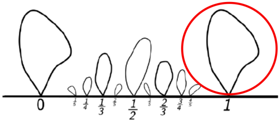
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f + a)$ is Diophantine $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f + a)$. Risler; Moldavskis
- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- $f + a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Endpoints of stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.



↑ Stairs

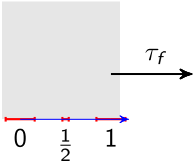


Limit values of τ_f on $\mathbb{R}$.

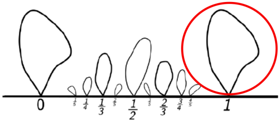
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f + a)$ is Diophantine $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f + a)$. Risler; Moldavskis
- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis;NG
- $f + a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Endpoints of stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.



↑ Stairs

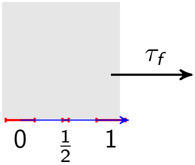


Limit values of τ_f on $\mathbb{R}$.

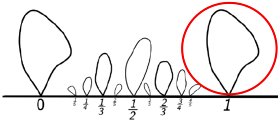
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is Diophantine $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. Risler; Moldavskis
- $f+a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis;NG
- $f+a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. J.Lacroix; NG
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^\xi}{q^2}$.



↑ Stairs

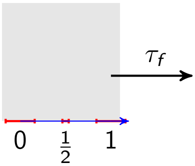


Limit values of τ_f on $\mathbb{R}$.

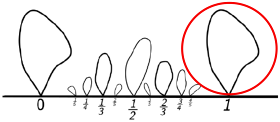
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. X.Buff, NG ← Risler; Moldavskis
- $f+a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis;NG
- $f+a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. J.Lacroix; NG
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.



↑ Stairs

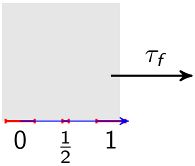


Limit values of τ_f on $\mathbb{R}$.

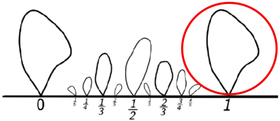
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. X.Buff, NG ← Risler; Moldavskis
- $f+a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis;NG
- $f+a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. J.Lacroix; NG
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^\xi}{q^2}$.



↑ Stairs

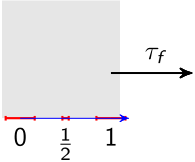


Limit values of τ_f on $\mathbb{R}$.

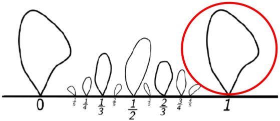
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. X.Buff, NG Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. X.Buff, NG ← Risler; Moldavskis
- $f+a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis;NG
- $f+a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. J.Lacroix; NG
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.



↑ Stairs

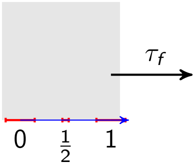


Limit values of τ_f on $\mathbb{R}$.

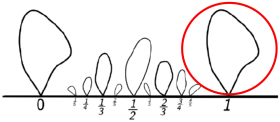
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. X.Buff, NG Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. X.Buff, NG $\leftarrow$ Risler; Moldavskis
- $f+a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis;NG
- $f+a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. J.Lacroix; NG
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.X.Buff, NG
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.



↑ Stairs

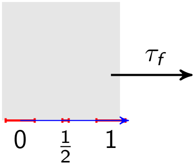


Limit values of τ_f on $\mathbb{R}$.

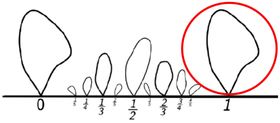
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. X.Buff, NG Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. X.Buff, NG $\leftarrow$ Risler; Moldavskis
- $f+a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis;NG
- $f+a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. J.Lacroix; NG
- Bubbles are (generically) self-similar near rational points. NG
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.X.Buff, NG
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller: $<\sim \frac{(dist(p/q,\alpha))^{\xi}}{q^2}$.



$\uparrow$ Stairs

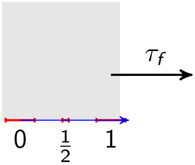


Limit values of τ_f on $\mathbb{R}$.

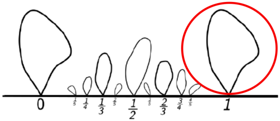
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. X.Buff, NG Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f + a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f + a)$. X.Buff, NG ← Risler; Moldavskis
- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis;NG
- $f + a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f + a)$. J.Lacroix; NG
- Bubbles are (generically) self-similar near rational points. NG
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.X.Buff, NG
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller. NG, I.Gorbovickis



↑ Stairs

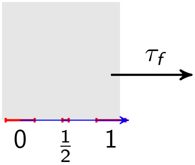


Limit values of τ_f on $\mathbb{R}$.

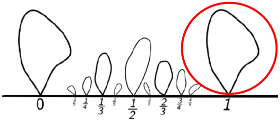
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. X.Buff, NG Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. X.Buff, NG $\leftarrow$ Risler; Moldavskis
- $f+a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis;NG
- $f+a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. J.Lacroix; NG
- Bubbles are (generically) self-similar near rational points. NG
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.X.Buff, NG
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller. NG, I.Gorbovickis



↑ Stairs

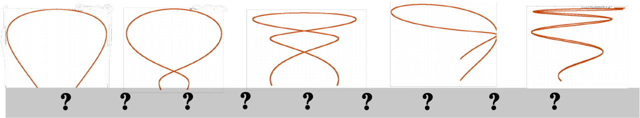


Limit values of τ_f on $\mathbb{R}$.

Bubbles: overview of results

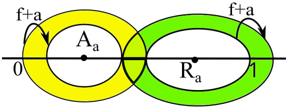
$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. X.Buff, NG Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $\mathrm{rot}(f+a)$ is irrational $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. X.Buff, NG $\leftarrow$ Risler; Moldavskis
- $f+a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis;NG
- $f+a$ is parabolic $\Rightarrow \hat{\tau}_f(a) = \mathrm{rot}(f+a)$. J.Lacroix; NG
- Bubbles are (generically) self-similar near rational points. NG
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.X.Buff, NG
- Near a Diophantine number α, the $\frac{p}{q}$-bubble is much smaller. NG, I.Gorbovickis



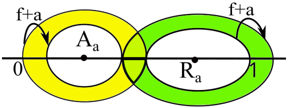
Zero bubbles for perturbations of $z \mapsto \frac{az+b}{cz+d}$, approximation.

On a bubble: let $f + a$ be a hyperbolic circle diffeomorphism and $\omega \to a$.



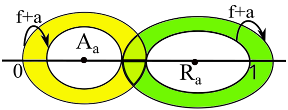
- T_ω is the quotient of a fundamental domain of $f + \omega$ via $f + \omega$. This domain degenerates as $\omega \to a$.
- $f + a$, $a \in \mathbb{R}$, has an annular fundamental domain Π as well!
- $\hat{\tau}_f(a) = \lim_{\omega \to a} \tau_f(\omega)$ is the modulus of the torus $\Pi/(f + a)$.

On a bubble: let $f + a$ be a hyperbolic circle diffeomorphism and $\omega \to a$.



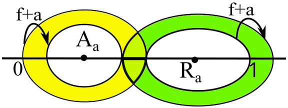
- T_ω is the quotient of a fundamental domain of $f + \omega$ via $f + \omega$. This domain degenerates as $\omega \to a$.
- $f + a$, $a \in \mathbb{R}$, has an annular fundamental domain Π as well!
- $\hat{\tau}_f(a) = \lim_{\omega \to a} \tau_f(\omega)$ is the modulus of the torus $\Pi/(f + a)$.

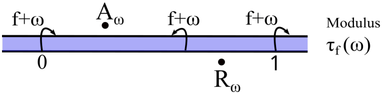
On a bubble: let $f + a$ be a hyperbolic circle diffeomorphism and $\omega \to a$.



- T_ω is the quotient of a fundamental domain of $f + \omega$ via $f + \omega$. This domain degenerates as $\omega \to a$.
- $f + a$, $a \in \mathbb{R}$, has an annular fundamental domain Π as well!
- $\hat{\tau}_f(a) = \lim_{\omega \to a} \tau_f(\omega)$ is the modulus of the torus $\Pi/(f + a)$.

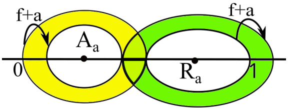
On a bubble: let $f + a$ be a hyperbolic circle diffeomorphism and $\omega \to a$.

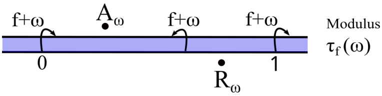




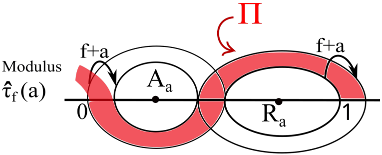
- T_ω is the quotient of a fundamental domain of $f + \omega$ via $f + \omega$. This domain degenerates as $\omega \to a$.
- $f + a$, $a \in \mathbb{R}$, has an annular fundamental domain Π as well!
- $\hat{\tau}_f(a) = \lim_{\omega \to a} \tau_f(\omega)$ is the modulus of the torus $\Pi/(f + a)$.

On a bubble: let $f + a$ be a hyperbolic circle diffeomorphism and $\omega \to a$.

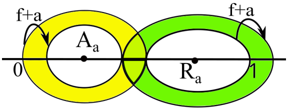


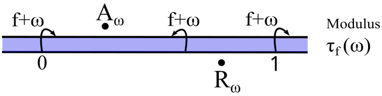


- T_ω is the quotient of a fundamental domain of $f + \omega$ via $f + \omega$. This domain degenerates as $\omega \to a$.
- $f + a$, $a \in \mathbb{R}$, has an annular fundamental domain Π as well!
- $\hat{\tau}_f(a) = \lim_{\omega \to a} \tau_f(\omega)$ is the modulus of the torus $\Pi/(f + a)$.

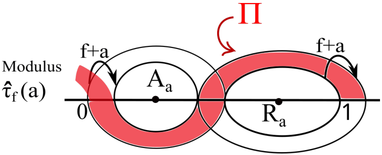


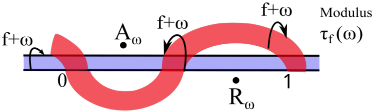
On a bubble: let $f + a$ be a hyperbolic circle diffeomorphism and $\omega \to a$.



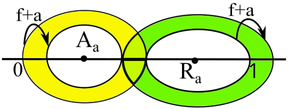


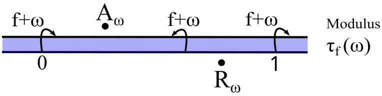
- T_ω is the quotient of a fundamental domain of $f + \omega$ via $f + \omega$. This domain degenerates as $\omega \to a$.
- $f + a$, $a \in \mathbb{R}$, has an annular fundamental domain Π as well!
- $\hat{\tau}_f(a) = \lim_{\omega \to a} \tau_f(\omega)$ is the modulus of the torus $\Pi/(f + a)$.



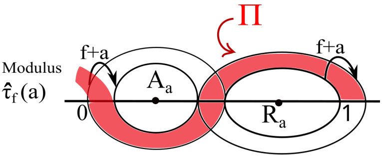


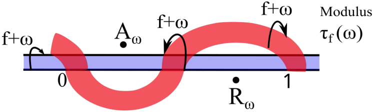
On a bubble: let $f + a$ be a hyperbolic circle diffeomorphism and $\omega \to a$.



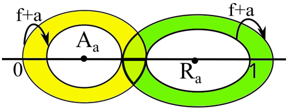


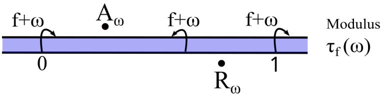
- T_ω is the quotient of a fundamental domain of $f + \omega$ via $f + \omega$. This domain degenerates as $\omega \to a$.
- $f + a$, $a \in \mathbb{R}$, has an annular fundamental domain Π as well!
- $\hat{\tau}_f(a) = \lim_{\omega \to a} \tau_f(\omega)$ is the modulus of the torus $\Pi/(f + a)$.



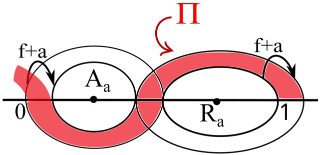


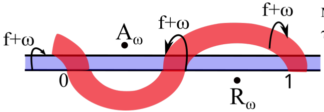
On a bubble: let $f + a$ be a hyperbolic circle diffeomorphism and $\omega \to a$.



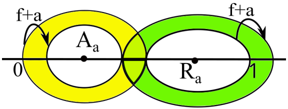


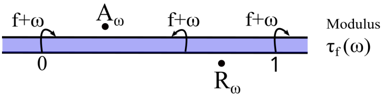
- T_ω is the quotient of a fundamental domain of $f + \omega$ via $f + \omega$. This domain degenerates as $\omega \to a$.
- $f + a$, $a \in \mathbb{R}$, has an annular fundamental domain Π as well!
- $\hat{\tau}_f(a) = \lim_{\omega \to a} \tau_f(\omega)$ is the modulus of the torus $\Pi/(f + a)$.



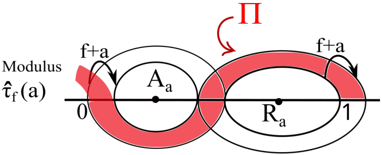


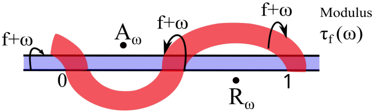
On a bubble: let $f + a$ be a hyperbolic circle diffeomorphism and $\omega \to a$.





- T_ω is the quotient of a fundamental domain of $f + \omega$ via $f + \omega$. This domain degenerates as $\omega \to a$.
- $f + a$, $a \in \mathbb{R}$, has an annular fundamental domain Π as well!
- $\hat{\tau}_f(a) = \lim_{\omega \to a} \tau_f(\omega)$ is the modulus of the torus $\Pi/(f + a)$.





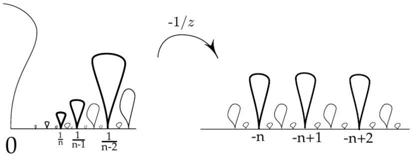
Self-similarity of bubbles

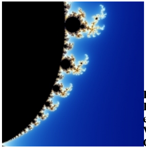
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

$$\operatorname{rot}(\mathcal{R}f) = -\frac{1}{\operatorname{rot} f} \mod 1; \qquad \tau(\mathcal{R}f) = -\frac{1}{\tau(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Fact: $\mathcal{R}(f+a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

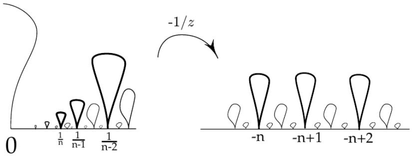
Self-similarity of bubbles

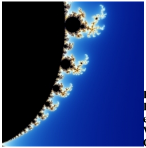
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

$$\mathrm{rot}(\mathcal{R}f) = -\frac{1}{\mathrm{rot}\, f} \mod 1; \qquad \tau(\mathcal{R}f) = -\frac{1}{\tau(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Fact: $\mathcal{R}(f+a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

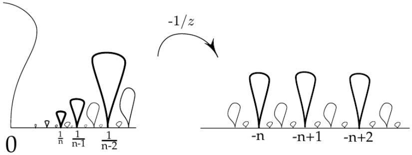
Self-similarity of bubbles

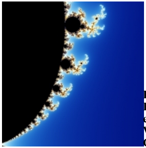
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

$$\operatorname{rot}(\mathcal{R}f) = -\frac{1}{\operatorname{rot} f} \mod 1; \qquad \tau(\mathcal{R}f) = -\frac{1}{\tau(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Fact: $\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





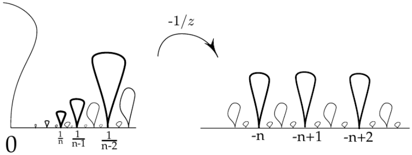
Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

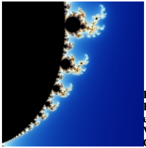
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

$$\operatorname{rot}(\mathcal{R}f) = -\frac{1}{\operatorname{rot} f} \mod 1; \qquad \tau(\mathcal{R}f) = -\frac{1}{\tau(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Fact: $\mathcal{R}(f+a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





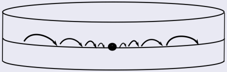
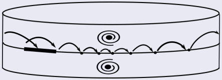
Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

Self-similarity of bubbles

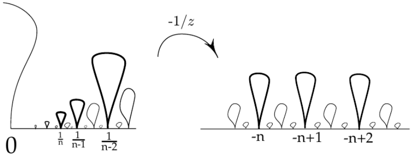
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

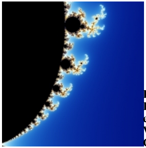
$$\operatorname{rot}(\mathcal{R}f) = -\frac{1}{\operatorname{rot} f} \mod 1; \qquad \tau(\mathcal{R}f) = -\frac{1}{\tau(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Dynamics of f  and $f + a$, $a > 0$: 

Fact: $\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





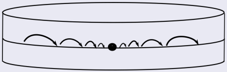
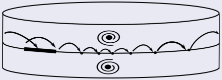
Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed
CC-BY-SA 3.0

Self-similarity of bubbles

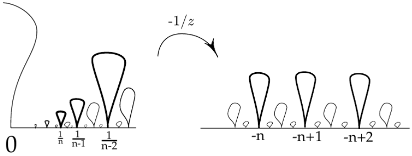
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

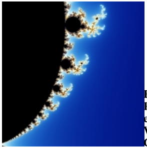
$$\mathrm{rot}(\mathcal{R}f) = -\frac{1}{\mathrm{rot}\, f} \mod 1; \qquad \tau(\mathcal{R}f) = -\frac{1}{\tau(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Dynamics of f  and $f + a$, $a > 0$: 

Fact: $\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.

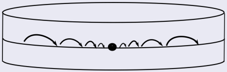
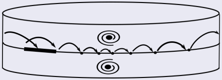




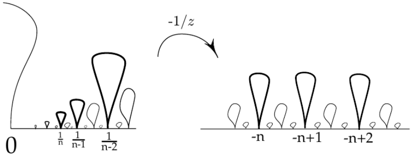
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

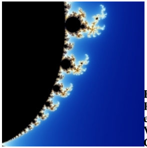
$$\operatorname{rot}(\mathcal{R}f) = -\frac{1}{\operatorname{rot} f} \mod 1; \qquad \tau(\mathcal{R}f) = -\frac{1}{\tau(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Dynamics of f  and $f + a$, $a > 0$: 

Fact: $\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





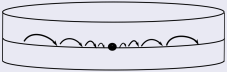
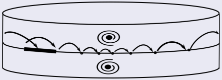
Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

Self-similarity of bubbles

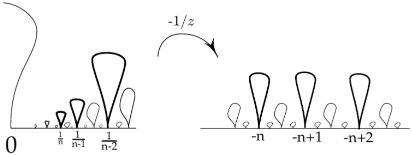
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

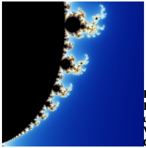
$$\operatorname{rot}(\mathcal{R}f) = -\frac{1}{\operatorname{rot} f} \mod 1; \qquad \tau(\mathcal{R}f) = -\frac{1}{\tau(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Dynamics of f  and $f + a$, $a > 0$: 

Fact: $\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





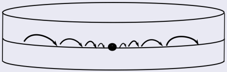
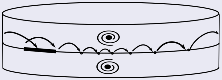
Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

Self-similarity of bubbles

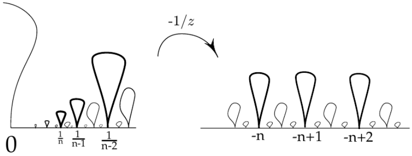
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

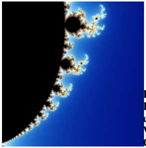
$$\operatorname{rot}(\mathcal{R}f) = -\frac{1}{\operatorname{rot} f} \mod 1; \qquad \tau(\mathcal{R}f) = -\frac{1}{\tau(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Dynamics of f  and $f + a$, $a > 0$: 

Fact: $\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





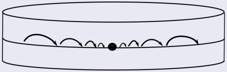
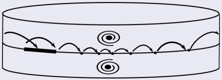
Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

Self-similarity of bubbles

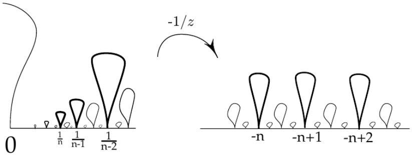
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

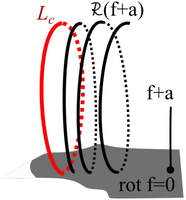
$$\operatorname{rot}(\mathcal{R}f) = -\frac{1}{\operatorname{rot} f} \mod 1; \qquad \tau(\mathcal{R}f) = -\frac{1}{\tau(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Dynamics of f  and $f + a$, $a > 0$: 

Fact: $\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.



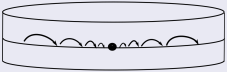
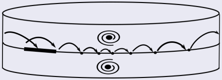


Self-similarity of bubbles

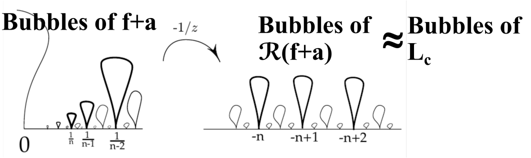
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

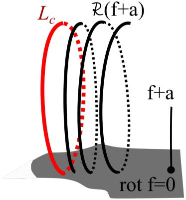
$$\operatorname{rot}(\mathcal{R}f) = -\frac{1}{\operatorname{rot} f} \mod 1; \qquad \tau(\mathcal{R}f) = -\frac{1}{\tau(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Dynamics of f  and $f + a$, $a > 0$: 

Fact: $\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.



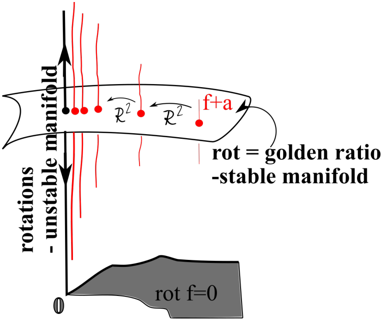


Renormalization (joint with M.Yampolsky) and bubbles

- Golden ratio rotation is a hyperbolic fixed point for $\mathcal{R}^2$
- $\Rightarrow$ bubbles are small near the golden ratio (Gorbovickis, NG; in progress).
- $\Rightarrow$ "rot f = Herman number" is an analytic condition (Risler's theorem).
- "rot f = golden ratio" are at least finitely smooth near critical maps (M.Yampolsky, NG; in progress).

QQ Are they only finitely smooth?

QQ Do critical maps have bubbles? How do they look like?

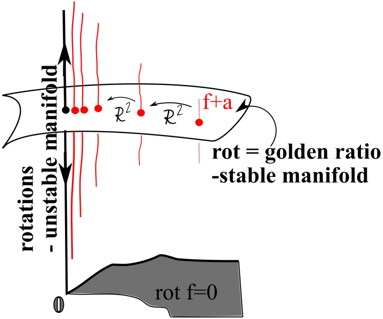


Renormalization (joint with M.Yampolsky) and bubbles

- Golden ratio rotation is a hyperbolic fixed point for $\mathcal{R}^2$
- $\Rightarrow$ bubbles are small near the golden ratio (Gorbovickis, NG; in progress).
- $\Rightarrow$ "rot f = Herman number" is an analytic condition (Risler's theorem).
- "rot f = golden ratio" are at least finitely smooth near critical maps (M.Yampolsky, NG; in progress).

QQ Are they only finitely smooth?

QQ Do critical maps have bubbles? How do they look like?

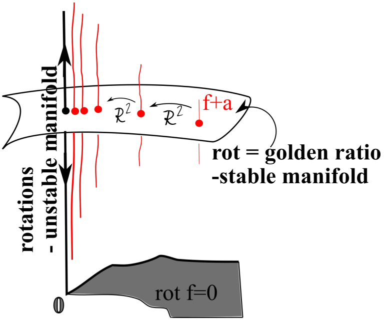


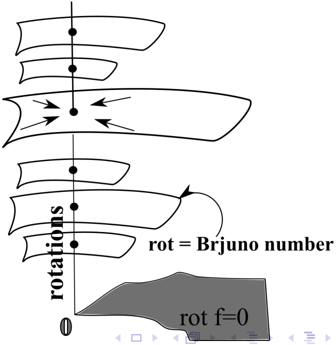
Renormalization (joint with M.Yampolsky) and bubbles

- Brjuno rotations are a hyperbolic set for $\mathcal{R}$ (joint with M. Yampolsky).
- $\Rightarrow$ bubbles are small near Herman numbers (Gorbovickis, NG; in progress).
- $\Rightarrow$ "rot f = Herman number" is an analytic condition (Risler's theorem).
- "rot f = golden ratio" are at least finitely smooth near critical maps (M.Yampolsky, NG; in progress).

QQ Are they only finitely smooth?

QQ Do critical maps have bubbles? How do they look like?



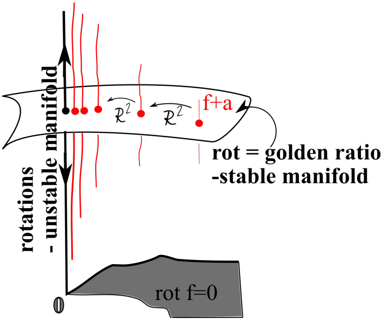


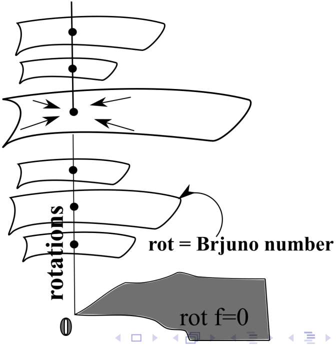
Renormalization (joint with M.Yampolsky) and bubbles

- Brjuno rotations are a hyperbolic set for $\mathcal{R}$ (joint with M. Yampolsky).
- $\Rightarrow$ bubbles are small near Herman numbers (Gorbovickis, NG; in progress).
- $\Rightarrow$ "rot f = Herman number" is an analytic condition (Risler's theorem).
- "rot f = golden ratio" are at least finitely smooth near critical maps (M.Yampolsky, NG; in progress).

QQ Are they only finitely smooth?

QQ Do critical maps have bubbles? How do they look like?



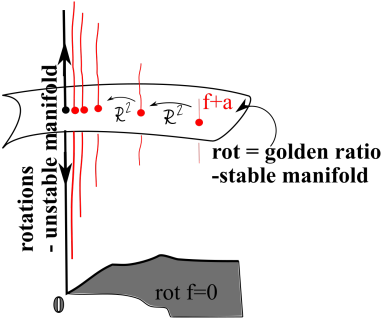


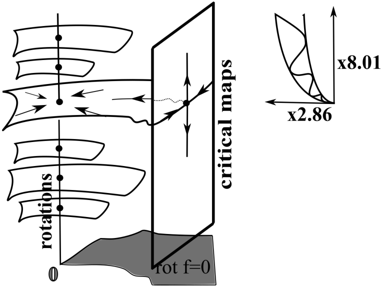
Renormalization (joint with M.Yampolsky) and bubbles

- Brjuno rotations are a hyperbolic set for $\mathcal{R}$ (joint with M. Yampolsky).
- $\Rightarrow$ bubbles are small near Herman numbers (Gorbovickis, NG; in progress).
- $\Rightarrow$ "rot f = Herman number" is an analytic condition (Risler's theorem).
- "rot f = golden ratio" are at least finitely smooth near critical maps (M.Yampolsky, NG; in progress).

QQ Are they only finitely smooth?

QQ Do critical maps have bubbles? How do they look like?

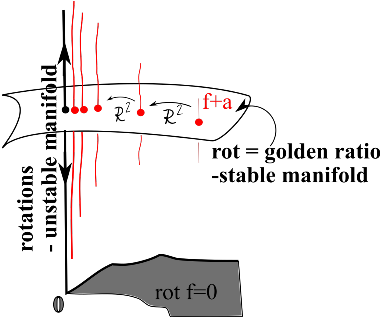


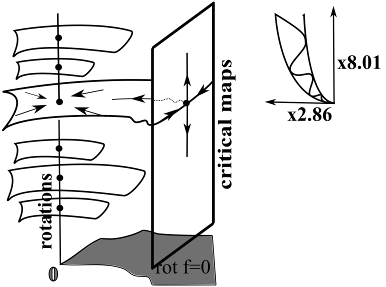


- Brjuno rotations are a hyperbolic set for $\mathcal{R}$ (joint with M. Yampolsky).
- $\Rightarrow$ bubbles are small near Herman numbers (Gorbovickis, NG; in progress).
- $\Rightarrow$ "rot f = Herman number" is an analytic condition (Risler's theorem).
- "rot f = golden ratio" are at least finitely smooth near critical maps (M.Yampolsky, NG; in progress).

QQ Are they only finitely smooth?

QQ Do critical maps have bubbles? How do they look like?



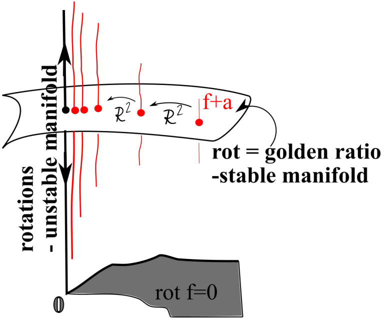


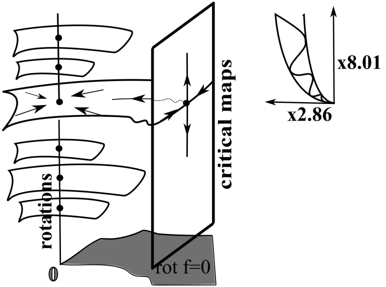
Renormalization (joint with M.Yampolsky) and bubbles

- Brjuno rotations are a hyperbolic set for $\mathcal{R}$ (joint with M. Yampolsky).
- $\Rightarrow$ bubbles are small near Herman numbers (Gorbovickis, NG; in progress).
- $\Rightarrow$ "rot f = Herman number" is an analytic condition (Risler's theorem).
- "rot f = golden ratio" are at least finitely smooth near critical maps (M.Yampolsky, NG; in progress).

QQ Are they only finitely smooth?

QQ Do critical maps have bubbles? How do they look like?





Renormalization (joint with M.Yampolsky) and bubbles

- Brjuno rotations are a hyperbolic set for $\mathcal{R}$ (joint with M. Yampolsky).
- $\Rightarrow$ bubbles are small near Herman numbers (Gorbovickis, NG; in progress).
- $\Rightarrow$ "rot f = Herman number" is an analytic condition (Risler's theorem).
- "rot f = golden ratio" are at least finitely smooth near critical maps (M.Yampolsky, NG; in progress).

QQ Are they only finitely smooth?

QQ Do critical maps have bubbles? How do they look like?