Geometry, universality and Beltrami complex structure for scaling limits of random dimer coverings.

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University of Helsinki

The Analysis and Geometry of Random Spaces
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Based on joint work with
Erik Duse, Istvan Prause and Xiao Zhong

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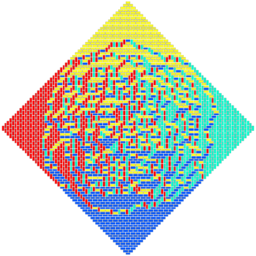
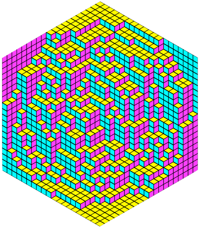
Random tilings and Complex Analysis

- Scaling limits of random tilings





- The Beltrami equation: $\overline{\partial} f(z) = f(z)\, \partial f(z), \quad f : \mathscr{L} \to \mathbb{D}.$

Pictures: Courtesy of R. Kenyon, L. Petrov, S. Chhita, I. Prause, E. Duse...

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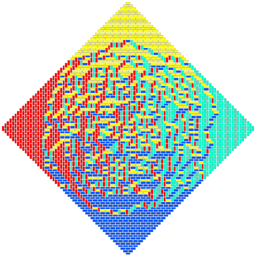

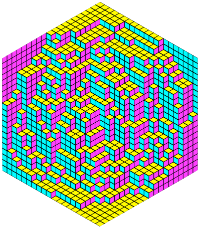

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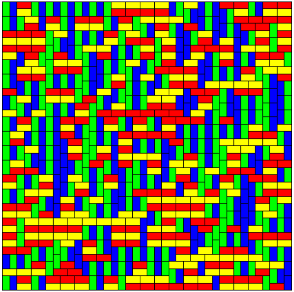
Random tilings (by Dominoes)

Tilings by dominoes (uniform probability):



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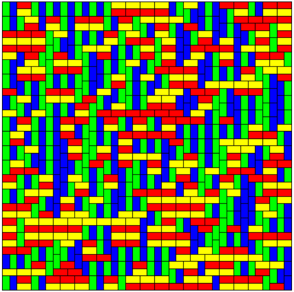

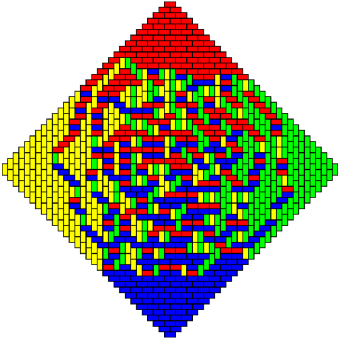

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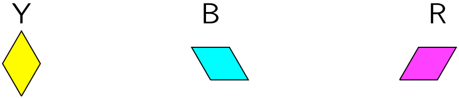

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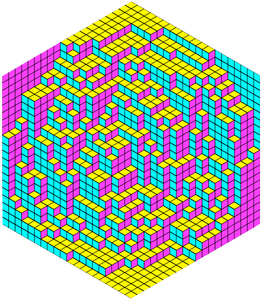
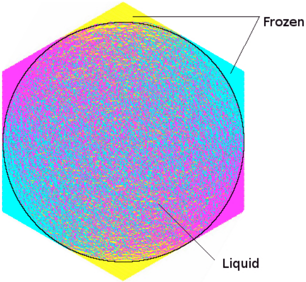


Scaling limit ?



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In scaling limits of random lozenges tilings of a regular hexagon,

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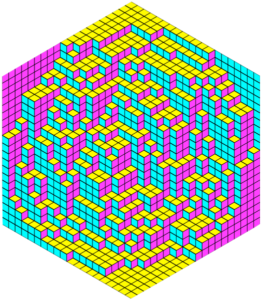
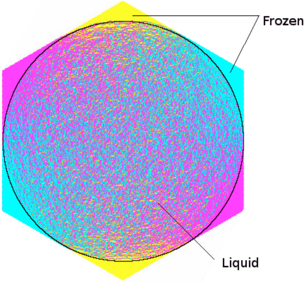


Scaling limit, with probability 1:



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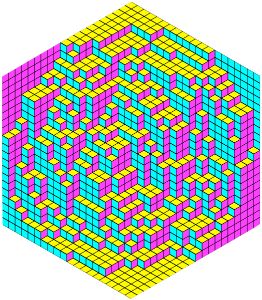

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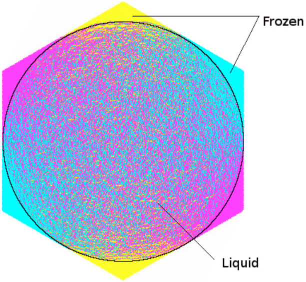

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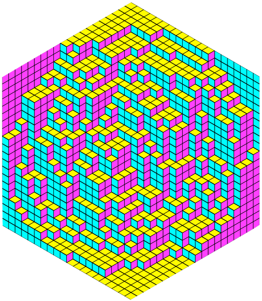

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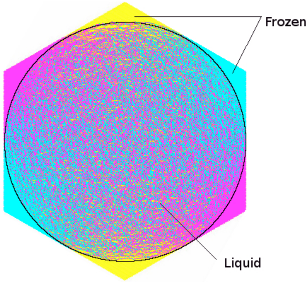
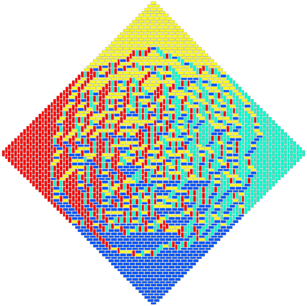
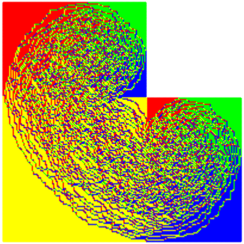


Random tilings and dimer models

[Cohn-Kenyon-Propp (2001)]; [Kenyon-Okounkov-Sheffield (2006)]:

In scaling limits, a.s. disordered (or "liquid") regions $\mathcal{L}$ and ordered (or "frozen") regions $\mathcal{F}$ in other polygons, too, for lozenge and domino tilings and more generally, for all dimer models.





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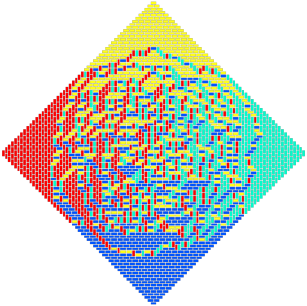

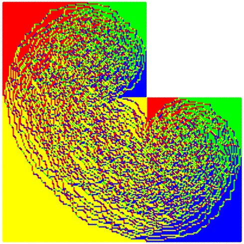

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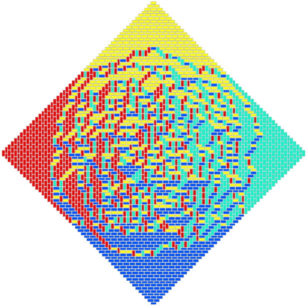
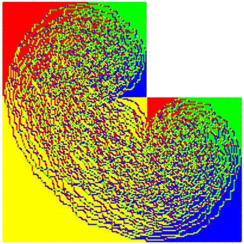
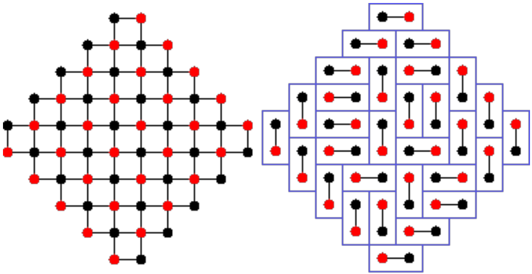




Random tilings and dimer models

- Dimer models: perfect matchings on a bipartite, doubly periodic planar graph.



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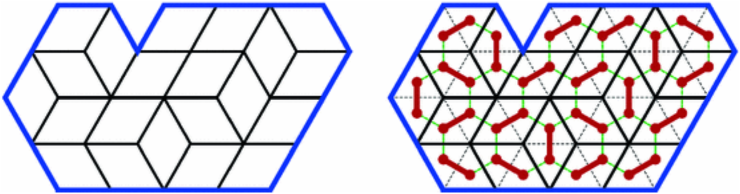

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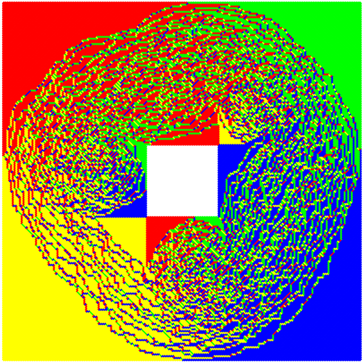
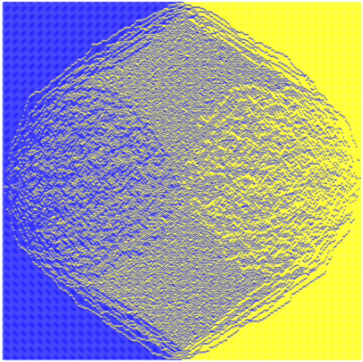


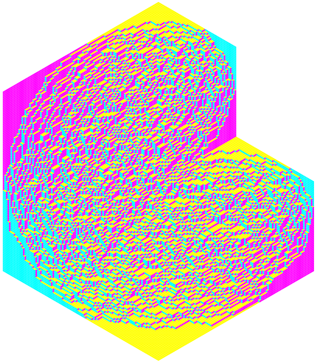


Problem: **Describe** the geometry of all frozen boundaries $\partial\mathcal{L} = \mathcal{F}$!

Random tilings (Losenges)

With algebraic geometry and complex Burgers equation, Kenyon-Okounkov (2007):

For losenges tilings, in polygons with $3n$ sides, directions cyclically repeated: Frozen boundary is (a specific) algebraic curve.

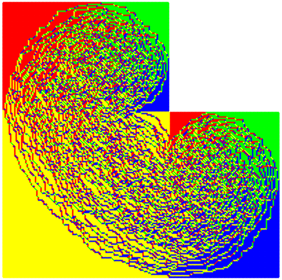


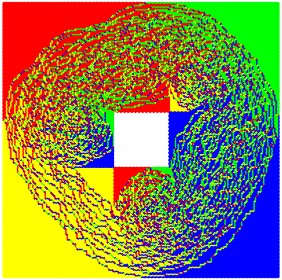
General dimer models

Question:

Geometry of Frozen boundary for dominoes (on square lattice) ?

For general dimer models ??





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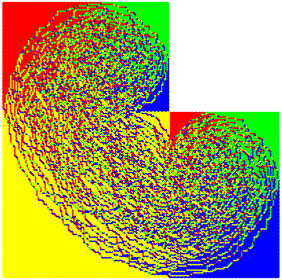
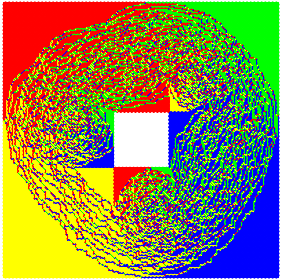
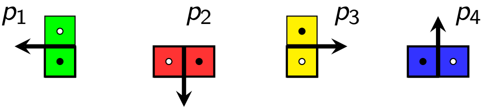
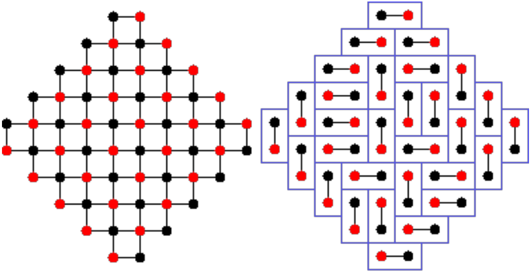




Random tilings and Thurston's discrete height function





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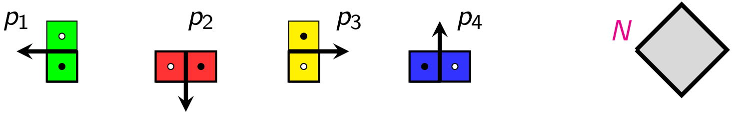
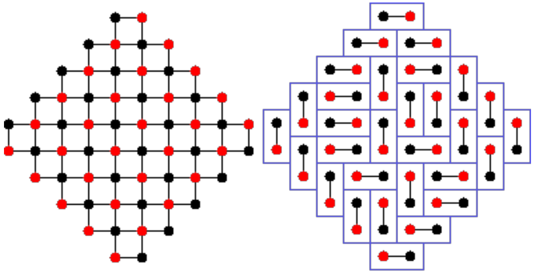


- Asymptotic height function h.
- N = Gradient constraint for ∇h [N depends on dimer model !]

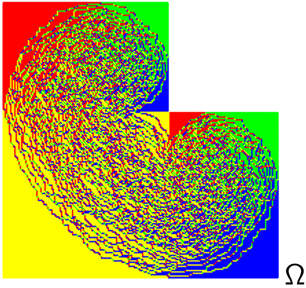


Random tilings and Thurston's discrete height function

[Cohn-Kenyon-Propp (2001)]; [Kenyon-Okounkov-Sheffield (2006)]:

Determining the frozen boundaries is equivalent to

a (non standard) variational problem for h !



Free boundary problem for Dimer Models [CKP], [KOS].

- Variational problem, with gradient constraint: $\Omega \subset \mathbb{C}$ open;

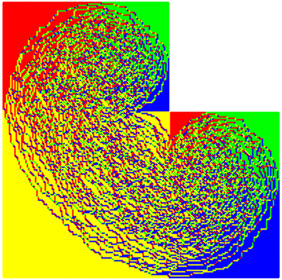
$$\int_\Omega \sigma(\nabla h) = \inf \left\{ \int_\Omega \sigma(\nabla v) : v_{|\partial\Omega} = h_0,\ \nabla v \in N_\sigma \right\}, \quad h_0 \in Lip_1(\partial\Omega),$$

- "Energy" or "Surface tension" σ has special form:

$\det D^2\sigma = 1 + \sum_{q\in\mathcal{G}} \delta_q$ in a convex polygon $N = N_\sigma$, (MA)

σ convex, with $\sigma_{|\partial N_\sigma}$ piecewise linear.

- Liquid domains:



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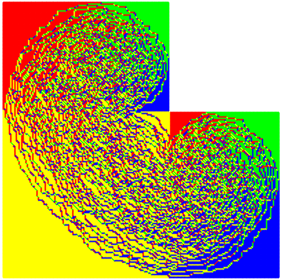

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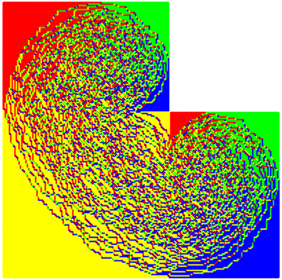

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Problem: Describe the geometric properties of $\partial\mathscr{L}$!

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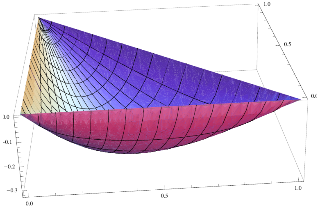
Lozenge model: surface tension

Let $N = N_{Lo} := \overline{\text{conv}}\{(0,0),(0,1),(1,0)\}$. Then $\sigma = \sigma_{Lo}$ where

$$\nabla\sigma_{Lo}(s,t) = \frac{1}{\pi}\left(\log\left(\frac{\sin(\pi s)}{\sin(\pi(t+s))}\right), \log\left(\frac{\sin(\pi t)}{\sin(\pi(t+s))}\right)\right).$$



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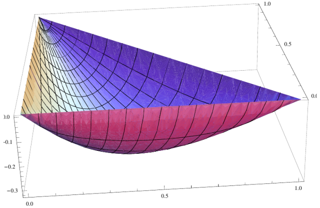

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Geometry of Frozen boundaries

Theorem 1 (A.-Duse-Prause-Zhong)

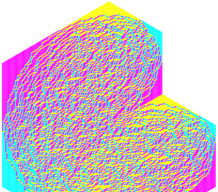
Let $\mathscr{L} \subset \mathbb{C}$ be bounded (simply- or multiply connected) domain.
If $\mathscr{L}$ liquid for any dimer model, with boundary $\partial\mathscr{L}$ frozen, then

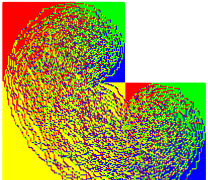
1) $\partial\mathscr{L}$ is the real locus of an algebraic curve.

2) The singularities of $\partial\mathscr{L}$ are all first order cusps or tacnodes.

3) For $\zeta \in \partial\mathscr{L}$, outside cusps and tacnodes, $\partial\mathcal{L}$ is locally convex:
$B(\zeta, \varepsilon) \cap \mathscr{L}$ is convex $\forall\, \varepsilon > 0$ small.

4) Characterisation of Rational maps that arise as parametrization of some $\partial\mathscr{L}$ (when $\mathscr{L}$ simply connected).





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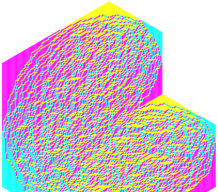

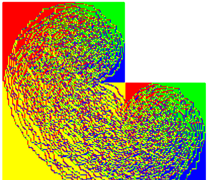

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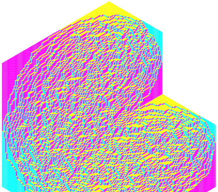

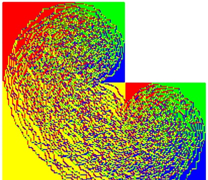

Geometry of Frozen boundaries

Theorem 1 (A.-Duse-Prause-Zhong)

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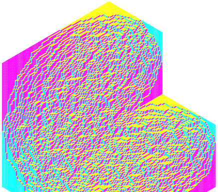

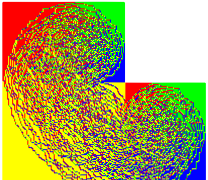

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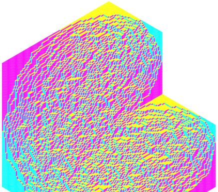

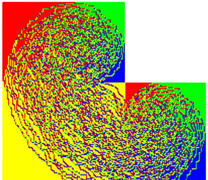

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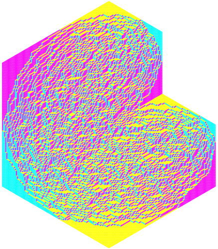

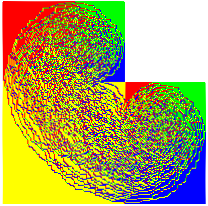

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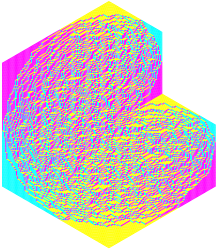

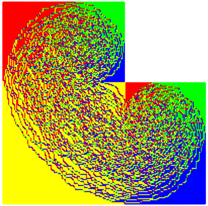

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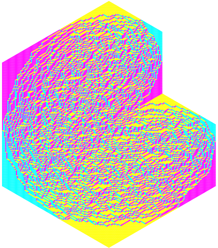

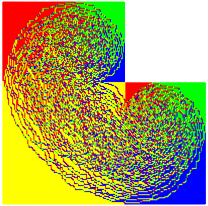

Free boundary problem for Dimer models: What to do ?

- Variational problem with Gradient constraint N.

$$\int_\Omega \sigma(\nabla h) = \inf\left\{\int_\Omega \sigma(\nabla v) : v_{|\partial\Omega} = h_0\right\}, \quad \det D^2\sigma = 1 + \sum_{q\in\mathscr{G}} \delta_q.$$

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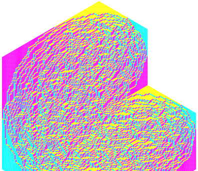
Thus $\operatorname{div} \nabla\sigma(\nabla h) = 0$ in $\mathscr{L}$

TASK: When $\partial\mathscr{L}$ frozen, describe its geometry !



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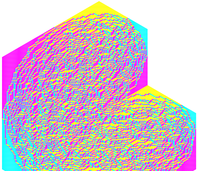

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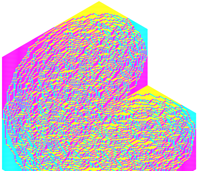

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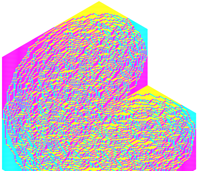

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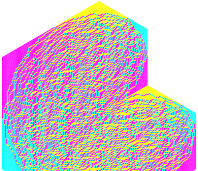

Beltrami equation from height function: Complex structure

$$F_{\bar{z}} = \mu F_z, \qquad |\mu|_\infty \leqslant 1.$$

- " Cauchy-Riemann eqn's " for $\operatorname{div} \nabla\sigma(\nabla h) = 0$:

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  $$L_\sigma^{-1} := (I + \nabla\sigma)^{-1} \quad \text{harmonic homeo !}$$

Beltrami equation from height function: First Conclusions

Theorem 3 (A-D-P-Z) Suppose:

σ convex, $\det D^2\sigma = 1$ in $N = co\{p_j\}_1^n$, $\sigma_{|\partial N}$ piecewise affine,

and $\mathrm{div}(\nabla\sigma \circ \nabla h) = 0$ in bdd domain $\mathscr{L}$ with $\partial\mathscr{L}$ frozen. Then

(1) $\nabla h(z) = \sum_{j=1}^{n} p_j\, \omega_{\mathbb{D}}\big(f(z); I_j\big)$ [$\omega_{\mathbb{D}}$ harmonic meas. in $\mathbb{D}$]

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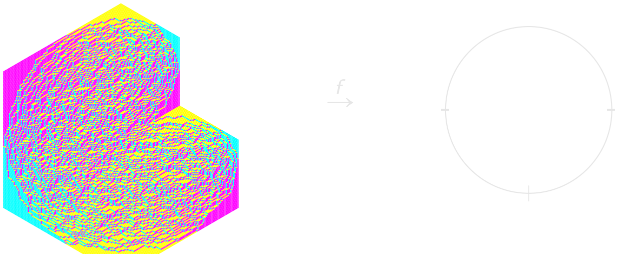
(2) $f : \mathscr{L} \to \mathbb{D}$ proper with $f_{\bar z} = \mathcal{H}'_\sigma(f)\, f_z$, $z \in \mathscr{L}$.



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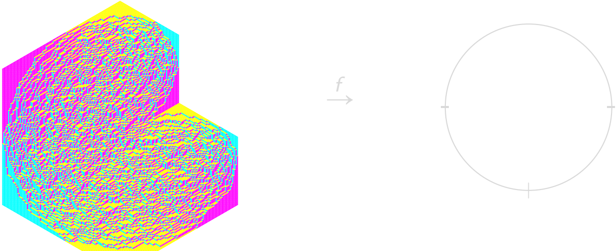

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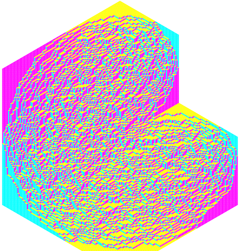

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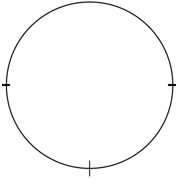

Beltrami equations and geometry of frozen boundaries

The "Universal Equation":

$$\overline{\partial} f(z) = f(z)\, \partial f(z) \qquad (*)$$

- If $f : \mathscr{L} \to \mathbb{D}$ proper and satisfies $f_{\bar{z}} = \mathcal{H}'_\sigma(f)\, f_z$,

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- $f : \mathscr{L} \to \mathbb{D}$ is continuous and proper; $\mathscr{L}$ bounded.
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Then

1) $f(x)$ extends continuously to $\partial\mathscr{L}$, f smooth in $\mathscr{L}$.

2) $\partial\mathscr{L}$ algebraic; if $\mathscr{L}$ simply connected, $\partial\mathscr{L} = r(\mathbb{S}^1)$; $r \in R(\mathbb{C})$.

3) $f \in Lip_{1/3}(\mathscr{L})$;

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3) $f \in Lip_{1/3}(\mathscr{L})$;

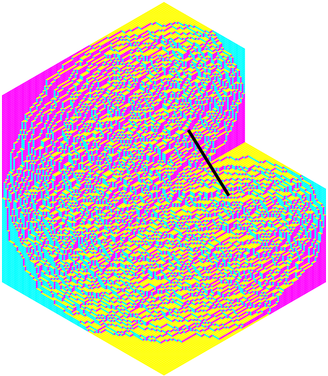
4) $f \in Lip_{1/2}(\mathscr{L})$ outside a finite set of singularities on $\partial \mathscr{L}$.

Beltrami equations and geometry of frozen boundaries

Singularities:

5) Boundary $\partial \mathscr{L}$ has only first order cusps.

6) $f \in Lip_{1/3}$ on line ℓ transversal to cusp; $f \in Lip_{1/2}$ in the cusp direction.



$f : \mathscr{L} \to \mathbb{D}$

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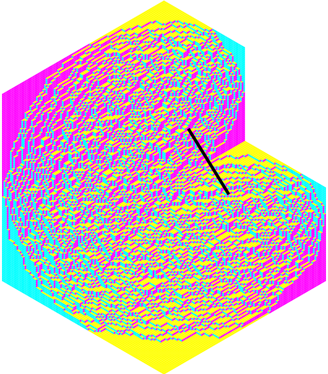

$$f : \mathscr{L} \to \mathbb{D}$$

Beltrami equation: Boundary continuity

$\overline{\partial} f(z) = f(z)\, \partial f(z)$, $\quad f \in W^{1,2}_{loc}$, $\quad f : \mathscr{L} \to \mathbb{D}$ cont. and proper

How to approach ? !

Hodograph transform: For $\mathscr{L}$ simply connected,

$$f = B \circ G$$

where

$G : \mathscr{L} \to \mathbb{D}$ homeo, solution to $\overline{\partial} G(z) = f(z)\, \partial G(z)$,

$B : \mathbb{D} \to \mathbb{D}$ analytic and proper $\Rightarrow B =$ finite Blaschke product.

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$g(z) = \Phi(z, 1/\overline{z})$, $\qquad \Phi(z, w)$ rational in $\mathbb{C}^2$, symmetric.

Beltrami equation: Boundary continuity

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$f = B \circ g^{-1}$, thus $f : \mathscr{L} \to \mathbb{D}$ continuous up to $\partial\mathscr{L}$. $\square$

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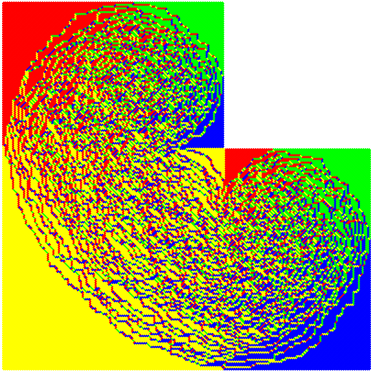
Corollary: Regularity of Height Function

Recall height function $h(z)$; $\text{div}\left(\nabla\sigma(\nabla h)\right) = 0$ in liquid region

$$\mathcal{L} = \left\{z \in \Omega : \nabla h(z) \in \text{int}(K)\right\}.$$

- $\nabla h \in C^{1/2}(\mathcal{L})$, outside a finite set of cusp singularities on $\partial\mathscr{L}$.
  (Pokrovsky-Talapov law)



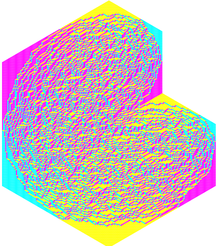
Height functions, Beltrami equation and Universality

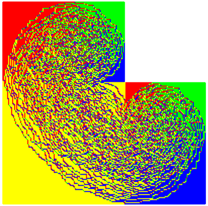
Conversely: Suppose $\sigma = \sigma_{|_{Lo}}$;

- Given: $f : \mathscr{L}_0 \to \mathbb{D}$ proper with $f_{\bar{z}} = f\, f_z$, $z \in \mathscr{L}_0$,
- Formula $\nabla h(z) = \sum_{j=1}^{3} p_j\, \omega_{\mathbb{D}}\big(f(z); I_j\big)$ defines $h \in Lip_1(\mathscr{L}_0)$.
- $\mathrm{div}(\nabla\sigma \circ \nabla h) = 0$ in $\mathscr{L}_0$.

Last: Extend h to polygonal $\Omega \supset \mathscr{L}_0$ s.t.

- $\int_\Omega \sigma_{_{Lo}}(\nabla h) = \inf\left\{\int_\Omega \sigma_{_{Lo}}(\nabla v) : v_{|\partial\Omega} = h_0,\ \nabla v \in N_{_{Lo}}\right\}$, and
- $\mathscr{L}_0 \equiv \{z : \nabla h(z) \in (N_{_{Lo}})^\circ\}$; $\nabla h(z) \to \partial N_{_{Lo}}$ as $z \to \partial\mathscr{L}_0$.





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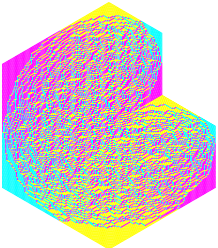

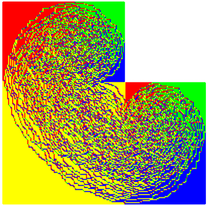

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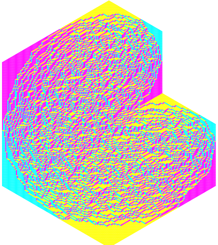

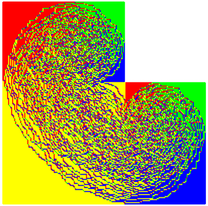

Height functions, Beltrami equation and Universality

Conversely: Suppose $\sigma = \sigma_{|_{Lo}}$;

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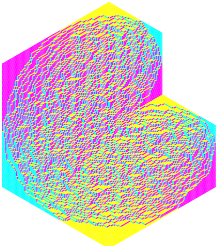

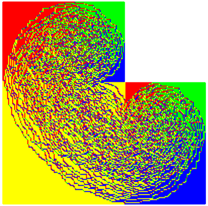

Height functions, Beltrami equation and Universality

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- $\mathrm{div}(\nabla\sigma \circ \nabla h) = 0$ in $\mathscr{L}_0$.

Last: (Nontrivial) Extend h to polygonal $\Omega \supset \mathscr{L}_0$ s.t.

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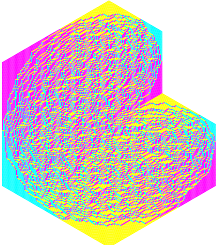

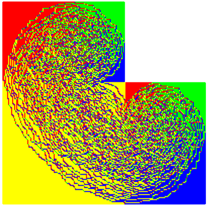

Thank you !