

Some progress on 3D Yang–Mills

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Yang–Mills action

- Let G be a compact Lie group contained in $U(n)$ for some n , and let \mathfrak{g} be its Lie algebra.
- Let \mathcal{A} be the space of **smooth G -connections** on the trivial principal bundle $\mathbb{R}^d \times G$ — that is, the set of all smooth $A : \mathbb{R}^d \rightarrow \mathfrak{g}^d$.
- The **curvature form** F of A is defined as

$$F_{jk}(x) = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} + [A_j(x), A_k(x)],$$

so that F is a $d \times d$ array of elements of \mathfrak{g} .

- The **Yang–Mills action** of A is defined as

$$S_{\text{YM}}(A) = - \int_{\mathbb{R}^d} \sum_{j,k=1}^d \text{Tr}(F_{jk}(x)^2) dx,$$

provided that the integral is well-defined.

Euclidean Yang–Mills theories

- Heuristically, a Euclidean Yang–Mills theory on \mathbb{R}^d with gauge group G is a probability measure on \mathcal{A} with density proportional to $e^{-\beta S_{\text{YM}}(A)}$ (with respect to a hypothetical Lebesgue measure on \mathcal{A}), where β is the **coupling parameter**. This hypothetical object is sometimes called the **Yang–Mills measure**.
- Construction of this measure for $d = 4$ would be a key step to the solution of the **Yang–Mills existence problem**, which is one of the Clay Millennium Prize problems.
- This would give a rigorous mathematical foundation for the **Standard Model** of quantum mechanics, which is our best available model for the quantum world.

- One common theme in the literature is to look at the problem in dimensions two and three, instead of four.
- The greatest amount of progress has been made in 2D. In fact, we now know how to construct the Yang–Mills measures in 2D.
- On the other hand, very few results are known in 3D and 4D.
- Many of the above results are for compact manifolds, such as tori, instead of \mathbb{R}^d .

- Many authors have made contributions to understanding 2D Euclidean Yang–Mills theories. An incomplete list, in chronological order: Brydges, Fröhlich & Seiler '79 '80, Klimek & Kondracki '87, Gross, King & Sengupta '89, Fine '90 '91, Sengupta '92 '93 '97, Lévy '03 '10, Chevyrev '19, Chandra, Chevyrev, Hairer & Shen '20.
- One of the most recent papers, Chevyrev '19, constructs a space of connections and a probability measure on this space of connections that can be interpreted as a 2D Yang–Mills measure. Previous works constructed measures on spaces of observables.
- Chandra, Chevyrev, Hairer and Shen '20 construct a state space and a Markov process on this state space, such that the unique invariant measure of this process can be interpreted as a 2D Yang–Mills measure. It is not known whether the unique invariant measure exists.

Results in 3D and 4D

- $U(1)$ theory is not difficult to construct in any dimension, since the Yang–Mills measure is Gaussian.
- The problem becomes far more challenging for non-Abelian theories.
- The state of the art for 3D and 4D, for a general (non-Abelian) Lie group, consists of the phase cell renormalization results of Błaban '83–'89, the phase cell renormalization results of Federbush '86–'90, and an alternative approach of Magnen, Rivasseau, and Sénéor '93.
- In a very recent preprint, Chandra, Chevyrev, Hairer and Shen '22 proved the short-time existence of a stochastic PDE whose invariant measure (if it exists and is unique) would be a candidate for 3D Yang–Mills measures. (This is the **stochastic quantization** approach of Parisi and Wu '81.)
- As of now, there is no construction of the Yang–Mills measure in 3D and 4D, in the way that was done by the previously cited works in the 2D case.

- To understand the main difficulty in constructing Euclidean Yang–Mills theories in $d \geq 3$, we have to first understand the following:
 - gauge transforms and gauge invariance,
 - connections on a principal bundle,
 - Wilson loop observables,
 - the massless Gaussian free field and some of its properties, and
 - the Yang–Mills heat flow.

Gauge transforms

- Take any $A \in \mathcal{A}$ and any differentiable $g : \mathbb{R}^d \rightarrow G$.
- For $1 \leq j \leq d$, let

$$A_j^g(x) = g(x)^{-1} A_j(x) g(x) + g(x)^{-1} \frac{\partial g}{\partial x_j}.$$

- The field $A^g = (A_1^g, \dots, A_d^g)$ is a **gauge transform** of A . Each choice of g induces a gauge transform.
- It is a fact that A is also a gauge transform of A^g . The fields A and A^g are **gauge equivalent**. This is an equivalence relation.
- The space \mathcal{A} of G -connections is not physically relevant. Rather, the quotient space \mathcal{A}/\mathcal{G} of **gauge equivalence classes** is the physically relevant space (where \mathcal{G} is the set of gauge transforms).

- A function f on \mathcal{A} is called **gauge invariant** if $f(A) = f(A')$ whenever A and A' are gauge equivalent.
- For example, it is not difficult to check that the Yang–Mills action S_{YM} is gauge invariant.
- Any physical observable must be gauge invariant.
- The most important gauge invariant observables are **Wilson loop observables**. Will be defined soon.

What is a G -connection?

- We have seen earlier that the Yang–Mills measure is supposed to be a probability measure on some space of G -connections on a principal bundle.
- To fix ideas, let's say our principal bundle is the trivial bundle $\mathbb{T}^d \times G$, where \mathbb{T}^d is the d -dimensional unit torus $\mathbb{R}^d/\mathbb{Z}^d$.
- One can think of the bundle as a copy of G sitting at each point in \mathbb{T}^d .
- The **main role** of G -connection A is to give a prescription for **change of coordinates** — it prescribes how an element $g \in G$ at a point $x \in \mathbb{T}^d$ transforms to some other element $h \in G$ at a different point $y \in \mathbb{T}^d$ as we move along a curve ℓ from x to y .
- We say that g is **parallel transported** to h along ℓ . It is defined as follows.

Parallel transport

- Let A be a **smooth** G -connection on \mathbb{T}^d .
- Given a piecewise C^1 path $\ell : [0, 1] \rightarrow \mathbb{T}^d$, let $\phi : [0, 1] \rightarrow G$ be a solution to the ODE

$$\phi'(t) = \phi(t) \sum_{j=1}^d A_j(\ell(t)) \ell'_j(t), \quad \phi(0) = \text{id}.$$

- From standard ODE theory, ϕ exists and is unique.
- The group element $\phi(1)$ is the **parallel transport of the identity element along the curve ℓ** .
- When $G = U(1)$, each A_j is a function from \mathbb{T}^d into the imaginary axis. Hence, the parallel transport is simply the exponential of the line integral $\int_{\ell} \sum A_j dx_j$.
- When G is non-Abelian, the A_j 's are non-Abelian matrix-valued functions.
- In this case, $\phi(1)$ is a **path ordered exponential of the line integral $\int_{\ell} \sum A_j dx_j$** , defined as the solution of the above ODE.

Observables

- Any physical theory should have **observables** — that is, functions on the state space whose values can be the result of experimental measurements.
- For gauge theories, the parallel transport maps defined by the connections are the natural observables.
- But they are not **gauge invariant**, and hence, unphysical.
- Even if we take the parallel transport $P_\ell(A)$ induced by a connection A along a **loop** ℓ , it is not gauge invariant.
- But, parallel transport along a loop is **gauge covariant**, meaning that $P_\ell(A^g) = g^{-1}P_\ell(A)g$.
- Thus, if χ is a **character** of G , then $W_{\ell,\chi}(A) := \chi(P_\ell(A))$ is gauge invariant.
- $W_{\ell,\chi}$ is called a **Wilson loop observable**. Besides being mathematically natural, they also have a great deal of physical importance. Calculations related to Wilson loop observables come up in the problems of **quark confinement** and **mass gap**.

Massless Gaussian free field

- For $d \geq 2$, the **massless Gaussian free field** ϕ on the torus \mathbb{T}^d is a **random distribution** with the property that for any smooth f and g , $\phi(f)$ and $\phi(g)$ are jointly Gaussian random variables, with mean zero and covariance $\int f(x)G(x,y)g(y)dxdy$, where G is the Green's function on \mathbb{T}^d .
- Taking $f = \delta_x$ and $g = \delta_y$, we get the formal expression $\text{Cov}(\phi(x), \phi(y)) = G(x, y)$, even though ϕ is not defined pointwise.
- Note that the covariance blows up to infinity as $y \rightarrow x$. This is consistent with the fact that ϕ is infinity at any given point.

A belief about Yang–Mills theories

- Recall that Euclidean Yang–Mills theories are supposed to describe random G -connections on principal bundles — let's say, the trivial bundle $\mathbb{T}^d \times G$.
- It is **believed** that these random connections are not really random functions, but rather, **random distributions**.
- Furthermore, it is believed that componentwise, these random distributions **have similar behavior as the massless Gaussian free field**.
- All this has been verified rigorously for 2D Euclidean Yang–Mills theories (e.g., in the recent work of Chevyrev '19), but not in higher dimensions.

Wilson loop observables for Yang–Mills theories

- So, assuming that Yang–Mills theories behave like massless free fields, let us consider the question of computing Wilson loop observables.
- Recall that Wilson loop observables are defined using **line integrals**.
- Let ℓ be a loop and suppose that we want to integrate a massless free field ϕ along ℓ .
- The Green's function $G(x, y)$ blows up like $\log(1/|x - y|)$ as $y \rightarrow x$ when $d = 2$, and like $|x - y|^{2-d}$ when $d \geq 3$.
- From this, one can show that the line integral of ϕ along ℓ is well-defined in 2D, but blows up in higher dimensions.
- Thus, we do not expect to be able to define Wilson loop observables directly in $d \geq 3$. Some indirect approach has to be taken.

Renormalization

- Since Wilson loop observables are unlikely to be definable for Euclidean Yang–Mills theories in $d \geq 3$, one idea is that we should first make the theories **smooth** by some kind of **regularization** (also called **renormalization**).
- The problem is that we need the regularization to be **gauge covariant**, meaning that if $R(A)$ is the regularized version of a distributional connection A , then $R(A^g)$ must equal $R(A)^g$.
- We need this so that the regularization is in fact acting on the **physical** space \mathcal{A}/\mathcal{G} , rather than the unphysical space \mathcal{A} .
- A simple regularization, such as taking convolution with a smooth kernel, is not going to work.
- The works on Yang–Mills in the 80s focused on lattice approximations of Yang–Mills theories and smoothing using a complicated gauge-covariant procedure known as **phase cell renormalization**. This program was not completed, which is why the Yang–Mills existence problem is still open.

Renormalization by Yang–Mills heat flow

- An idea that has emerged in the last fifteen years is that Yang–Mills theories can be regularized by the **Yang–Mills heat flow**, which is the **gradient flow of the Yang–Mills action**. (Will be defined soon.)
- The Yang–Mills heat flow has a long and illustrious history in mathematics, playing an important role in the profound work of Donaldson, among other things.
- The idea of renormalization by the Yang–Mills heat flow appeared in the works of physicists (Narayanan & Neuberger '06, Lüscher '10).
- The advantage of the Yang–Mills heat flow over the ordinary heat flow is that it is gauge covariant, which makes it a legitimate smoothing apparatus for Yang–Mills theories.

The Yang–Mills heat flow

- Given a G -connection A on \mathbb{T}^d , recall that the curvature form F is defined as

$$F_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} + [A_j, A_k].$$

- The Yang–Mills heat equation is the following nonlinear PDE, that describes the evolution of a time-varying connection $A(t)$:

$$\frac{\partial A_j}{\partial t} = - \sum_{k=1}^d \frac{\partial F_{jk}}{\partial x_k} - \sum_{k=1}^d [A_k, F_{jk}],$$

holding for $j = 1, \dots, d$.

The Zwanziger–DeTurck–Donaldson–Sadun flow

- It is convenient to work with a variant of the Yang–Mills heat flow, discovered independently by Zwanziger '81, DeTurck '83, Donaldson '85, and Sadun '87. This is the flow

$$\begin{aligned}\partial_t A_i(t) = & \Delta A_i(t) \\ & + \sum_{j=1}^d [A_j(t), 2\partial_j A_i(t) - \partial_i A_j(t) + [A_j(t), A_i(t)]],\end{aligned}$$

holding for $i = 1, \dots, d$. Let us call this the **ZDDS flow**.

- It is related to the Yang–Mills heat flow as follows.
- If $C(t)$ solves the ZDDS equation and $A(t)$ solves the Yang–Mills heat equation with the same initial data, then $C(t) = A(t)^{g(t)}$ for a time-varying gauge transform $g(t)$.
- Thus, the Yang–Mills heat flow and the ZDDS flow **are the same on the quotient space \mathcal{A}/\mathcal{G}** .
- For this reason, the ZDDS flow is also known as the **Yang–Mills heat flow in the DeTurck gauge**.

Yang–Mills heat flow with rough initial data

- Recall that Yang–Mills theories are believed to be random distributions with similar features as the massless Gaussian free field.
- So, for renormalization using the Yang–Mills heat flow to be a valid procedure, we need that solutions of the Yang–Mills heat flow, or equivalently, the ZDDS flow, should exist for initial data that behaves like a free field.
- The nonlinearity of the PDEs make this a challenging problem.
- Available PDE results do not allow initial data that is rougher than H^1 . In particular, distributional initial data is not allowed.

The program of Charalambous and Gross

- Leonard Gross and Nelia Charalambous, in a series of papers between 2010 and 2017, increased the allowed roughness of the initial data to $H^{1/2}$ for the 3D Yang–Mills heat flow on a compact manifold.
- The goal was to **construct 3D Euclidean Yang–Mills theories as trajectories of the Yang–Mills heat flow.**
- But $H^{1/2}$ is still **functional**, rather than **distributional**, initial data. The massless free field has regularity $H^{-1/2-\epsilon}$, for any $\epsilon > 0$.
- It is possible that a general existence theorem cannot do better than $H^{1/2}$ initial data in 3D.

Taking the program to completion: Our contributions

Theorem (Cao & C., 2021a. Rough statement.)

It is possible to define the Zwanziger–DeTurck–Donaldson–Sadun variant of the Yang–Mills heat flow on the 3D torus starting from free field initial data. Moreover, the initial data need not be exactly the free field; it suffices for it to be a random distribution having certain features of the free field.

Theorem (Cao & C., 2021b. Rough statement.)

It is possible to construct a state space for Euclidean Yang–Mills theories on the 3D torus as a set of trajectories of the Yang–Mills heat flow, and have a criterion for the existence of a convergent subsequence of a given sequence of probability measures on this space (i.e., an analogue of Prokhorov's theorem).

PDEs with random initial data

- The main idea in the proof of our first theorem is that the randomness of the initial data helps ensure the existence of a solution, which may not hold for deterministic initial data of comparable roughness.
- The general idea to exploit the effects of probabilistic smoothing was first used by Bourgain '96 '99 to analyze the nonlinear Schrödinger equation with GFF initial data.
- A similar idea was later used by Da Prato and Debussche '02 '03 in the stochastic PDE setting.
- There is by now a wide body of work building on this idea in many different settings.
- As far as we can tell, our work is the first to carry out such a program for the Yang–Mills heat flow.

Proof sketch: Step 0

- Write the ZDDS equation as

$$\partial_t A_i(t) = \Delta A_i(t) + X_i(A(t)), \quad 1 \leq i \leq d,$$

where

$$X_i(A(t)) := \sum_{j=1}^d [A_j(t), 2\partial_j A_i(t) - \partial_i A_j(t) + [A_j(t), A_i(t)]],$$

- Given initial data $A_0 = (A_{0,1}, \dots, A_{0,d})$, it is equivalent to the integral equation

$$A_i(t) = e^{t\Delta} A_{0,i} + \int_0^t e^{(t-s)\Delta} X_i(A(s)) ds, \quad 1 \leq i \leq d.$$

Proof sketch: Step 1

- This can be formulated as a fixed point equation $A = W(A)$, where $A = (A_1, \dots, A_d)$ and

$$W(A)_i(t) := e^{t\Delta}A_{0,i} + \int_0^t e^{(t-s)\Delta}X_i(A(s))ds.$$

- The idea is to apply a contraction mapping argument.
- For that, we need to (a) find a set of connections which is closed under a norm topology and mapped into itself by W , and (b) show that W is a contraction mapping on this set.
- Finally, we need to show that steps (a) and (b) can be executed when the components of A_0 are massless free fields.

- Fixed point equations can be set up in many different ways.
- In fact, this is how proofs of the existence of solutions to the Yang–Mills heat equation and the ZDDS equation have been carried out in all prior work.
- However, the techniques of those papers do not allow distributional initial data, so we need to set up the fixed point problem in a different way.

Proof sketch: Step 2 — defining the Banach space

- For a C^1 connection A , define $\|A\|_{C^0}$ to be the maximum of the supremum norms of the components of A , and let $\|A\|_{C^1}$ be the maximum of the supremum norms of the components of A and all of their first order derivatives.
- Next, for a continuous flow $\{A(t)\}_{0 < t \leq T}$ on the space of C^1 connections, and given some $\gamma \geq 0$, define

$$\|A\|_{Q_T^\gamma} := \sup_{0 < t \leq T} t^\gamma \|A(t)\|_{C^0} + \sup_{0 < t \leq T} t^{1/2+\gamma} \|A(t)\|_{C^1}.$$

- It turns out that this norm is **complete** on the space of continuous flows of C^1 connections in the time interval $(0, T]$, and hence defines a **Banach space**.
- Note that time 0 is intentionally left out, to allow for the possibility that $A(t)$ approaches a distribution as $t \downarrow 0$.
- Finiteness of the Q_T^γ norm means that $\|A(t)\|_{C^0} = O(t^{-\gamma})$ and $\|A(t)\|_{C^1} = O(t^{-1/2-\gamma})$ as $t \downarrow 0$.

Proof sketch: Step 3 — the contraction criterion

- Given initial data A_0 , let $B(t) := e^{t\Delta}A_0$ be the solution of the ordinary heat equation with initial data A_0 .
- Here A_0 may be a distribution, as long as $B(t) \in C^1$ for $t > 0$.
- Define

$$\mathcal{B}_{T,R}^\gamma := \{A : \|A - B\|_{\mathcal{Q}_T^\gamma} \leq R\}.$$

- A lengthy calculation shows that if for some $\gamma_1 \in [0, 1/2)$ and $\gamma_2 \in [0, 1/4)$ such that $\gamma_1 + \gamma_2 < 1/2$, we have

$$\|B(t)\|_{\mathcal{Q}_1^{\gamma_1}} \leq R \quad \text{and} \quad \|W(B(t)) - B(t)\|_{\mathcal{Q}_1^{\gamma_2}} \leq R,$$

then W is a **contraction mapping** on $\mathcal{B}_{T,3R}^{\gamma_2}$ for sufficiently small T (depending on R , γ_1 and γ_2).

- Thus, it has a **fixed point** in this set.
- Any fixed point of W is a solution of the ZDDS equation with initial data A_0 .

What we have shown so far:

- Let A_0 be our initial data, which may be a distributional connection.
- Let $B(t) := e^{t\Delta}A_0$ be the solution of the ordinary heat equation with initial data A_0 .
- Let $C(t) := W(B(t)) - B(t)$, where W is the map (defined earlier) whose fixed points are solutions of the ZDDS equation.
- Suppose that for some $\gamma_1 \in [0, 1/2)$ and $\gamma_2 \in [0, 1/4)$ such that $\gamma_1 + \gamma_2 < 1/2$, as $t \downarrow 0$, we have

$$\|B(t)\|_{C^0} = O(t^{-\gamma_1}), \quad \|B(t)\|_{C^1} = O(t^{-1/2-\gamma_1})$$

$$\|C(t)\|_{C^0} = O(t^{-\gamma_2}), \quad \|C(t)\|_{C^1} = O(t^{-1/2-\gamma_2}).$$

- Then, the ZDDS equation has a short-time solution with initial data A_0 .
- This is valid in any dimension.

Proof sketch: Step 4 — random distributional initial data

- Suppose now that our initial data A_0 is a random distribution.
- It is not hard to see that for any $x \in \mathbb{T}^d$, $B(t)(x)$ is a **linear form in the Fourier coefficients of A_0** , and $C(t)(x)$ is a **cubic form in the Fourier coefficients of A_0** .
- Thus, we can hope to obtain upper bounds on the C^0 and C^1 norms of $B(t)$ and $C(t)$ using **chaining arguments** from probability theory (Fernique, Talagrand, Dudley, ...).
- Let us specialize to the case where A_0 is the **d -dimensional massless free field on the space of G -connections**.
- In this case, chaining arguments show that the four conditions listed in the previous slide are satisfied for any $\gamma_1 > (d - 2)/4$ and $\gamma_2 > (d - 3)/2$.
- But we also need $\gamma_1 < 1/2$, $\gamma_2 < 1/4$, and $\gamma_1 + \gamma_2 < 1/2$. Thus, this works for $d = 3$, but not $d = 4$.

Constructing the state space

- Having shown that short-time solutions of the 3D Yang–Mills heat flow exist for free field initial data, the next step is to generalize the result to initial data that is ‘similar to a free field’.
- We have a generalization of that sort, that needs only exponential tail bounds for linear and quadratic forms of the initial data A_0 and some features of two-point and four-point correlations.
- The space of **distributional gauge orbits** is then constructed, following an idea of Charalambous & Gross, as a space of trajectories of the YM heat flow on the quotient space \mathcal{A}/\mathcal{G} in the time interval $(0, \infty)$ (leaving out $t = 0$, to allow for distributional initial conditions).
- A criterion for tightness of probability measures on this space is then proved using **Uhlenbeck compactness**.

What's left to do:

- The results presented here complete about half of the program initiated by Charalambous and Gross for constructing 3D Euclidean Yang–Mills theories.
- The main thing that's remaining to show is that a sequence of approximations of a 3D non-Abelian Yang–Mills theory (such as, a sequence of lattice gauge theories) does indeed behave like the free field in the limit.
- This would require proving various correlation decay estimates for the approximate theories.
- A number of estimates already exist in the papers from the 80s by Bataban and others.
- Hopefully, it can be done.

Thanks for your attention!