On log-CFT for UST and SLE(8)

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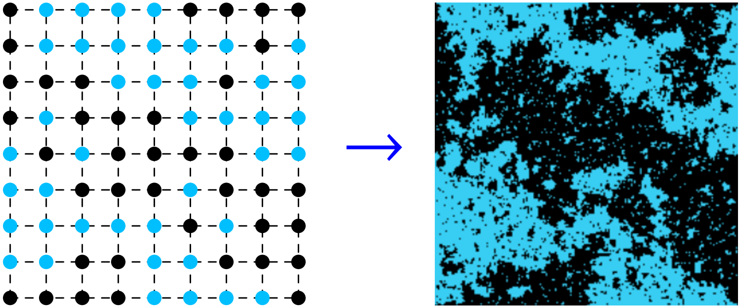
Joint work w/ Mingchang Liu & Hao Wu (Vienna / Tsinghua)

Conformal invariance "conjecture"

Any (reasonable) **critical lattice model** converges in the scaling limit to **a conformal field theory (CFT)**.

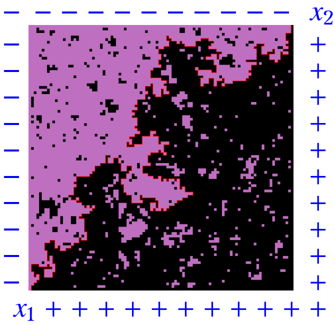


CFT: *conformally invariant* quantum field theory

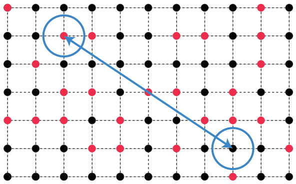
What is this supposed to mean?

Conformal invariance in terms of observables

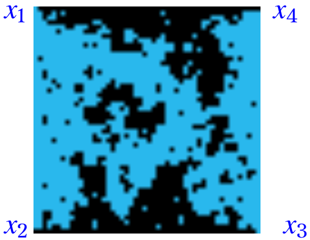
interfaces: random curves



$\mathbb{E}[\sigma_{x_1}\sigma_{x_2}]$



correlations (e.g. between spins)



probabilities of topological events

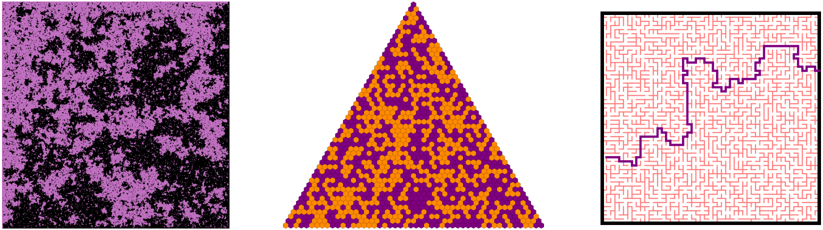
Scaling limit $\delta \to 0$

at critical temperature $T = T_c$

$\Rightarrow$ **conformal invariance?**

Lattice correlations — CFT correlation functions?

Convergence results use model-specific tools. Many open questions.

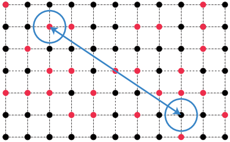


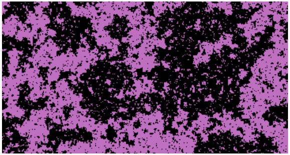
Some correlation functions have explicit formulas, but this is rare.

Problem: CFT not understood mathematically!

Lattice correlations — CFT correlation functions?

- Consider models with *continuous phase transition* at critical point.
- E.g. *Ising* $\mathbb{P}\left[(\sigma_x)_{x\in V}\right] \propto \exp\Big(\beta_c \sum_{x\sim y} \sigma_x \sigma_y\Big)$ for random spins $\sigma_x = \pm 1$, nearest neighbor interaction ($x \sim y$) on finite graph (V, E) on $\delta\mathbb{Z}^2$





Thm. (spins in Ising model) $\delta^{-n/8}\, \mathbb{E}\left[\sigma_{z_1} \cdots \sigma_{z_n}\right] \xrightarrow{\delta\to 0} F(z_1, \ldots, z_n)$

[Hongler, Smirnov '13 (energy); Chelkak, Hongler, Izyurov '15 (spin); CHI '21 (gen.)]

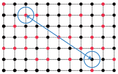
Proof relies on solutions of Riemann bdry value problems (discrete holomorphicity). Uses specific fermionic structures and techniques. □

Lattice correlations — CFT correlation functions?

- Certain correlations in critical models satisfy **(BPZ) PDEs**: e.g.

$$\left\{ \frac{8}{3} \frac{\partial^2}{\partial z_j^2} + \sum_{i \neq j} \left(\frac{2}{z_i - z_j} \frac{\partial}{\partial z_i} - \frac{1/8}{(z_i - z_j)^2} \right) \right\} F(z_1, \ldots, z_n) = 0 \qquad \forall\, 1 \leq j \leq n$$

for "Ising CFT" spin correlations ($\kappa = 3$)



[Izyurov '20]

$$F(z_1, \ldots, z_n) \;=\; \lim_{\delta \to 0} \; \delta^{-n/8} \, \mathbb{E}\,[\sigma_{z_1} \cdots \sigma_{z_n}]$$

- They also satisfy **conformal covariance** rule of primary fields:

$$F(f(z_1), \ldots, f(z_n)) = \Big(\prod_{j=1}^{n} \big| f'(z_j) \big|^{-1/8} \Big) \, F(z_1, \ldots, z_n) \qquad \forall\, f \text{ conformal map}$$

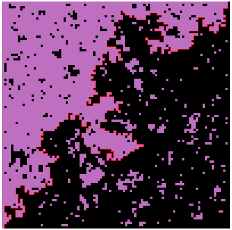
These are general features of so-called
CFT primary fields that are *degenerate* at level two.

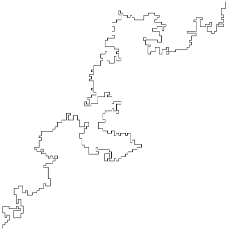
- Also: SLE(κ) partition functions, certain Liouville correlators, ...
- (Higher order (level) PDEs arise in *fusion*.)

Scaling limits of critical interfaces — SLE(κ) curves

- $\kappa > 0$ labels *universality class*
- convergence weakly for proba. measures on curves

(in some topology to be specified)





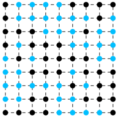
(critical) interface $\xrightarrow{\delta \to 0}$ Schramm-Loewner evolution, SLE(κ)

Usual proof strategy:

1. *tightness* (e.g. control via crossing estimates)
2. *identification* of the limit (e.g. via discrete holomorphic observable) □

Conformal invariance "conjecture" in SLE context

Boundary conditions:
"*disorder*" fields in CFT?



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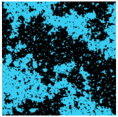


Any (reasonable) **critical lattice model** converges in the scaling limit to **a conformal field theory (CFT)**.

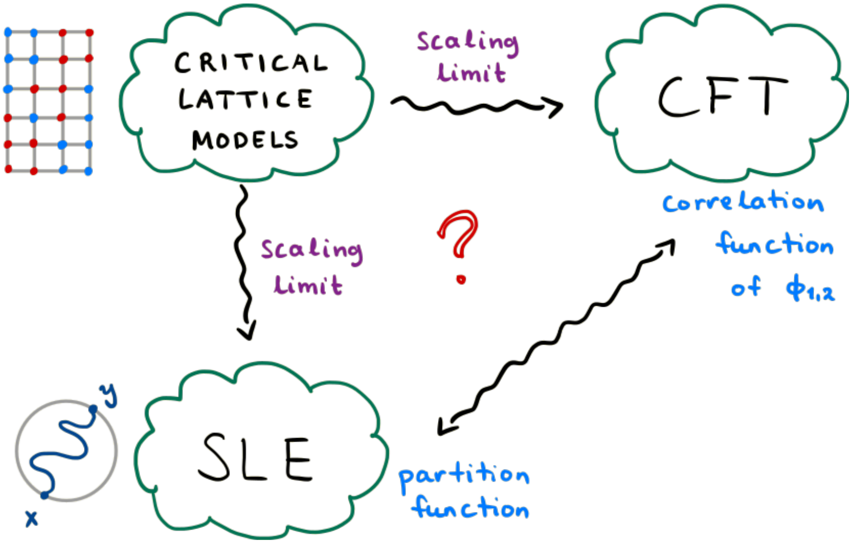
- [Cardy 84'; Bauer & Bernard '02]: SLE(κ) curve started at x should be generated by *certain CFT primary field* denoted $\phi_{1,2}(x)$.
- **Why?** Correlation functions give rise to (local) SLE(κ)-mgles:

$$M_t(x; z_1, \ldots, z_n) = \Big(\prod_{j=1}^{n} g_t'(z_j)^{\Delta_j} \Big) \, F(W_t; g_t(z_1), \ldots, g_t(z_n))$$

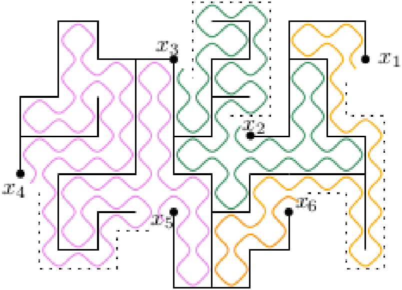
 has zero drift iff F solves BPZ PDE related to $\phi_{1,2}$.
- Generally, $s-1$ curves at x conjecturally related to $\phi_{1,s}(x)$ (fusion).

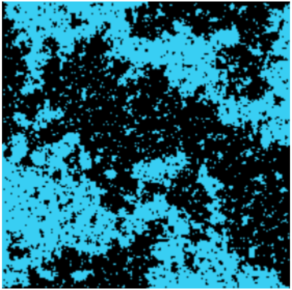
see Nam-Gyu's talk! [e.g. Duplantier & Saleur 87'; Bauer & Saleur '89]

Planar models



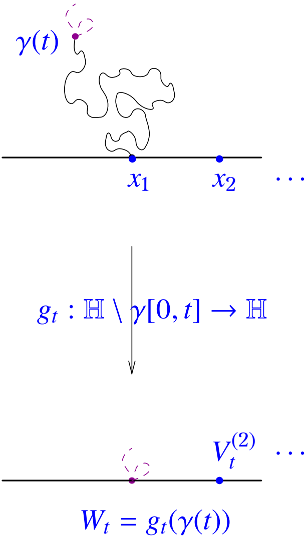
CROSSING PROBABILITIES GIVEN BY MULTIPLE $\mathrm{SLE}(\kappa)$ PURE PARTITION FUNCTIONS





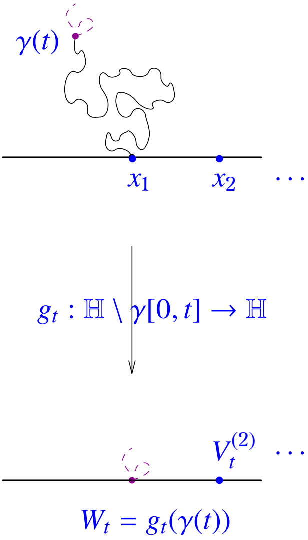
$$\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{Z}(W_t, V_t^{(2)}, V_t^{(3)}, \ldots, V_t^{(N)})\,\mathrm{d}t$$
$$\mathrm{d}V_t^{(i)} = \frac{2\,\mathrm{d}t}{V_t^{(i)} - W_t}$$

Growing multiple SLEs via Loewner equation



- re-sampling symmetry (*Dubédat's comm. relations*): can grow one curve at a time
  [Dubédat '06–'07]
- *driving process* of one curve γ: image of tip
- *interaction* encoded in **partition function** $\mathcal{Z}$
  - $\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{Z}(W_t, V_t^{(2)}, V_t^{(3)}, \ldots)\,\mathrm{d}t$
  - $\mathrm{d}V_t^{(i)} = \frac{2\,\mathrm{d}t}{V_t^{(i)} - W_t}$
  - $W_0 = x_1$, and $V_0^{(i)} = x_i$ for $i \neq 1$

Growing multiple SLEs via Loewner equation



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  - $\mathrm{d}V_t^{(i)} = \frac{2\,\mathrm{d}t}{V_t^{(i)} - W_t}$
  - $W_0 = x_1$, and $V_0^{(i)} = x_i$ for $i \neq 1$
- $\mathcal{Z}$ must satisfy system of (BPZ) PDEs $\forall j$

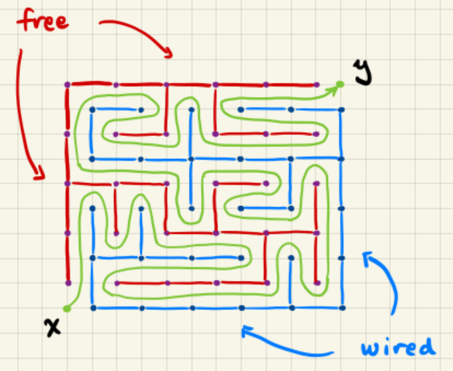
$$\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left(\frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{6/\kappa - 1}{(x_i - x_j)^2} \right) \right\} \mathcal{Z}(x_1, \ldots, x_{2N}) = 0$$

& Möbius covariance:

(for $f : \mathbb{H} \to \mathbb{H}$ s.t. $f(x_1) < \cdots < f(x_{2N})$)

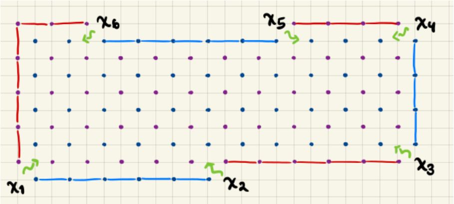
$$\mathcal{Z}(f(x_1), \ldots, f(x_{2N})) = \Big(\prod_{1 \leq j \leq 2N} |f'(x_j)|^{\frac{\kappa-6}{2\kappa}} \Big)\, \mathcal{Z}(x_1, \ldots, x_{2N})$$

Uniform spanning tree (UST) Peano curve – SLE(8)

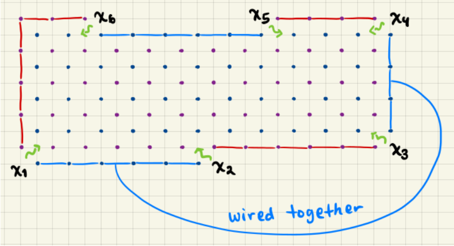


Thm. [Lawler & Schramm & Werner '04]
UST Peano curve on $(\Omega^{\delta,\diamond}; x^{\delta\diamond}, y^{\delta\diamond})$ converges weakly to SLE(8) in the scaling limit $\delta \to 0$ (in certain curve topology).

More general *boundary conditions* labeled by planar link patterns (planar pair partitions) $\beta \in \mathrm{LP}_N$:



$\beta : \{1, 2\}, \{3, 4\}, \{5, 6\}$



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Internal connectivities of Peano curves labeled similarly by $\alpha \in \mathrm{LP}_N$

Crossing probabilities for UST on $\Omega^\delta \subset \delta\mathbb{Z}^2$

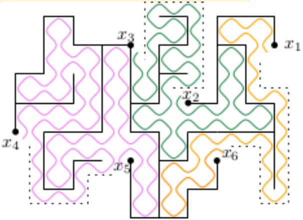
- $(\Omega^{\delta,\diamond}; x_1^{\delta\diamond}, \ldots, x_{2N}^{\delta\diamond}) \xrightarrow{\delta\to 0} (\Omega; x_1, \ldots, x_{2N})$ (assuming C^1-Jordan domain)

Thm. [Liu & P. & Wu '21] [arXiv:2108.04421]

Connectivities of UST Peano curves on $(\Omega^{\delta,\diamond}; x_1^{\delta\diamond}, \ldots, x_{2N}^{\delta\diamond})$ with b.c. β converge: for each possible $\alpha \in \mathrm{LP}_N$,

$$\lim_{\delta\to 0} \mathbb{P}_\beta^\delta[\text{ connectivity} = \alpha] = \frac{\mathcal{Z}_\alpha^{(\kappa=8)}(\Omega; x_1, \ldots, x_{2N})}{\mathcal{F}_\beta^{(\kappa=8)}(\Omega; x_1, \ldots, x_{2N})}.$$

- $\{\mathcal{Z}_\alpha^{(\kappa=8)} : \alpha \in \mathrm{LP}_N\}$ "pure partition functions"
- $\{\mathcal{F}_\beta^{(\kappa=8)} : \beta \in \mathrm{LP}_N\}$ partition functions for b.c.



also [Dubédat '07]

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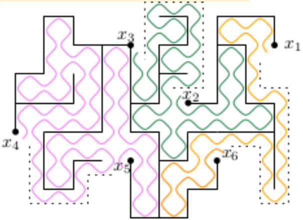
- $\{\mathcal{Z}_\alpha^{(\kappa=8)} : \alpha \in \mathrm{LP}_N\}$ "pure partition functions"
- $\{\mathcal{F}_\beta^{(\kappa=8)} : \beta \in \mathrm{LP}_N\}$ partition functions for b.c.

Main inputs to the proof:

- convergence of Peano curves to SLE(8) variants
- combinatorial formulas for $\mathbb{P}_\beta^\delta[\alpha]$ by Kenyon & Wilson
- martingale argument to identify with $\mathcal{Z}_\alpha/\mathcal{F}_\beta$



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UST Peano curves on $\Omega^\delta \subset \delta\mathbb{Z}^2$

Thm. [Liu & P. & Wu '21] (one curve [LSW04]; two curves: [HLW20])

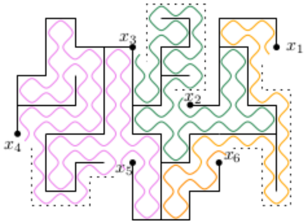
UST Peano curves on $(\Omega^{\delta,\diamond}; x_1^{\delta\diamond}, \ldots, x_{2N}^{\delta\diamond})$ with b.c. β

$\xrightarrow{\delta \to 0}$ SLE(8) with partition function $\mathcal{F}_\beta^{(\kappa=8)}$: e.g. curve starting at x_1

$$\mathrm{d}W_t = \sqrt{8}\,\mathrm{d}B_t + 8\,\partial_1 \log \mathcal{F}_\beta^{(\kappa=8)}(\Omega; W_t, V_t^2, V_t^3, \ldots, V_t^{2N})\,\mathrm{d}t, \qquad \mathrm{d}V_t^i = \frac{2\mathrm{d}t}{V_t^i - W_t}$$

"$\mathcal{F}_\beta^{(\kappa=8)} = \langle \Phi_{1,2}(x_1) \cdots \Phi_{1,2}(x_{2N}) \rangle_\beta$" lin. independent BCFT correlations

- BPZ PDEs, *simultaneous positivity*
- explicit *logarithmic* fusion rules!



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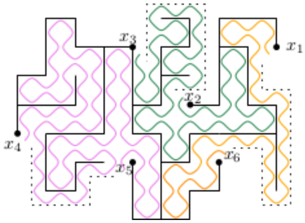
- BPZ PDEs, *simultaneous positivity*
- explicit *logarithmic* fusion rules!

Main inputs to the proof:

- discrete holomorphic multi-point observable
  ⇝ discrete bdry value problem
- solution for it related to $\mathcal{F}_\beta^{(\kappa=8)}$
- standard tightness arguments (cf. [LSW04, HLW20])



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UST Peano curves on $(\Omega^{\delta,\diamond}; x_1^{\delta\diamond}, \ldots, x_{2N}^{\delta\diamond})$

conditioned to form connectivity α (regardless of b.c.!)

$\xrightarrow{\delta\to 0}$ SLE(8) with partition function $\mathcal{Z}_\alpha^{(\kappa=8)}$: e.g. curve starting at x_1

$$dW_t = \sqrt{8}\, dB_t + 8\, \partial_1 \log \mathcal{Z}_\alpha^{(\kappa=8)}(\Omega; W_t, V_t^2, V_t^3, \ldots, V_t^{2N})\, dt, \qquad dV_t^i = \frac{2dt}{V_t^i - W_t}$$

"$\mathcal{Z}_\alpha^{(\kappa=8)} = \langle \Phi_{1,2}(x_1) \cdots \Phi_{1,2}(x_{2N}) \rangle_\alpha$" lin. independent BCFT correlations

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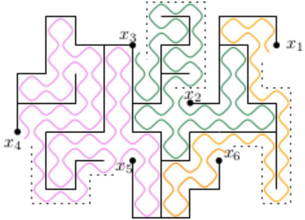

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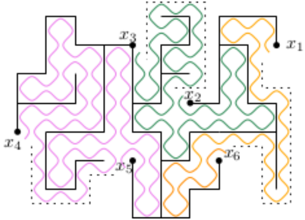
"$\mathcal{Z}_\alpha^{(\kappa=8)} = \langle \Phi_{1,2}(x_1) \cdots \Phi_{1,2}(x_{2N}) \rangle_\alpha$" lin. independent BCFT correlations

- BPZ PDEs, explicit *logarithmic* fusion

Main inputs to the proof:

- know all connection probabilities $\mathbb{P}_\beta[\alpha] = \mathcal{Z}_\alpha / \mathcal{F}_\beta$
- know limit of unconditioned Peano curves: p.f. $\mathcal{F}_\beta$
- thus: conditioned p.f. is $\mathbb{P}_\beta[\alpha]\mathcal{F}_\beta = \mathcal{Z}_\alpha$
- tightness as before



Partition functions for $\kappa = 8$

Construction of solutions in integral form (Coulomb gas formalism):

- $N = 2$: four-point function [Cardy's formula]

$$\mathcal{F}_{\frown\,\frown}(x_1, x_2, x_3, x_4) = \pi^2 \, (x_4 - x_1)^{1/4}(x_3 - x_2)^{1/4}\Big(\frac{x_{21}x_{43}}{x_{31}x_{42}}\Big)^{1/4} \, {}_2\mathrm{F}_1\Big(\frac{1}{2}, \frac{1}{2}, 1; \frac{x_{21}x_{43}}{x_{31}x_{42}}\Big)$$

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- general N: multi-point function

$$\mathcal{F}_\beta\,(x_1, \ldots, x_{2N}) = \prod_{1\le i<j\le 2N}(x_i - x_j)^{1/4}\int_{\Gamma_\beta}\prod_{1\le r\le N}\ \prod_{1\le j\le 2N}(w_r - x_j)^{-1/2}\prod_{1\le r<s\le N}(w_r - w_s)\,\mathrm{d}w_1\cdots\mathrm{d}w_N$$

where Γ_β are certain integration contours determined from β cf. [Dubédat '07]

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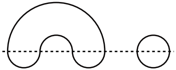
where Γ_β are certain integration contours determined from β cf. [Dubédat '07]

- pure partition functions: $\mathcal{Z}_\alpha := \sum_{\beta \in \mathrm{LP}_N} \mathcal{M}^{-1}_{\alpha,\beta} \, \mathcal{F}_\beta$

$(\alpha, \beta) =$  evaluates to $\mathcal{M}_{\alpha,\beta} := \begin{cases} 1 \text{ if } (\alpha,\beta) \text{ has 1 loop;} \\ 0 \text{ else} \end{cases}$

(meander matrices [DiFrancesco-Golinelli-Guitter 90s; Flores-Kleban-Simmons-Ziff '17])

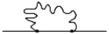
Logarithmic fusion for SLE(8) boundary field $\Phi_{1,2}$

Thm. [Han & Liu & Wu '20; Liu & P. & Wu '21]
Explicit fusion rules from SLE(8) *pure partition functions* $\mathcal{Z}_\alpha^{(\kappa=8)}$:

- "singlet channel"

$$\frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{|x_{j+1}-x_j|^{1/4}\log|x_{j+1}-x_j|}$$

$$\xrightarrow{x_j,x_{j+1}\to\xi} \quad \mathcal{Z}_{\alpha\setminus\{j,j+1\}}(x_1,\ldots,x_{j-1},x_{j+2},\ldots,x_{2N}) \qquad \text{if } \{j,j+1\}\in\alpha$$



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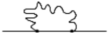
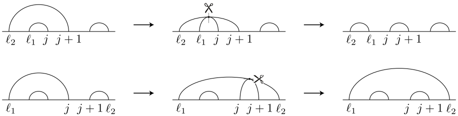


- "triplet channel" (NB: limit is **independent of** ξ)

$$\frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{|x_{j+1}-x_j|^{1/4}} \xrightarrow{x_j,x_{j+1}\to\xi} \pi\,\mathcal{Z}_{\wp(\alpha)\setminus\{j,j+1\}}(x_1,\ldots,x_{j-1},x_{j+2}\ldots,x_{2N}) \qquad \text{if } \{j,j+1\}\notin\alpha$$

where $\wp(\alpha)$ is obtained from α by "tying" the points j and $j+1$:



Logarithmic fusion for SLE(8) boundary field $\Phi_{1,2}$

Consequence. [Liu & P. & Wu '21]

For *any* CFT boundary fields describing SLE(8) curves, OPE product has *explicit form*

$$\Phi_{1,2}(z)\,\Phi_{1,2}(w) \;\sim\; (z-w)^{-1/4}\left(\pi\,\Phi_{1,1}(z) - \log(z-w)\,\tilde{\Phi}_{1,3}(z)\right).$$

Compare with fusion of two simple Virasoro modules with $c = -2$:

- for simple module $S_{1,2}$ (corresponding to $\Phi_{1,2}$ with $\kappa = 8$):

$$0 \longrightarrow S_{1,1} \xrightarrow{\iota} S_{1,2} \boxtimes S_{1,2} \xrightarrow{\pi} S_{1,3} \longrightarrow 0$$

  where $S_{1,2} \boxtimes S_{1,2}$ is so-called *staggered module* (not semisimple)
  - $S_{1,1}$ corresponding to $\Phi_{1,1}$ and $S_{1,3}$ corresponding to its "log-partner" $\tilde{\Phi}_{1,3}$

  [Gurarie '93; Gaberdiel & Kausch '96; Rohsiepe '96; Kytölä & Ridout '09]
- also agree with bdry-arm exponents for SLE(8) [Wu & Zhan '17]

Heuristics: Fusion and OPE in CFT

- "multiplication of fields" given by *operator product expansion* (OPE)
- e.g. for primary fields $\Phi_{1,2}$,

$$\text{“}\ \Phi_{1,2}(z)\,\Phi_{1,2}(w) \sim \frac{c_1}{(w-z)^{\Delta_{1,1}}}\,\Phi_{1,1}(w) + \frac{c_2}{(w-z)^{\Delta_{1,3}}}\,\Phi_{1,3}(w)\ \text{”}, \qquad \text{as } |z-w| \to 0,$$

  - $c_1, c_2 \in \mathbb{C}$ *structure constants*
  - $\Delta_{1,1} = 2h_{1,2}(\kappa) - h_{1,1}(\kappa) = \frac{6-\kappa}{\kappa}$ and $\Delta_{1,3} = 2h_{1,2}(\kappa) - h_{1,3}(\kappa) = -\frac{2}{\kappa}$

Works well for generic $\kappa \notin \mathbb{Q}$. However:

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Works well for generic $\kappa \notin \mathbb{Q}$. However:

- anomaly: $\kappa = 8 \implies \Delta_{1,1} = \Delta_{1,3}$

$\rightsquigarrow$ should use less restrictive form

$$\text{“}\ \Phi_{1,2}(z)\,\Phi_{1,2}(w) \ \sim\ c_1(z,w)\,\Phi_{1,1}(w) \ +\ c_2(z,w)\,\Phi_{1,3}(w)\ \text{”}, \qquad \text{as } |z-w| \to 0,$$

for some functions $c_1(z_1, z_2)$ and $c_2(z_1, z_2)$ *allowing logarithms*

Heuristics: logarithmic fields

- *Virasoro algebra* $\mathfrak{Vir}$: Lie algebra generated by $(L_n)_{n\in\mathbb{N}}$ and C

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12}n(n^2-1)\,\delta_{n+m,0}, \qquad [C, L_n] = 0$$

- *Verma module* $\mathrm{V}_{h,c} = \mathfrak{Vir}.v$ universal highest weight module

$$L_0 v = hv, \qquad L_n v = 0 \text{ for } n \geq 1, \qquad C.v = c\,v,$$

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- e.g. in $0 \longrightarrow \mathrm{S}_{1,1} \xrightarrow{\iota} \mathrm{S}_{1,2} \boxtimes \mathrm{S}_{1,2} \xrightarrow{\pi} \mathrm{S}_{1,3} \longrightarrow 0$ we have

$$L_0 v_{1,3} = v_{1,1}, \qquad L_n v_{1,3} = 0 \text{ for } n \geq 1$$

$\Longrightarrow$ $\{v_{1,1}, v_{1,3}\}$ form *Jordan cell* for $L_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ in $\mathrm{S}_{1,2} \boxtimes \mathrm{S}_{1,2}$

Comments and questions

History:

- Dubédat '06–'07: Euler integrals & Commuting SLEs
- Kenyon & Wilson '11: combinatorial method to calculate UST crossing probabilities in the discrete, while it doesn't relate to CFT.
- Predictions for crossing formulas in the physics literature.

[Cardy 80s, Flores-Kleban-Simmons-Ziff '17]

What's new?

- Proba construction of bdry correlations with any number points.
- Explicit structure constants and (log!) fusion rules for primaries.
- In contrast to Liouville or unitary minimal models, CFT with $c = -2$ is non-unitary (so Osterwalder-Schrader axioms will fail).
- **Q**: UST natural for imaginary geometry. [Miller-Sheffield '12] Understand the boundary "fields" from that framework?
- **NB**: **Found correlation functions are explicit!**


Thank you!