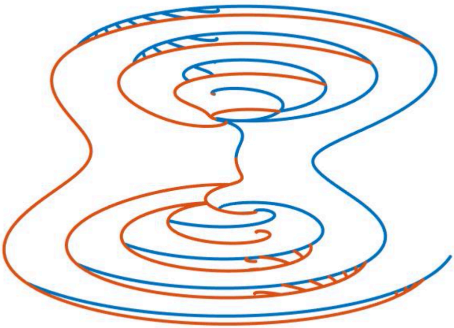
The Loewner equation with complex-valued driving functions



Joan Lind – University of Tennessee

joint work with Jeffrey Utley

Outline

- Introduction to the Loewner equation
- Loewner hulls with complex-valued driving functions
- Phase transition for complex-driven Loewner hulls
- Differences between complex-driven Loewner hulls and real-driven Loewner hulls

Brief history

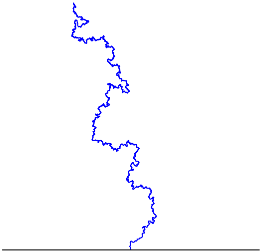
Bieberbach conjecture (1916):
For $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ conformal on $\mathbb{D}$, then $|a_n| \leq n$.

Charles Loewner introduced the Loewner equation in 1923 to prove the $n = 3$ case.

Louis des Branges again used the Loewner equation when he proved the conjecture in 1985.

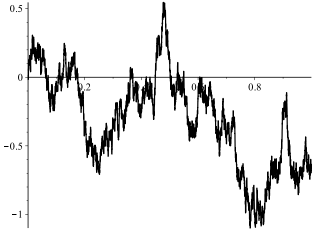
In 2000, Oded Schramm introduced Schramm-Loewner Evolution, SLE_κ.

The chordal Loewner equation



L Equ

$\longleftrightarrow$



growing families of 2-d sets

real-valued functions

From functions to growing families of sets

For a continuous, real-valued function $\lambda(t)$ and $z \in \mathbb{H}$, consider

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z$$

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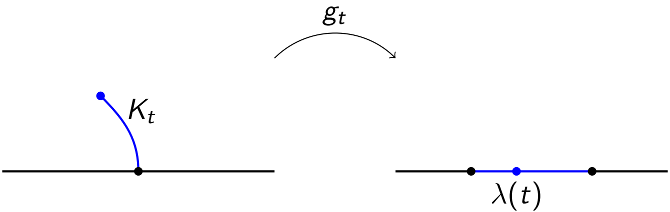
Theorem: g_t is a conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$.

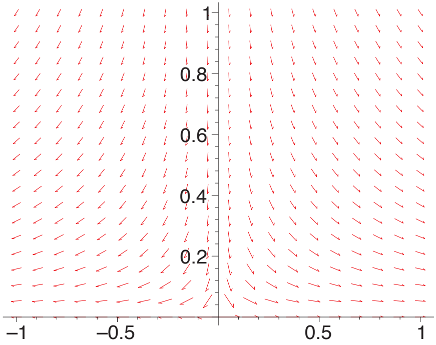
Loewner equation visual



$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \lambda(t)}$$

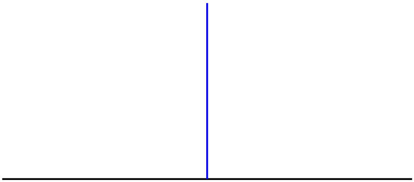
Loewner flow

$$\frac{\partial}{\partial t} g_t(z) = 2\,\frac{\text{Re } g_t(z) - \lambda(t)}{|g_t(z) - \lambda(t)|^2} - 2i\,\frac{\text{Im } g_t(z)}{|g_t(z) - \lambda(t)|^2}$$



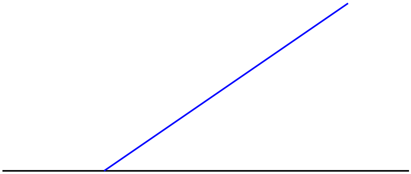
Example 1

Loewner hull generated by $\lambda(t) \equiv 0$.



Example 2

Loewner hull generated by $\lambda(t) = c\sqrt{t}$.



Simple curve Loewner hulls

Question: When are the Loewner hulls a simple curve?

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Lip(1/2) functions:

$$|\lambda(t) - \lambda(s)| \leq M\,|t-s|^{1/2}$$

for all t, s is the domain of λ. The smallest such M is $||\lambda||_{1/2}$.

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Answer: (Marshall, Rohde) There exists $C_0 > 0$ so that for $\lambda \in$ Lip(1/2) with $||\lambda||_{1/2} < C_0$, then the Loewner hull is a quasislit γ.

Simple curve Loewner hulls

Question: Do all Lip(1/2) driving functions generate simple curves?

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Answer: (Marshall, Rohde) No. There is a non-simple example (a curve that spirals around a disc) that is generated by a Lip(1/2) driving function.

Simple curve Loewner hulls

Question: What is the optimal value of C_0 for the Marshall-Rohde theorem?

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Answer: (L) $C_0 = 4$.

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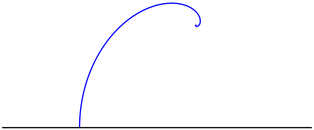
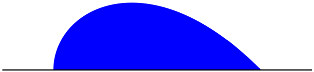
Key examples: $\lambda(t) = -c\sqrt{1-t}$

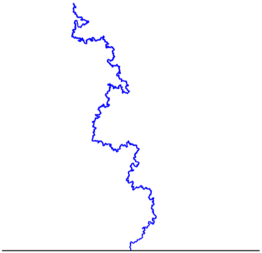


$c = 3$



$c = 5$

Loewner equation recap



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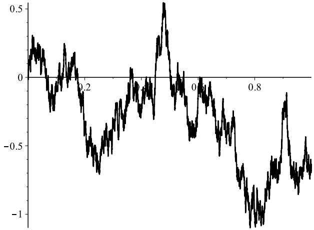

growing families of 2-d sets

real-valued functions

Complex-valued driving functions

Question: What happens with complex-valued driving functions?

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Previous definition:

For a real-valued function $\lambda(t)$ and $z \in \mathbb{H}$, consider

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Loewner hulls: $K_t = \{z \in \mathbb{H} : g_s(z) = \lambda(s) \text{ for some } s \leq t\}$.

Complex-valued driving functions

Question: What happens with complex-valued driving functions?

New definition:

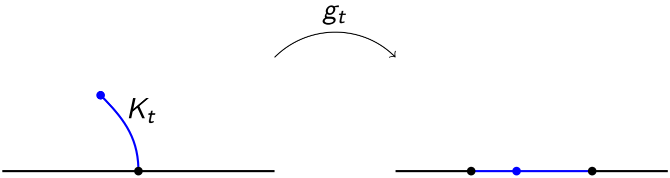
For a complex-valued function $\lambda(t)$ and $z \in \mathbb{C}$, consider

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z$$

Loewner hulls: $L_t = \{z \in \mathbb{C} : g_s(z) = \lambda(s) \text{ for some } s \leq t\}$.

Complex-valued driving functions

With real-valued driving functions: $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$



Complex-valued driving functions

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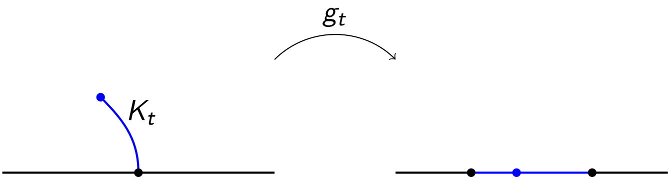
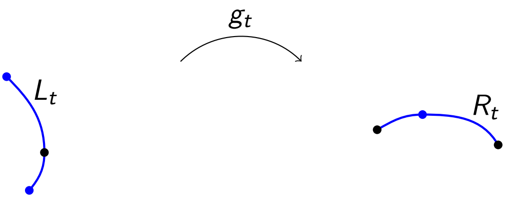
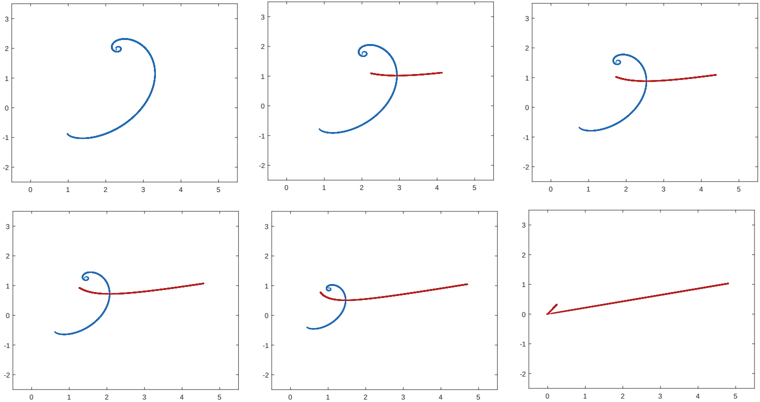


With complex-valued driving functions: $g_t : \mathbb{C} \setminus L_t \to \mathbb{C} \setminus R_t$



Loewner flow

The Loewner flow transforms L_t into R_t.



Properties

Duality Property: The right hull R_t driven by $s \mapsto \lambda(s)$ is a rotation of the left hull driven by $s \mapsto -i\lambda(t-s)$.

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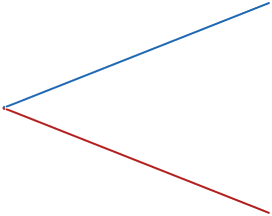
Still have familiar properties:

- Scaling Property
- Translation Property
- Reflection Property
- Concatenation Property

Examples of left hulls L_t



$\lambda(t) = 3\sqrt{t}$

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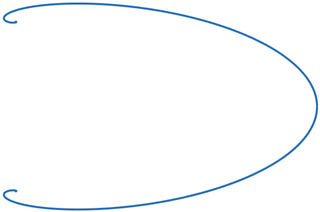
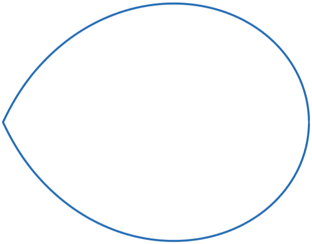
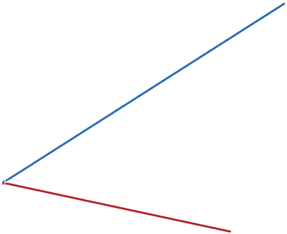


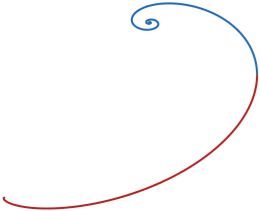
$\lambda(t) = 3\sqrt{1-t}$



$\lambda(t) = 4.5\sqrt{1-t}$

Examples of left hulls L_t





$$\lambda(t) = 2e^{i\pi/4}\sqrt{t} \qquad\qquad \lambda(t) = (3.31 + 1.15i)\sqrt{1-t}$$

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Answer: (Tran) There exists $C_0 > 0$ so that for complex-valued $\lambda \in$ Lip(1/2) with $||\lambda||_{1/2} < C_0$, then the Loewner hull is a quasi-arc γ.

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Question: What is the optimal value of C_0 for Tran's theorem?

Tran's approach

Fix complex-valued λ with small Lip(1/2) norm.

Consider the family of driving functions $\alpha\lambda$ for $\alpha \in \mathbb{D}$.

Question: What can we say about the Loewner hulls driven by $\alpha\lambda$?

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Answer (Tran) There exists $\gamma : \mathbb{D} \times [0,1] \to \mathbb{C}$ so that

- For fixed $t \in [0,1]$, the map $\alpha \mapsto \gamma(\alpha, t)$ is holomorphic.
- For fixed $\alpha \in \mathbb{D}$, the map $t \mapsto \gamma(\alpha, t)$ is injective.
- $\gamma(0, t) = 2i\sqrt{t}$.
- For $\alpha \in \mathbb{D} \cap \mathbb{R}$, then $\gamma^\alpha = \gamma(\alpha, \cdot)$ is generated by $\alpha\lambda$ from the Loewner equation.

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Hence $\gamma^\alpha(t) = \gamma(\alpha, t)$ is a quasiarc.

Think of γ^α as the "top curve" generated by $\alpha\lambda$ from the Loewner equation.

A key idea of Tran's work

What could prevent a simple curve hull?

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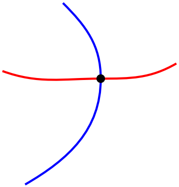
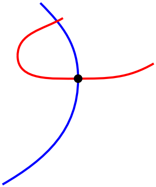
What could prevent a simple curve hull?

- Interaction between right hulls and left hulls during the Loewner flow.



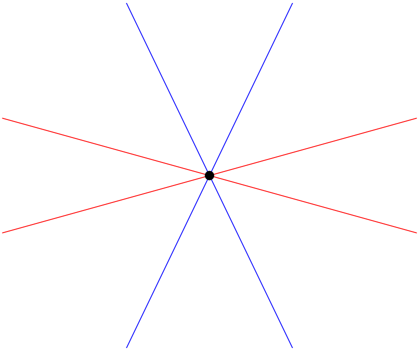
Simple curve



No simple curve

A key idea of Tran's work

Tran prevents this interaction with a cone condition



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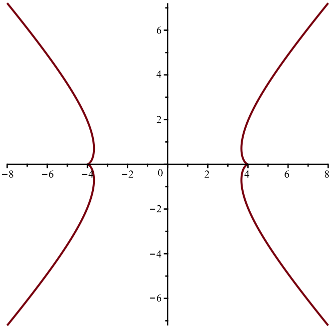
Determined from hulls driven by $c\sqrt{1-t}$ for $c \in \mathbb{C}$.

Hulls driven by $c\sqrt{1-t}$

Three regimes for hulls driven by $c\sqrt{1-t}$ for $c \in \mathbb{C}$ (L, Utley):

- Simple curve
- Bubble
- Transitional



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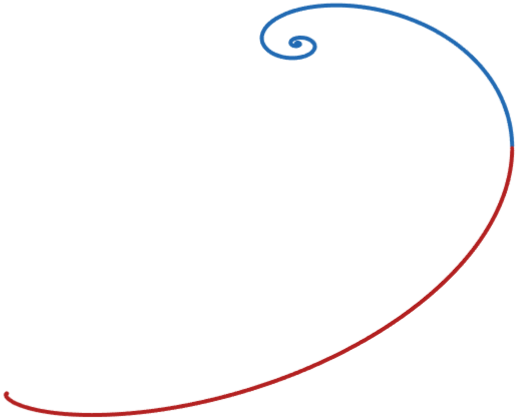
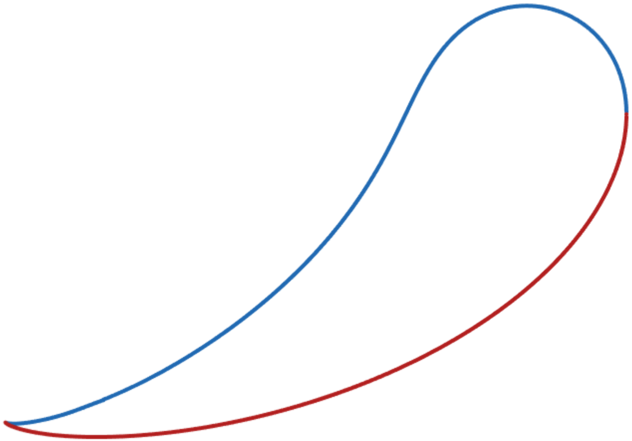
Simple curve regime



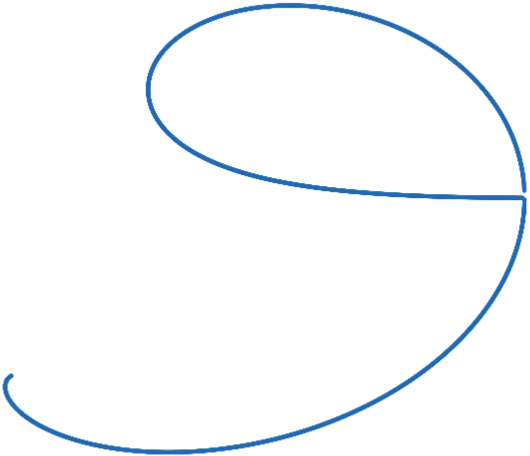
Hulls driven by $c\sqrt{1-t}$

Bubble regime



Hulls driven by $c\sqrt{1-t}$

Transitional regime



Hulls driven by $c\sqrt{t}$

Regimes for hulls driven by $c\sqrt{t}$ for $c \in \mathbb{C}$ (L, Utley):

- 2 line segments
- 1 line segment

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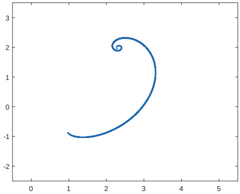
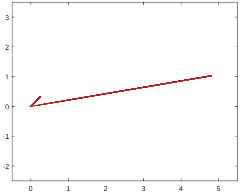
Note: The left hulls driven by $c\sqrt{t}$ are related to the right hulls driven by $\hat{c}\sqrt{1-t}$.





Hulls driven by $c\sqrt{\tau + t}$

Regimes for hulls driven by $c\sqrt{\tau + t}$ for $\tau > 0, c \in \mathbb{C}$ (L, Utley):

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Regimes for hulls driven by $c\sqrt{\tau + t}$ for $\tau > 0, c \in \mathbb{C}$ (L, Utley):

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Note: For c in the transitional regime, the left hull L_{t_0} generated by $c\sqrt{\tau + t}$ will be non-simple for t_0 large enough.

Complex-driven hulls versus real-driven hulls

- Real-valued: $\mathbb{H} \setminus K_t$ will always be simply connected.
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- Real-valued: If K_t is a simple curve, the corresponding L_t grows from two ends.
  Complex-valued: L_t may be a simple curve growing from only one end.

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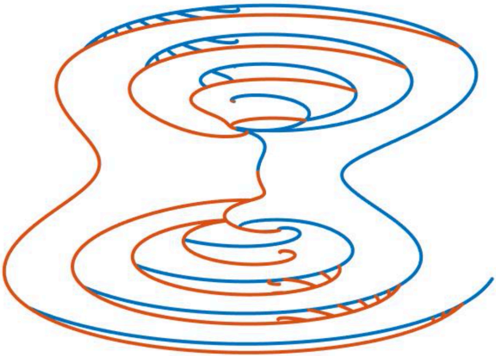
- Real-valued: If K_t is a simple curve, the corresponding L_t grows from two ends.
  Complex-valued: L_t may be a simple curve growing from only one end.

- Real-valued: K_t grows continuously
  Complex-valued: L_t may not grow continuously

Another example

Hull driven by $\lambda(t) = te^{2\pi i \cdot t}$



Tools

Tools used to analyze Loewner hulls driven by $c\sqrt{1-t}$:

- Implicit solution
- Tran's work
- Concatenation property
- Duality between left hulls and right hulls
- Holomorphic motions

Other work with complex-valued drivers

Ewain Gwynne and Joshua Pfeffer have studied Loewner evolution driven by complex Brownian motion.

Thank you