Integrability of SLE via conformal welding of random surfaces

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Collaboration with Morris Ang and Xin Sun

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Gaussian free field (GFF)

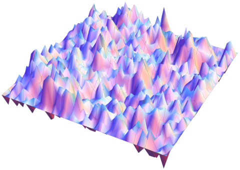
- Free boundary Gaussian free field h in $\mathbb{D}$ with mean zero on $\partial\mathbb{D}$: Gaussian random field with mean zero and covariance

$$\mathrm{Cov}(h(z), h(w)) = G(z, w),$$

where $G : \mathbb{D} \times \mathbb{D} \to [0, \infty)$ is the Neumann Green's function

$$G(z, w) = \log |z - w|^{-1} + \log |1 - z\overline{w}|^{-1}.$$

- h **not** well defined as a function since $G(z, z) = \infty$.
- h well-defined as a **random generalized function (distribution)**.
  - $\int_{\mathbb{D}} hf\, d^2z$ is well-defined for f a smooth test function.



Discrete GFF

Liouville quantum gravity (LQG)

- Let $\gamma \in (0,2)$ and let h be the Gaussian free field in $\mathbb{D}$.
- LQG surface: $e^{\gamma h}(dx^2 + dy^2)$

$$\text{Area measure: } \mu = \text{“}e^{\gamma h} d^2 z\text{”},$$

$$\text{Boundary measure: } \nu = \text{“}e^{\gamma h/2} dz\text{”},$$

$$\text{Distance: } D = \text{“}e^{\gamma h/d_\gamma} |dz|\text{”}, \quad d_\gamma = \text{dimension} > 2.$$

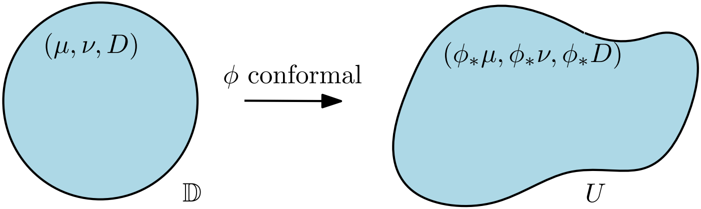
- The definition of an LQG surface does not make literal sense since h is a distribution and not a function.
- μ, ν, D defined rigorously via regularized version h_ϵ of h, e.g.

$$\mu(U) = \lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} \int_U e^{\gamma h_\epsilon(z)} d^2 z, \quad U \subset \mathbb{D}.$$

- References:
  - μ, ν: Hoegh-Krohn'71, Kahane'85, Duplantier-Sheffield'08, Rhodes-Vargas'13, Berestycki'15, etc.
  - D: Ding-Dubedat-Dunlap-Falconet'19, Gwynne-Miller'19
  - See also talks of Powell, Sturm, Kupiainen, Chen, Sun

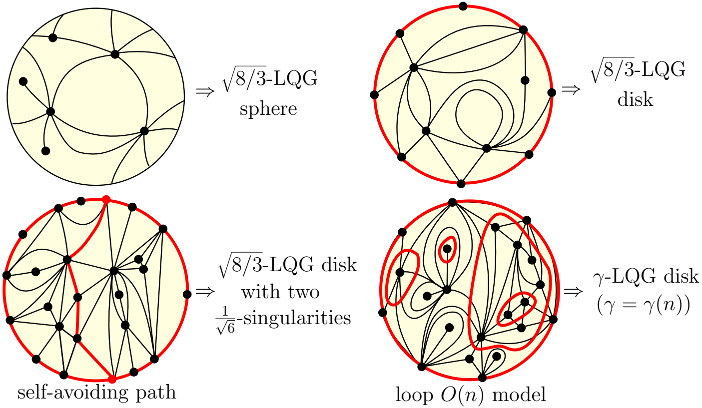
Liouville quantum gravity (LQG) surface

The tuple (μ, ν, D) describes the geometry of the γ-**LQG surface** $(\mathbb{D}, h)$.



Two different embeddings of the same γ-LQG surface

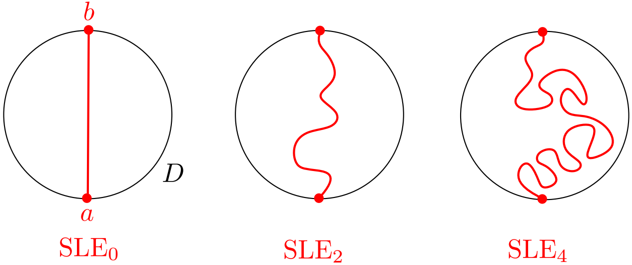
LQG as a scaling limit of random planar maps



- If boundary length and area is random, then we have **infinite measures** (not probability measures) on the space of LQG surfaces.
  - Example: Law of γ-LQG disk boundary length is $c\ell^{-2-\frac{4}{\gamma^2}}\,d\ell$ for $c > 0$.
- See e.g. Le Gall'11, Miermont'11, Duplantier-Miller-Sheffield'14, Miller-Sheffield'16, H.-Sun'19 for scaling limit results.

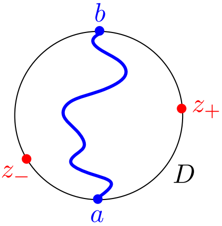
The Schramm-Loewner evolution (SLE)

- Let $\kappa \geq 0$, let $D \subset \mathbb{C}$ be simply connected, and let $a, b \in \partial D$.
- A **Schramm-Loewner evolution with parameter κ on** (D, a, b) is a random fractal curve from a to b in D.
- The Schramm-Loewner evolution describes the scaling limit of interfaces in statistical physics models.
- Introduced by Oded Schramm in 1999.
- Uniquely characterized by **conformal invariance** and the **domain Markov property**.



The Schramm-Loewner evolution with force points

- Let $\kappa \geq 0$, $D \subset \mathbb{C}$, $a, b, z_-, z_+ \in \partial D$, and $\rho_-, \rho_+ > -2$.
- $\mathrm{SLE}_\kappa(\rho)$, $\rho = (\rho_-; \rho_+)$, is the variant of SLE_κ where we keep track of two **force points** on the domain boundary.
- In remainder of talk: $z_- = a^-, z_+ = a^+$.
- Force point attractive (resp. repulsive) for $\rho_\pm < 0$ (resp. $\rho_\pm > 0$).
- Special case: $\rho_- = \rho_+ = 0$ gives SLE_κ.
- Studied in e.g. Lawler-Schramm-Werner'02, Dubedat'03, Miller-Sheffield'12.
- Arises in a variety of settings: conditioned SLE_κ, boundary data, chordal restriction, imaginary geometry, Liouville quantum gravity, ...



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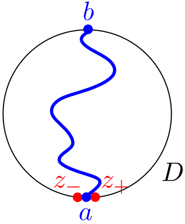

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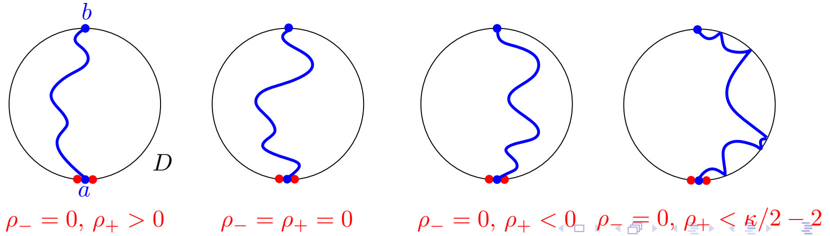
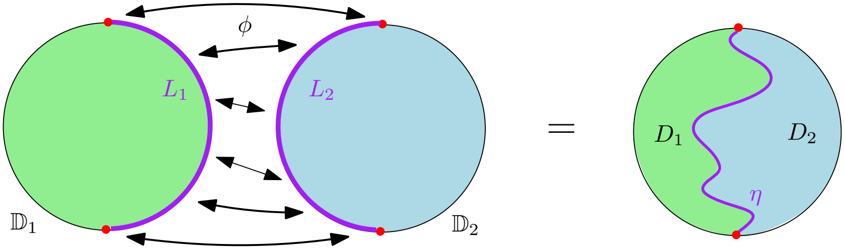
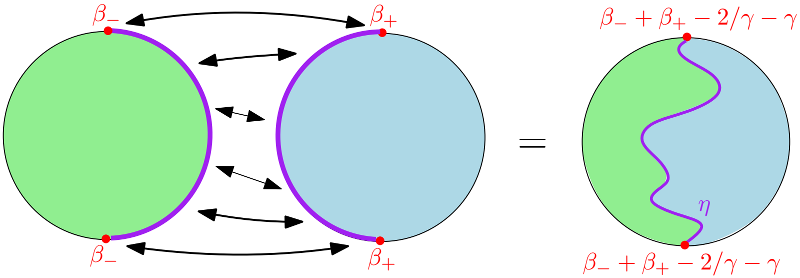


The conformal welding probl.: disk + disk = disk + curve

- $\mathbb{D}_1, \mathbb{D}_2$ copies of the unit disk; $\phi : L_1 \to L_2$ a homeomorphism.
- Conformal welding: a conformal structure on the disk $\mathbb{D}$ obtained by identifying L_1 and L_2 according to ϕ.
  - More precisely, we are interested in a curve η and conformal maps $\psi_j : \mathbb{D}_j \to D_j$, $j = 1, 2$, such that $\phi = \psi_2^{-1} \circ \psi_1|_{L_1}$.
- Does there exist a conformal welding? If so, is it unique?
- Existence and uniqeness may fail, but sufficient regularity of ϕ or η guarantees a unique solution.



Disk + disk = disk + SLE

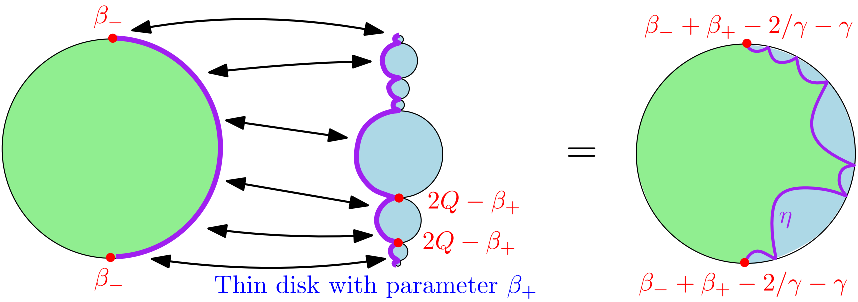


$\beta \in \mathbb{R}$ next to $z \in \partial\mathbb{D}$ means the field looks locally like GFF$+\beta \log|\cdot - z|^{-1}$

η has law $\mathrm{SLE}_\kappa(\rho_-;\rho_+)$, $\kappa=\gamma^2$, $\rho_\pm = \gamma^2 - \gamma\beta_\pm$

- Welding homeomorphism given by LQG boundary length
- Green & blue disks **independent** cond. on matching bdy lengths
- SLE and disk in left figure **independent**
- Ang-H.-Sun'20, building on Sheffield'10 & Duplantier-Miller-Sheff.'14

Conformal welding with thin LQG disks

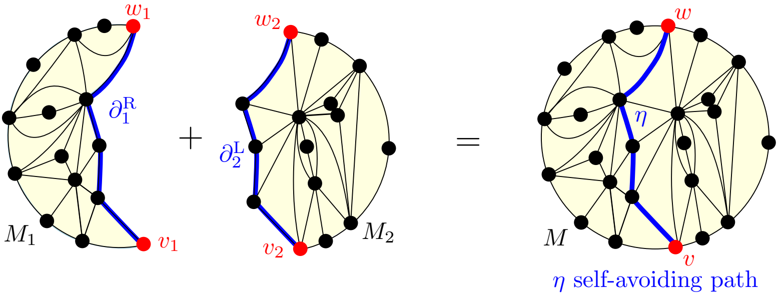


$\beta \in \mathbb{R}$ next to $z \in \partial\mathbb{D}$ means the field looks locally like GFF$+\beta \log|\cdot - z|$

η has law $\mathrm{SLE}_\kappa(\rho_-;\rho_+)$, $\kappa = \gamma^2$, $\rho_\pm = \gamma^2 - \gamma\beta_\pm$

- The thin disk is defined via a PPP $\{(t, S_t)\}$, where S_t is a two-pointed LQG disk and $t > 0$ indicates the relative ordering.
- Background charge $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.
- The thin disk is the natural extension of the thick disk for $\beta > Q$.

Discrete motivation for conformal welding



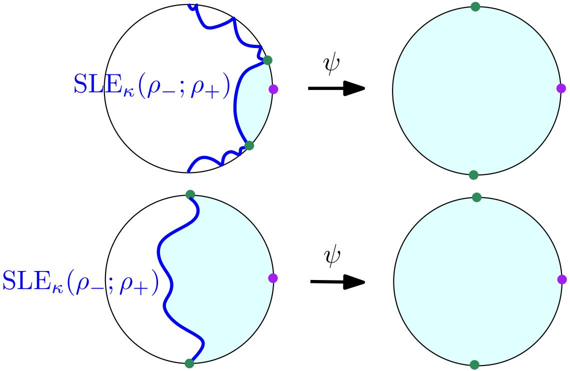
Bijection: $\big((M_1, v_1, w_2), (M_2, v_2, w_2)\big), \#\partial_1^R = \#\partial_2^L$ and (M, v, w, η)

Continuum result inspired by planar maps, but proof is purely continuum.

Moments for uniformizing map of $\mathrm{SLE}_\kappa(\rho_-;\rho_+)$

Theorem 1 (Ang-H.-Sun'21)

$$\mathbb{E}[\psi'(1)^\lambda] = \frac{F(\alpha(\lambda),\kappa,\rho_-,\rho_+)}{F(\sqrt{\kappa},\kappa,\rho_-,\rho_+)} \qquad \text{for } \lambda < \lambda_0.$$



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$$Q = 2/\sqrt{\kappa} + \sqrt{\kappa}/2,$$

$$\log \Gamma_{\frac{\sqrt{\kappa}}{2}}(z) = \int_0^\infty \frac{1}{t}\Big(\frac{e^{-zt} - e^{-Qt/2}}{(1-e^{-\frac{\sqrt{\kappa}}{2}t})(1-e^{-\frac{\sqrt{\kappa}}{2}t})} - \frac{(\frac{Q}{2}-z)^2}{2}e^{-t} + \frac{z-\frac{Q}{2}}{t}\Big)dt.$$

$$F(x,\kappa,\rho_-,\rho_+) = \frac{\Gamma_{\frac{\sqrt{\kappa}}{2}}(\frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2} + \frac{\rho_+}{\sqrt{\kappa}} + \frac{x}{2})\ \Gamma_{\frac{\sqrt{\kappa}}{2}}(\frac{4}{\sqrt{\kappa}} + \frac{\rho_+}{\sqrt{\kappa}} - \frac{x}{2})}{\Gamma_{\frac{\sqrt{\kappa}}{2}}(\frac{4}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2} + \frac{\rho_-+\rho_+}{\sqrt{\kappa}} + \frac{x}{2})\Gamma_{\frac{\sqrt{\kappa}}{2}}(\frac{6}{\sqrt{\kappa}} + \frac{\rho_-+\rho_+}{\sqrt{\kappa}} - \frac{x}{2})}.$$

$$\lambda = 1 - \frac{\alpha(\lambda)}{2}\Big(\frac{\sqrt{\kappa}}{2} + \frac{2}{\sqrt{\kappa}} - \frac{\alpha(\lambda)}{2}\Big).$$

$$\lambda_0 = (\rho_+ + 2)(\rho_+ + 4 - \kappa/2)/\kappa.$$

Moments for uniformizing map of $\mathrm{SLE}_\kappa(\rho_-;\rho_+)$

Theorem 1 (Ang-H.-Sun'21)

$$\mathbb{E}[\psi'(1)^\lambda] = \frac{F(\alpha(\lambda),\kappa,\rho_-,\rho_+)}{F(\sqrt{\kappa},\kappa,\rho_-,\rho_+)} \qquad \textit{for } \lambda < \lambda_0.$$

- $\psi'(1)$ quantifies how close the $\mathrm{SLE}_\kappa(\rho_-;\rho_+)$ gets to the point 1.
- The formula is the same for all $\kappa > 0$ although the proof is very different for $\kappa \in (0,4)$ and $\kappa > 4$.
  - Our proof for $\kappa > 4$ uses SLE duality between κ and $16/\kappa$.

Moments for uniformizing map of $\text{SLE}_\kappa(\rho_-;\rho_+)$

Theorem 1 (Ang-H.-Sun'21)

$$\mathbb{E}[\psi'(1)^\lambda] = \frac{F(\alpha(\lambda),\kappa,\rho_-,\rho_+)}{F(\sqrt{\kappa},\kappa,\rho_-,\rho_+)} \qquad \textit{for } \lambda < \lambda_0.$$

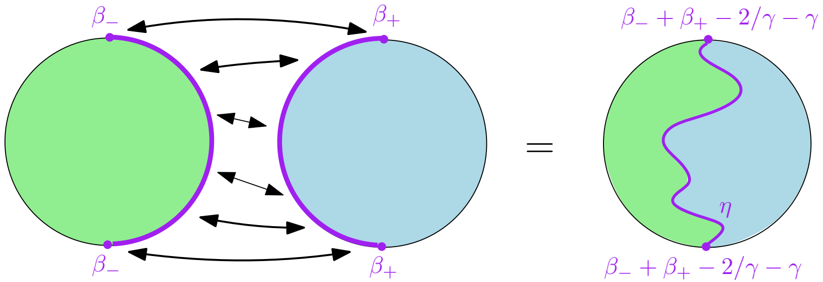
- Proof uses conformal welding and two sources of integrability in LQG:
  - Liouville conformal field theory (LCFT)
  - Random planar maps (RPM) and Brownian motion
- The theorem is difficult to approach via classical Loewner chain and Itô calculus methods.

Exact formulas for SLE via LQG: Other examples

- KPZ formula: relates exponents/dim. in Eucl. & LQG environments
  - E.g. Duplantier, Gwynne-H.-Miller'15
- Chen-Curien-Maillard'17: CLE conf. radius via RPM with $O(n)$ model
- Miller-Sheffield-Werner'20: trunk of $\text{SLE}_\kappa(\rho)$-processes, continuum Edwards-Sokal coupling, fuzzy Potts model arm exponent
- Ang-Sun'21: CLE electrical thickness and three-point function
- Kavvadias-Miller-Schoug'21: regularity of SLE_4 and SLE_8
- Ang-Remy-Sun'22: CLE modulus, $\text{SLE}_{8/3}$ loop part. func. (Sun's talk)

Related techniques used for **permutons** in recent Borga-H.-Sun-Yu'22.

Proof of SLE moment formula via welding



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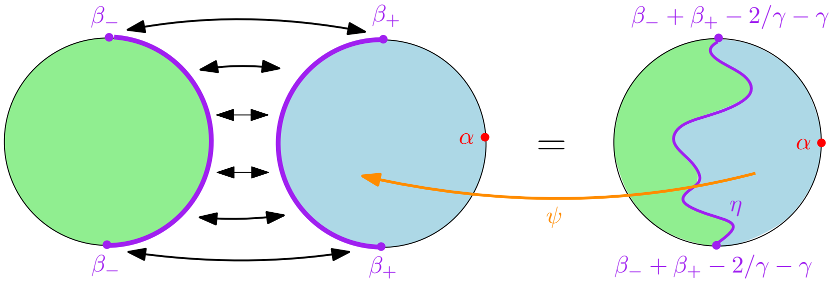
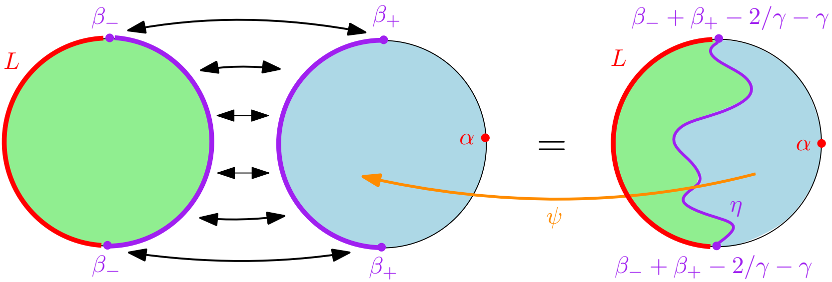


η has law $|\psi'(1)|^{1-\frac{\alpha}{2}(Q-\frac{\alpha}{2})} d\mathrm{SLE}_\kappa(\rho_-;\rho_+)$

Proof of SLE moment formula via welding

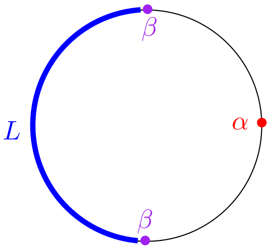


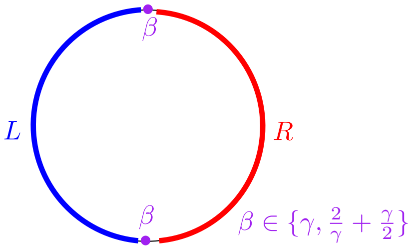
η has law $|\psi'(1)|^{1-\frac{\alpha}{2}(Q-\frac{\alpha}{2})} d\mathrm{SLE}_\kappa(\rho_-;\rho_+)$

First step in proof of moment formula:

1. Compute law of $\nu(L)$ using left and right sides of welding identity.
2. Formula using right side involves $\mathbb{E}[|\psi'(1)|^{1-\frac{\alpha}{2}(Q-\frac{\alpha}{2})}]$.
3. Set formulas in 1. equal to each other; solve for $\mathbb{E}[|\psi'(1)|^{1-\frac{\alpha}{2}(Q-\frac{\alpha}{2})}]$.
4. This strategy works for $\beta_- \in \{\gamma, \frac{2}{\gamma} + \frac{\gamma}{2}\}$ ($\rho_- \in \{0, \frac{\gamma^2}{2} - 2\}$).

Two inputs for computing law of LQG length $\nu(L)$ of L





	Law of $\nu(L)$	Joint law of $\nu(L)$ & $\nu(R)$
Method:	LCFT	RPM/Brownian motion

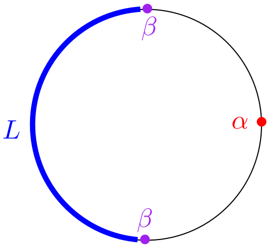
Boundary length of disks with three singularities

Remy-Zhu'20:

$$\nu(L) \sim \mathfrak{c}(\beta,\alpha)\ell^{\frac{2}{\gamma}(\beta+\frac{1}{2}\alpha-Q)-1}d\ell,$$

where $\nu(L)$ is the γ-LQG length of L and $\mathfrak{c}(\beta,\alpha)$ is equal to

$$\frac{1}{(Q-\beta)^2}\left(\frac{2\pi}{(\frac{\gamma}{2})^{\frac{\gamma^2}{4}}\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{2}{\gamma}(Q-\beta-\frac{1}{2}\alpha)}\frac{\Gamma_{\frac{\gamma}{2}}(\frac{1}{2}\alpha)^2\Gamma_{\frac{\gamma}{2}}(Q-\beta+\frac{1}{2}\alpha)\Gamma_{\frac{\gamma}{2}}(\beta+\frac{1}{2}\alpha-\frac{\gamma}{2})}{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma})\Gamma_{\frac{\gamma}{2}}(Q-\beta)^2\Gamma_{\frac{\gamma}{2}}(\alpha)}.$$



γ-LQG disk with singularities β,β,α

Thick-thin disk duality: disks with two marked points

Remy-Zhu'20: For thick disks with two β-singularities,

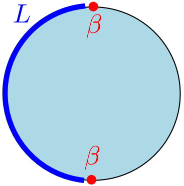
$$\nu(L) \sim \overline{R}(\beta)\ell^{-2-\frac{4}{\gamma^2}+\frac{2\beta}{\gamma}}\, d\ell. \tag{1}$$

Ang-H.-Sun'21: The thin disk also satisfies (1).

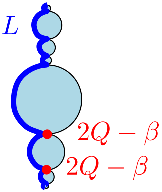
The proof uses:

- The PPP definition of the thin disk.
- Reflection principle:

$$R(\beta)R(2Q-\beta)=1, \qquad R(\beta) = -\Gamma(1-2(Q-\beta)/\gamma)\overline{R}(\beta).$$



Thick disk with parameter $\beta \le Q$



Thin disk with parameter $\beta > Q$

Thick-thin disk duality: disks with three marked points

Remy-Zhu'20: For thick disks with two β-singularities and one α-sing.,

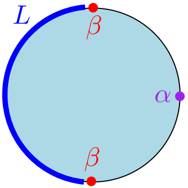
$$\nu(L) \sim \mathfrak{c}(\beta,\alpha)\ell^{\frac{2}{\gamma}(\beta+\frac{1}{2}\alpha-Q)-1}d\ell. \quad (2)$$

Ang-H.-Sun'21: The thin disk with an α-singularity also satisfies (2).

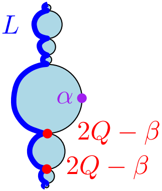
The proof uses:

- The PPP definition of the thin disk with an α-singularity.
- Reflection principle:

$$H_{(0,1,0)}^{(\beta,\beta,\alpha)} = R(\beta)^2 H_{(0,1,0)}^{(2Q-\beta,2Q-\beta,\alpha)}, \quad H_{(0,1,0)}^{(\beta,\beta,\alpha)} = c_{\gamma,\beta,\alpha}\mathfrak{c}(\beta,\alpha).$$

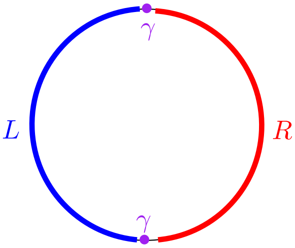


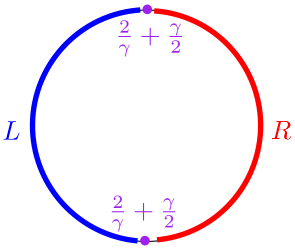
Thick disk with parameter $\beta \leq Q$ and an α-singularity



Thin disk with parameter $\beta > Q$ and an α-singularity

Joint boundary lengths of disks with two singularities





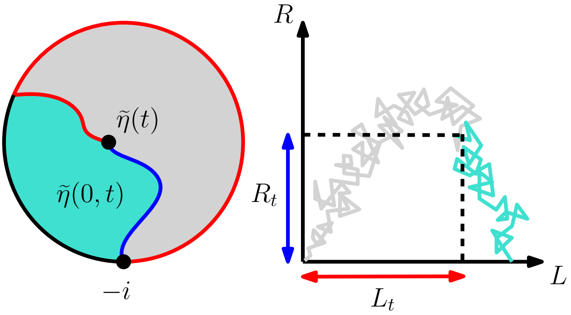
$$C(\ell + r)^{-\frac{4}{\gamma^2}-1}$$

$$C\frac{(\ell r)^{4/\gamma^2-1}}{(\ell^{4/\gamma^2} + r^{4/\gamma^2})^2}$$

$\ell = \nu(L),\ r = \nu(R),\ \nu =$LQG length measure

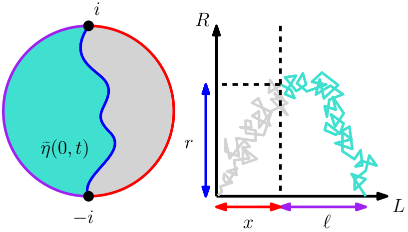
Proof uses encoding of LQG disks in terms of planar Brownian motion (mating of trees; Duplantier-Miller-Sheffield'14).

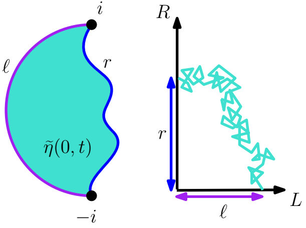
Planar Brownian motion and SLE on LQG



- $\widetilde{\eta}$ is a space-filling $\mathrm{SLE}_{16/\gamma^2}$ on an independent γ-LQG disk.
- Boundary length process given by correlated Brownian cone excursion.
- See Duplantier-Miller-Sheffield'14 and Ang-Gwynne'19.

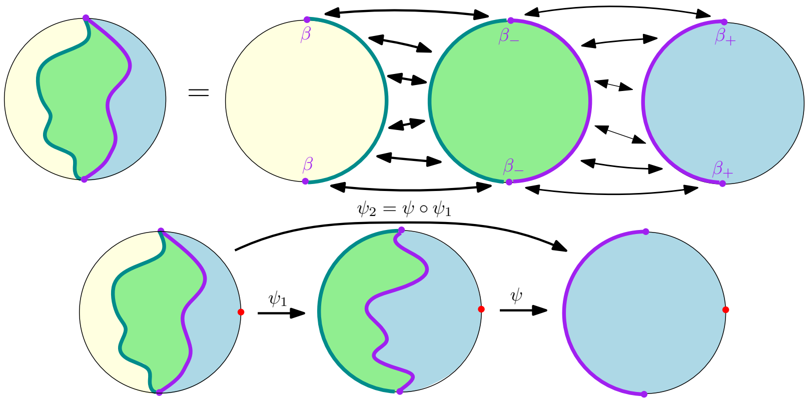
Planar Brownian motion and SLE on LQG





- $\tilde{\eta}$ is a space-filling $\mathrm{SLE}_{16/\gamma^2}$ on an independent γ-LQG disk.
- Boundary length process given by correlated Brownian cone excursion.
- The turquoise domain defines a disk with $\frac{2}{\gamma}+\frac{\gamma}{2}$ singularities.
- The time t corresponds to a running infimum for $L \Rightarrow$ The disk with $\frac{2}{\gamma}+\frac{\gamma}{2}$ singularities and boundary lengths ℓ, r encoded by Brownian path in first quadrant from $(\ell, 0)$ to $(0, r)$.
- Brownian motion estimates give partition function of such excursions.

General case of SLE moment theorem via shift equations



$$\psi_2'(1) = \psi_1'(1)\psi'(1) \quad \text{(product rule)}$$

$$f\left(\beta + \beta_- - \frac{2}{\gamma} - \gamma, \beta_+\right) = f\left(\beta, \beta_- + \beta_+ - \frac{2}{\gamma} - \gamma\right) f(\beta_-, \beta_+), \quad f(\beta_-, \beta_+) := \mathbb{E}[\psi'(1)^\lambda].$$

General case of SLE moment theorem via shift equations

$$f\left(\beta+\beta_- - \frac{2}{\gamma} - \gamma, \beta_+\right) = f\left(\beta, \beta_- + \beta_+ - \frac{2}{\gamma} - \gamma\right) f(\beta_-, \beta_+), \quad f(\beta_-, \beta_+) := \mathbb{E}[\psi'(1)^\lambda],$$

$$\frac{f(\beta_- - \frac{2}{\gamma}, \beta_+)}{f(\beta_-, \beta_+)} = f\left(\gamma, \beta_- + \beta_+ - \frac{2}{\gamma} - \gamma\right) \text{ known explicitly (1st shift equation),}$$

$$\frac{f(\beta_- - \frac{\gamma}{2}, \beta_+)}{f(\beta_-, \beta_+)} = f\left(\frac{2}{\gamma} + \frac{\gamma}{2}, \beta_- + \beta_+ - \frac{2}{\gamma} - \gamma\right) \text{ known explicitly (2nd shift equation).}$$

- The shift equations uniquely characterize f.
- Shift equations also play an essential role in LCFT (Teschner'95, Kupianen-Rhodes-Vargas'17, Remy-Zhu'20)

Thanks for your attention!