Tropical Complex Dynamics

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Tropical Complex Dynamics

Piece-wise linear maps on metric trees arising from complex dynamics in several ways.

Together with mappings between *small spheres* corresponding to vertices of the tree.

(Always remember they come from maps on a sphere.)

1. Multiply connected Fatou components and stretching deformation

def, Newton's method, inverse construction by qc-surgery, where?

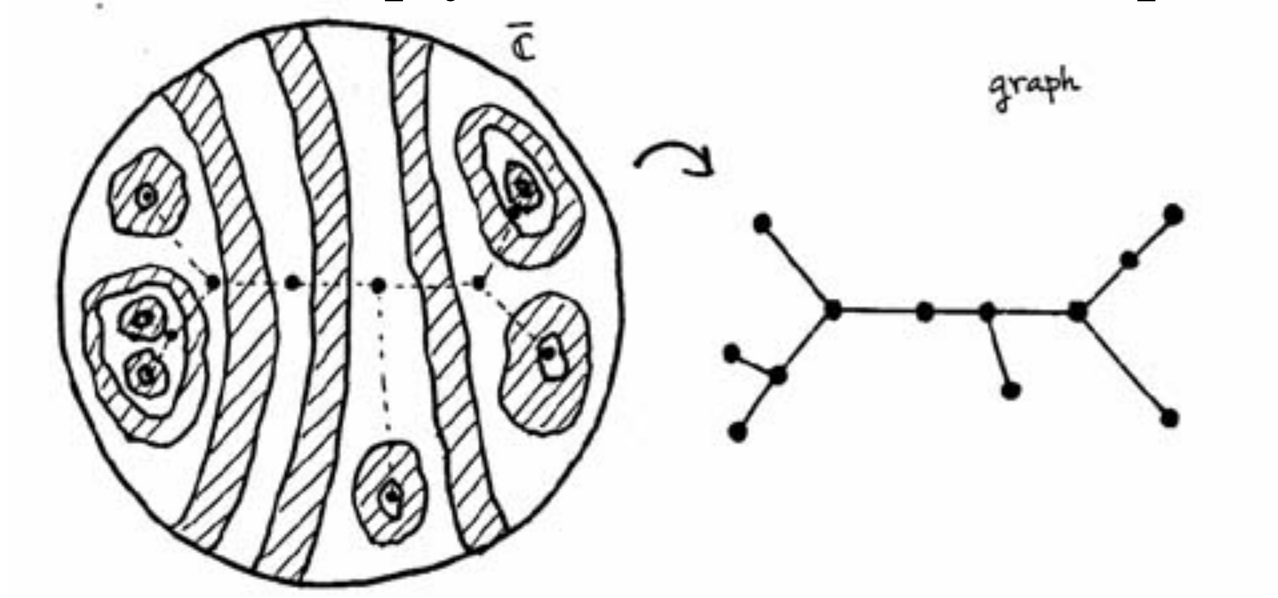
2. Thurston obstructions or invariant multicurves

Thurston's theorem, def, Levy cycle theorems, matings, inverse construction

3. Degeneration of rational maps, scaling limits, Arfeux's trees of spheres

inverse construction = arithmetic surgery

1. Tree for multiply connected Fatou components



Definition. Let $\mathcal{A}$ be a collection on *disjoint* annuli of $\widehat{\mathbb{C}}$. Each annulus A is canonically foliated by topological circles. For $x, y \in \widehat{\mathbb{C}}$, let $A[x, y]$ be the union of leaves which separate x and y. So $A[x, y]$ is either a subannulus of A or an emptyset. Define $d : \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \to [0, +\infty]$ by

$$d(x, y) = \sum_{A \in \mathcal{A}} mod\ A[x, y].$$

Let

$$T = T_{\mathcal{A}} = \widehat{\mathbb{C}}/_{\sim_{\mathcal{A}}},$$

where $x \sim_{\mathcal{A}} y$ if and only if $d(x, y) = 0$. Then T is a tree and d induces a (generalized) metric, which may take value ∞.

A tree and a piecewise linear map for a rational map

Let f be a rational map with a periodic Fatou component, which is not a parabolic basin. In each periodic Fatou component other than parabolic basin, i.e., Attracting basin, superattracting basin, Siegel disk or Herman ring, there is a canonical foliation by circles. (Use linearization, Böttcher etc.) Remove the leaves containing the grand orbits of critical points, and take all the inverse images. We obtain a collection of disjoint annuli $\mathcal{A}_f$ on $\widehat{\mathbb{C}}$. From $\mathcal{A}_f$, the tree $T = T_f = T_{\mathcal{A}_f}$ is defined.

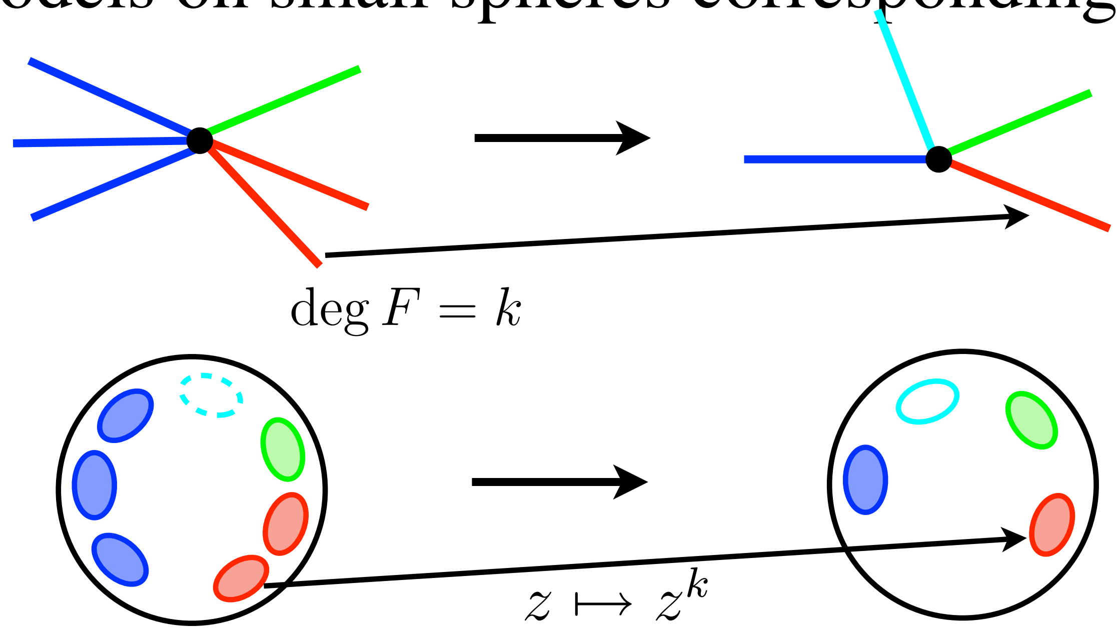
The map f defined a continuous map $F : T \to T$ and the degree function $\deg F : T \setminus Sing(T) \to \mathbb{N}$ with the following properties:

if $J = [x, y]$ is an arc in T joining x and y such that $F|_J$ is injective and $\deg F$ is constant on J, then $d(F(x), F(y)) = (\deg F|_J)\, d(x, y)$.

There is a dense open set in T such that any orbit from this set eventually lands on a periodic arcs on which the return map is identity (Siegel disks, Herman rings), a translation or an expansion towards a point at infinity (attracting or superattracting basin).

(cf. DeMarco-McMullen for polynomials

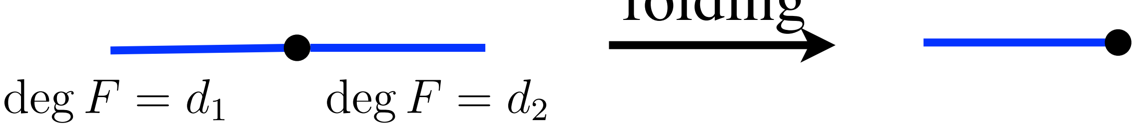
Local models on small spheres corresponding to vertices



If the point on the tree is periodic, a quasiconformal surgery can be carried out so that periodic isometric branches correspond to Siegel disks and periodic expanding branches correspond to superattracting basin.

A periodic orbit not intersecting $Sing(T)$ corresponds to a quasicircle.

An estimate on the number of critical points



$$\# \text{ of crit pts} \geq 2(d_1 + d_2) - 2 - (d_1 - 1) - (d_2 - 1) = d_1 + d_2$$

Weakly repelling fixed point and the complexity of F

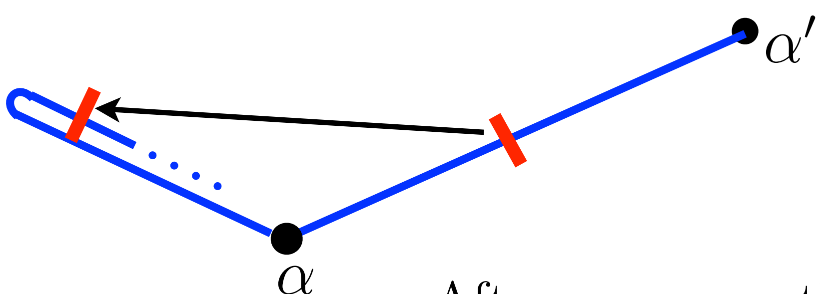
Theorem (Fatou): Every rational map of degree ≥ 1 has a weakly repelling fixed point (repelling or parabolic with multiplier 1).

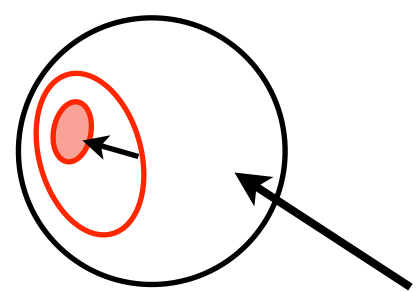
Theorem (S.) The Julia set of the Newton's method of a polynomial is connected. More generally if a Julia set of a rational map is not connected, there are two weakly repelling fixed points which are separated by a Fatou component.

Typical argument in the proof:

Suppose the Julia set is not connected. There exists a fixed point α of F, which is the projection of a weakly repelling fixed point and α' an inverse image of α, $\alpha' \neq \alpha$.

In the case where the branch of $[\alpha, \alpha']$ at α is not fixed ...

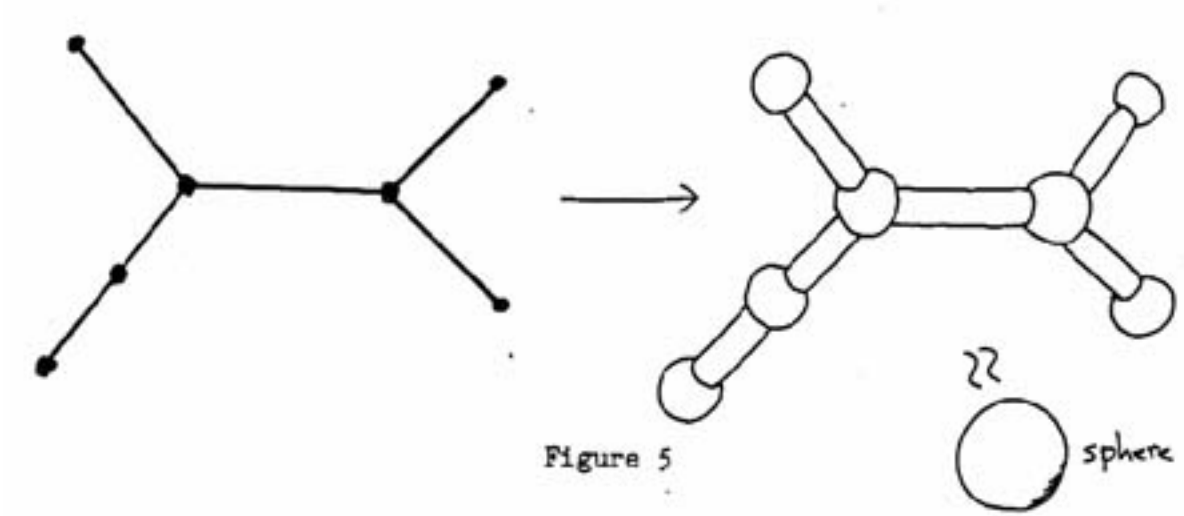




After a surgery, there must be another weakly repelling fixed point separated by a Fatou component.

Inverse problem

Construction of a rational map from a tree map and local models



If $Sing(T)$ has finite orbit, a surgery can be carried out.

Use quasiconformal surgery

Where can you find the map you constructed?

Look at the limit of stretching deformation, which should be where the degree drops and check the scaling limit of some iterates. The information on the scaling can be read off from the tree. (length becomes the power of the scaling parameter.)

2. Thurston's theorem on the characterization of rational maps among self-branched covering of 2-sphere.

Let $f : S^2 \to S^2$ be a branched covering. (locally like $z \mapsto z^k$)

$Crit(f) = \{\text{critical pts of } f\}$, $P_f = \bigcup_{n=1}^{\infty} f^n(Crit(f))$ (post-crit. set)

Assume $\sharp P_f < \infty$. (*Post-critically finite*, PCF)

Two PCF branched coverings f and g are *equivalent*, $f \sim g$, if there exist two orientation preserving homeomorphisms $\theta_1, \theta_2 : S^2 \to S^2$ such that

$\theta_i(P_f) = P_g$ $(i = 1, 2)$, $\theta_1 = \theta_2$ on P_f, θ_1 and θ_2 are isotopic relative to P_f, and the following diagram commutes:

$$\begin{array}{ccc} S^2 & \xrightarrow{\theta_1} & S^2 \\ f \downarrow & & \downarrow g \\ S^2 & \xrightarrow{\theta_2} & S^2. \end{array}$$

Q. Given f as above, when is it equivalent to a rational map?

A multicurve Γ is a collection of disjoint simple closed curves in $S^2 \setminus P_f$ such that they are noy homotopic to a point or to a puncture, not homotopic to each other.

Define $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ by $f_\Gamma : (m_\gamma)_{\gamma\in\Gamma} \mapsto (m'_\gamma)_{\gamma\in\Gamma}$ Thurston matrix

where $m'_\gamma = \sum_{\delta\in\Gamma} \sum_{\substack{\delta'\subset f^{-1}(\delta) \\ \delta'\sim\gamma}} \frac{m_\delta}{deg(f : \delta' \to \delta)}$.

$\lambda_\Gamma =$ leading eigenvalue of f_Γ (Thurston eigenvalue)

A multicurve Γ with $\lambda_\Gamma \geq 1$ is called Thurston obstruction.

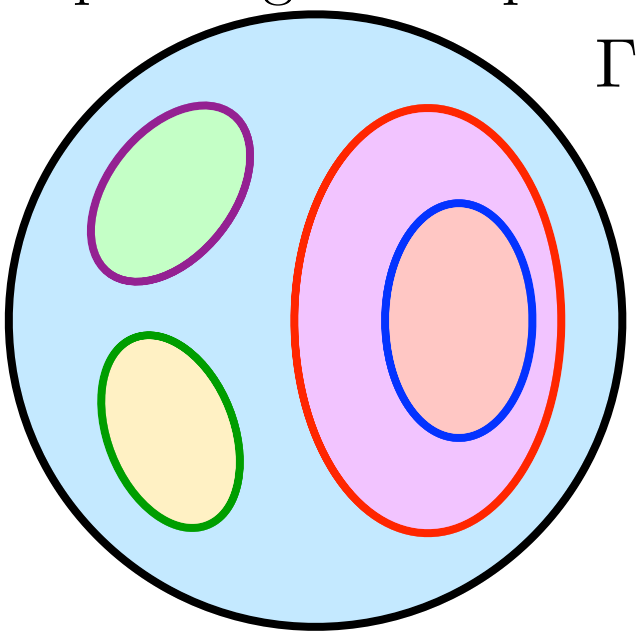
Theorem (Thurston). (Published by Douady-Hubbard 1993)
A PCF branched covering $f : S^2 \to S^2$ (with $\sharp P_f \geq 5$) is equivalent to a rational map if and only if it has no Thurston obstruction, i.e., any invariant multicurve $\Gamma \subset S^2 \setminus P_f$ satisfies $\lambda_\Gamma < 1$.
Moreover when this condition holds, the equivalent rational map is unique. (Rigidity)

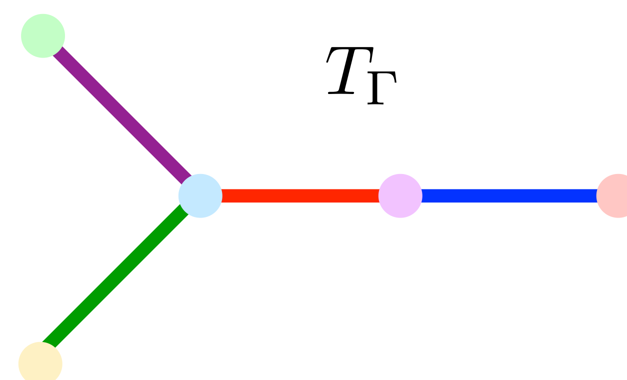
negative criterion: find one obstruction
positive criterion: check for all multicurves (cf. Dylan Thurston)

Tree associated to an invariant multicurve of a PCF map

Given a multicurve Γ on S^2, one can associate a tree $T = T_\Gamma$ so that each connected component of $S^2 \setminus \cup\Gamma$ corresponds to a vertex of T; each γ corresponds to an edge of T which connects the two vertices corresponding to components of $S^2 \setminus \cup\Gamma$ sharing γ as boundary.

Γ ("dual graph") cf. Work by Pilgrim





T_Γ

induced map $f : T_{f^{-1}(\Gamma)} \to T_\Gamma$ $\qquad T_\Gamma \xleftarrow{r} T_{f^{-1}(\Gamma)} \xrightarrow{f} T_\Gamma$

Assume that Γ is "irreducible in the sense of graph".

Take a positive eigenvector of the Thurston matrix for λ_Γ, one can define an eigenvector metric on the tree T_Γ.

With respect to this metric, the induced map $f : T_\Gamma \to T_\Gamma$ is piecewise linear with local Lischitz const $= \lambda_\Gamma \times \deg(f : \delta' \to f(\delta'))$

Application to Levy cycle theorems

A multicurve $\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_{p-1}\}$ is called a *Levy cycle* if each γ_i has an inverse image $\gamma'_{i-1} \subset f^{-1}(\gamma_i)$ which is homotopic to γ_{i-1} with $deg(f : \gamma'_{i-1} \to \gamma_i) = \pm 1$ ($i = 1, \dots, p$ with $\gamma_p = \gamma_0$). By taking inverse images, Γ can be extended to be a Thurston obstruction.

Levy cycles Theorem (Levy) If f is a postcritically topological polynomial, then any Thurston obstruction is a degenerate (removable) Levy cycle, i.e. bounds disks on which the map is homeomorphic.

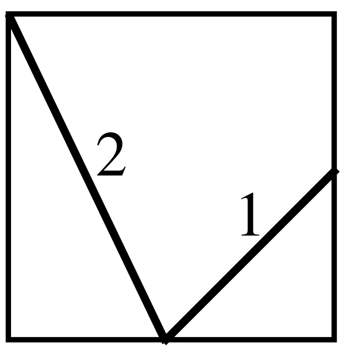
Levy cycles Theorem (Rees) If f is a postcritically branched covering with only two critical points, then any Thurston obstruction is either a degenerate Levy cycle or an essential Levy cycle, i.e. the Levy cycle bounds a single component on which the map is homeomorphic.

This argument fails as soon as there are 3 critical points.

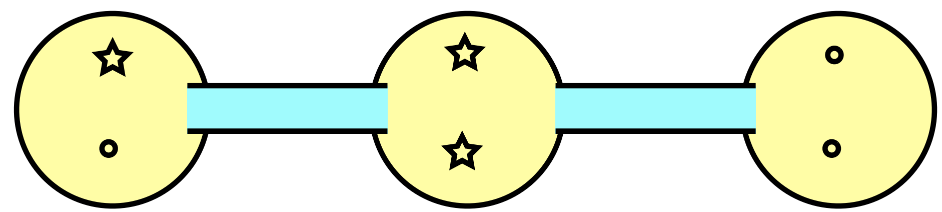
Counter example to Levy cycle theorem with 3 crit pts

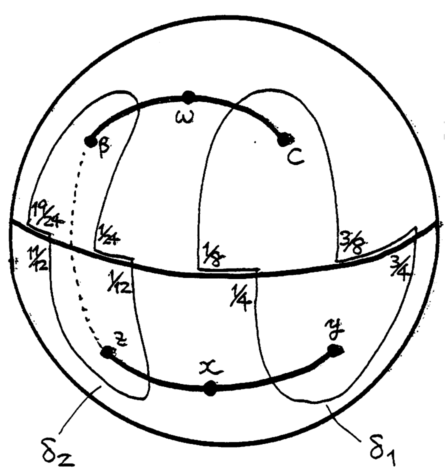
One can construct a Thurston obstruction by giving a tree map and adding an information on local models. When the number of branches is small, it is easy to create appropriate local models.

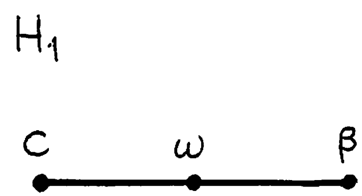
Theorem (S.-Tan). *There exists a mating of cubic polynomials such that it has a Thurston obstruction, but has no Levy cycle.*



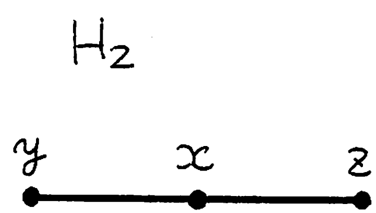
$\lambda = 1$







$\omega = 0 \xrightarrow{3} c \to \beta \to \beta$



$x \xrightarrow{2} y \xrightarrow{2} z \to x$

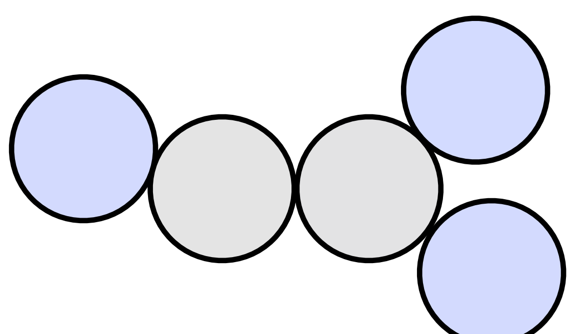
3. Degeneration, scaling limits, Arfeux's trees of spheres

Let $\{f_n\}$ be a sequence of rational maps of degree $d > 1$. It may happen that f_n converges to a limit f with degree $d' < d$ in the complement of a neighborhood of a fixed finite set.

If there exist an integer $k \geq 1$ and a sequence of Möbius transformations $\{\varphi_n\}$ such that $\varphi_n \circ f_n^k \circ \varphi_n^{-1}$ converge to a non-trivial limit g in the same sense as above, g is called a scaling limit of $\{f_n\}$. (Kiwi)

It is more natural to look at the dynamics along "moving frames".

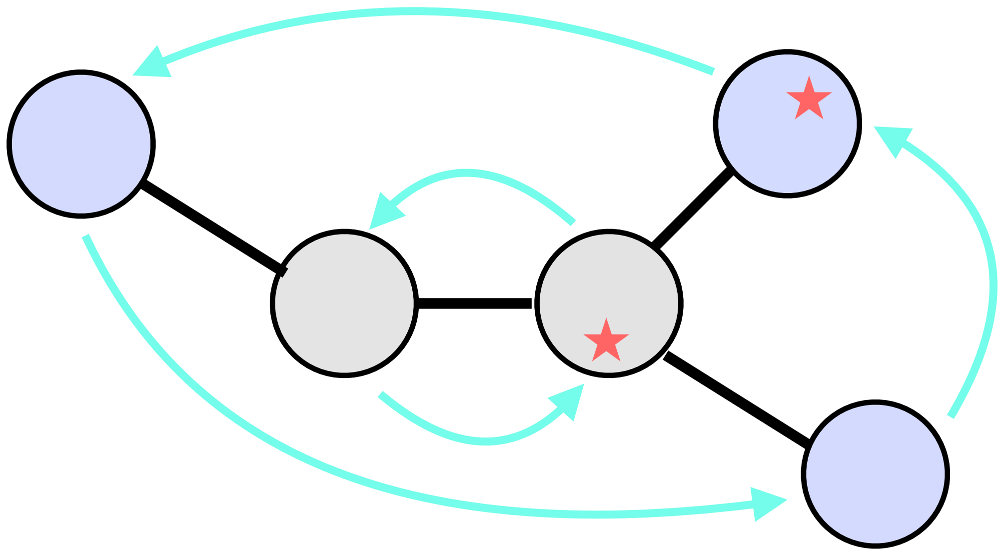
Arfeux: scaling limits are organized as a tree of spheres, on which the scaling limits are defined.



Inverse problem = realization problem

Q. When a tree of spheres is given, can it be realized for a family of rational maps $\{f_t\}_{t\in\mathbb{D}^*}$ (as $t \to 0$)?

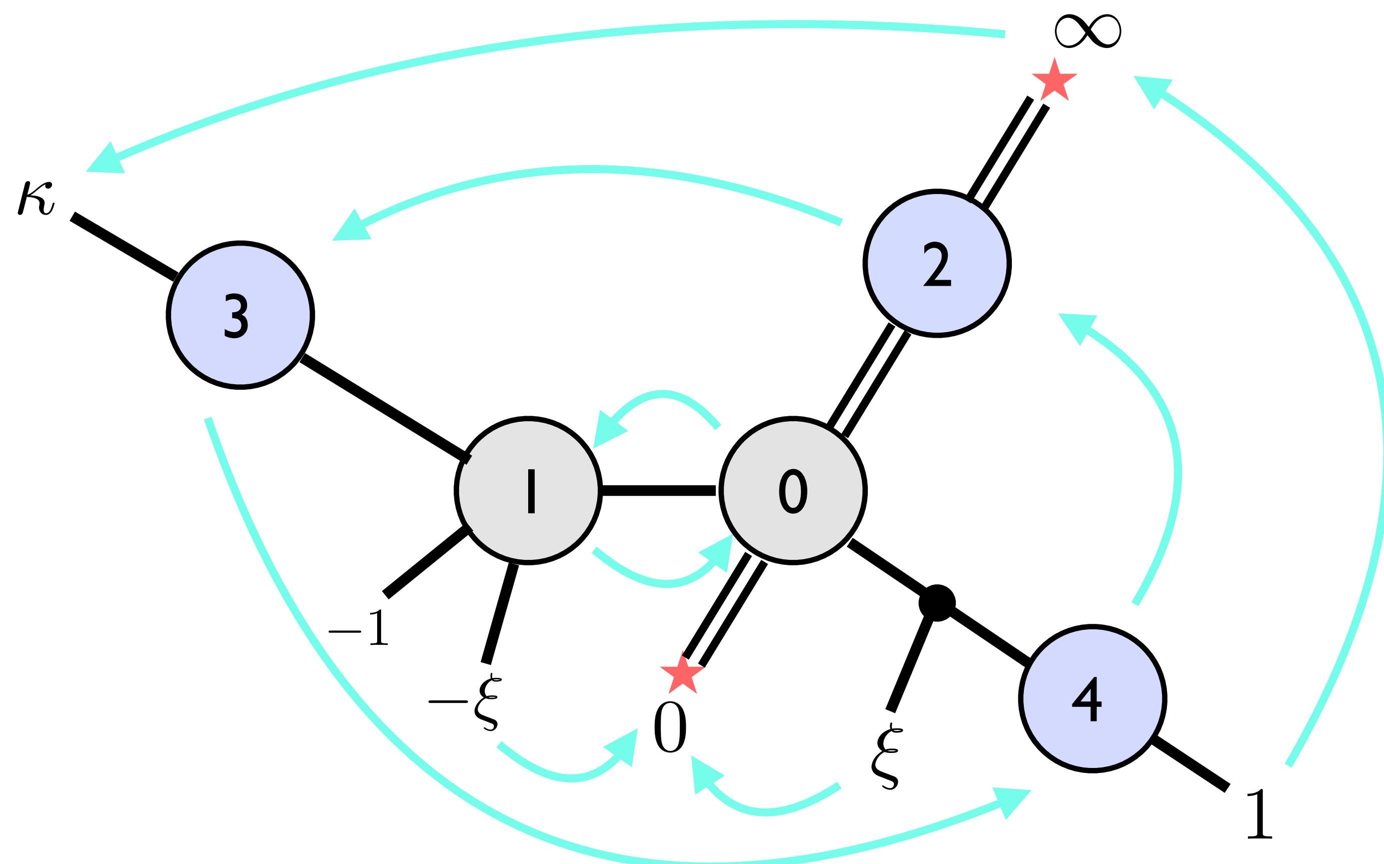
One should add edges between spheres with length, which give rise to a compatibility condition (necessary condition).



The question translates to an embedding problem of the tree into the Berkovich space for $k = \mathbb{C}((t))$ (or Puiseux series, completion etc).

Work in progress with M. Arfeux and J. Kiwi. Worked out examples for degree 2.

Specific example: degree 2, period 2 and 3 spheres



Global: 0 and ∞ are critical points. $f^{-1}(\infty) = \{\pm 1\}$.

$f^{-1}(0) = \{\pm\xi\}$, $\kappa = f(\infty)$.

$$f(z) = f_t(z) = \kappa \frac{z^2 - \xi^2}{z^2 - 1}.$$

$\xi = 1 + \xi_1 t$

$\kappa = -1 + \kappa_1 t + \kappa_2 t^2 + \kappa_3 t^3$

coordinates φ_j focusing on S_j

$\varphi_0(z) = z$, $\varphi_1(z) = t^{-1}(z+1)$

$\varphi_2(z) = tz$, $\varphi_4(z) = t^{-2}(z-1)$

$\varphi_3(z) = t^{-3}(z - f(\infty)) = t^{-3}(z - \kappa)$

The map f viewed in the new coordinates should have limits

$$(t \to 0)$$

$$f_0(\zeta) = \varphi_1 \circ f_t \circ \varphi_0^{-1}(\zeta) \to \kappa_1 + \frac{2\xi_1}{\zeta^2 - 1}$$

$$f_1(\zeta) = \varphi_0 \circ f_t \circ \varphi_1^{-1}(\zeta) \to -1 - \frac{\xi_1}{\zeta}$$

$$f_2(\zeta) = \varphi_3 \circ f_t \circ \varphi_2^{-1}(\zeta) \to \frac{2\xi_1}{\zeta^2}$$

$$f_4(\zeta) = \varphi_2 \circ f_t \circ \varphi_4^{-1}(\zeta) \to \frac{\xi_1}{\zeta}$$

$$f_3(\zeta) = \varphi_4 \circ f_t \circ \varphi_4^{-1}(\zeta) \to \frac{t(4\kappa_1 + 2\xi_1) + t^2(4\kappa_2 - 4\kappa_1^2 - 2\kappa_1\xi_1 + \xi_1^2) + O(t^3)}{-2\kappa_1 t^3 + O(t^2)}$$

To have a limit, we need $4\kappa_1 + 2\xi_1 = 4\kappa_2 - 4\kappa_1^2 - 2\kappa_1\xi_1 + \xi_1^2 = 0$

Then $f_3(\zeta) \to -\frac{2}{\kappa_1}(\zeta + \kappa_3)$

Return maps $f_1 \circ f_0(\zeta) \to \dfrac{\zeta^2 + 3}{\zeta^2 - 5}$ (conjugate to $z \mapsto \dfrac{z}{(1-z)^2}$)

$$f_4 \circ f_3 \circ f_2(\zeta) \to \frac{\zeta^2}{\frac{\kappa_3}{\kappa_1}\zeta^2 - 4} \quad \text{(conjugate to } z \mapsto z^2 - \tfrac{4\kappa_3}{\kappa_1^2}\text{)}$$

These are two scaling limits.

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Thank you!