Critical Liouville quantum gravity and Brownian half-plane excursions

Analysis and geometry of random spaces workshop, MSRI (March 2022)

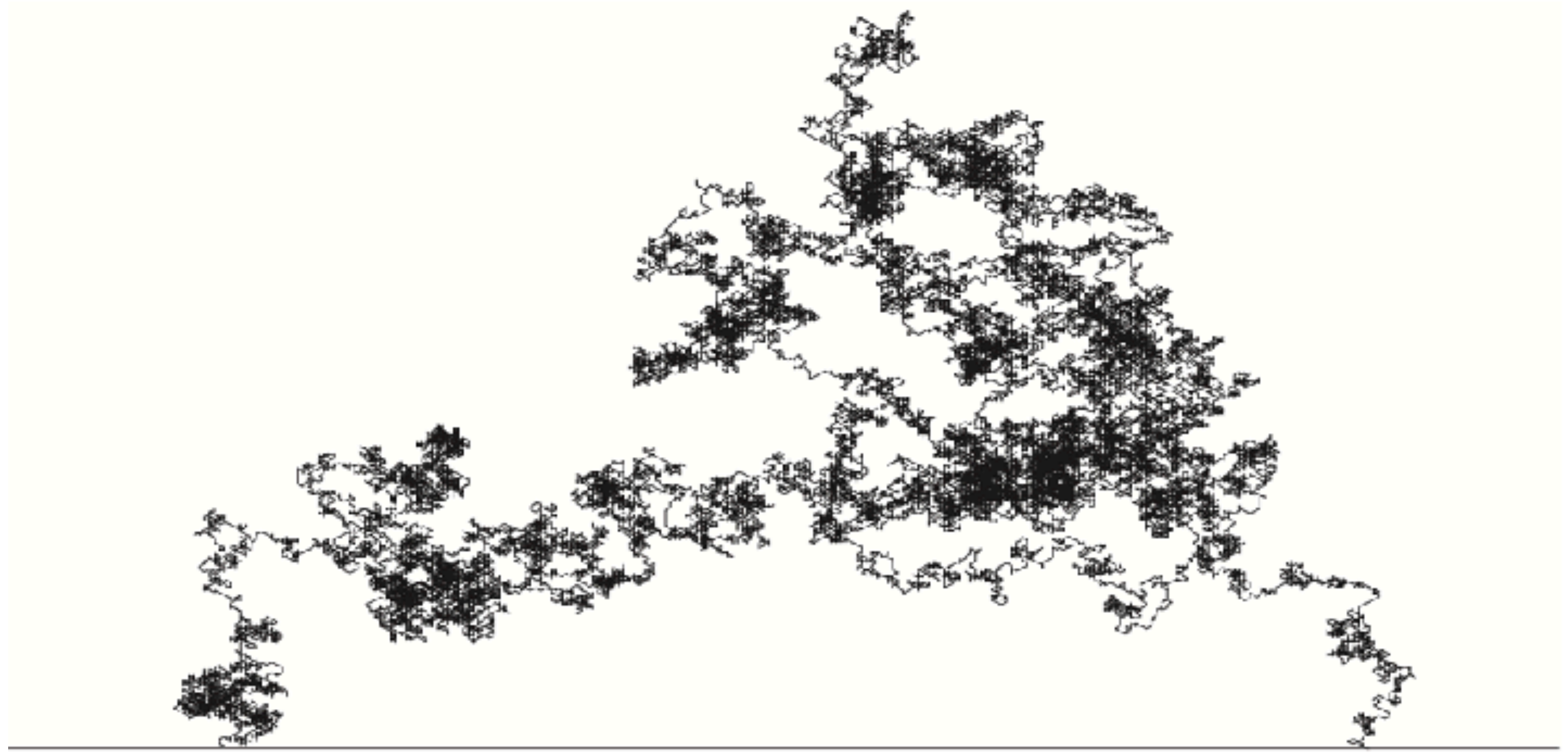
Ellen Powell, Durham University. Based on joint work with Juhan Aru, Nina Holden & Xin Sun

Aim of the talk

Critical Mating of Trees?

Correspondence between the following objects:

Brownian half-plane excursion



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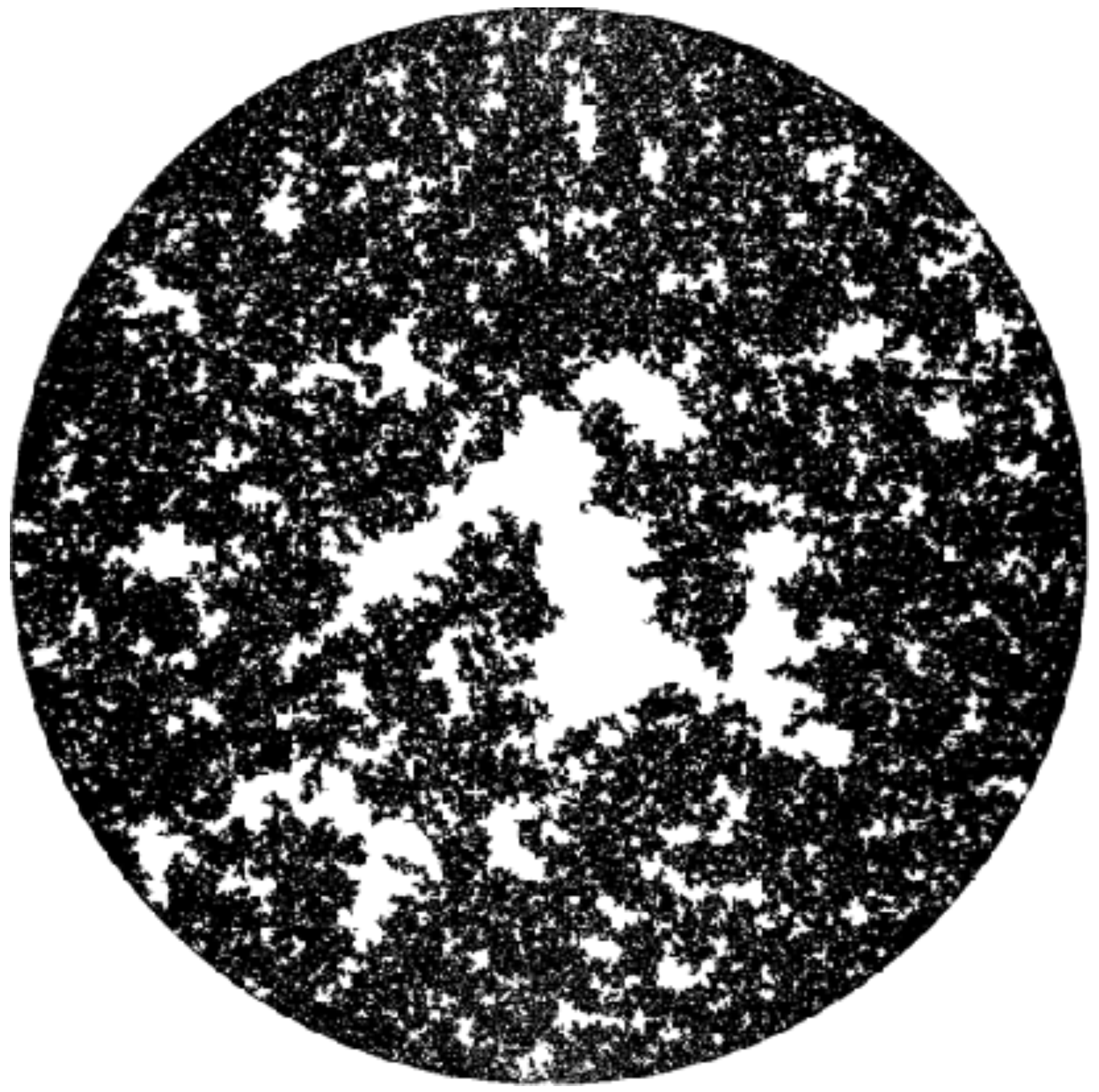
CLE_4



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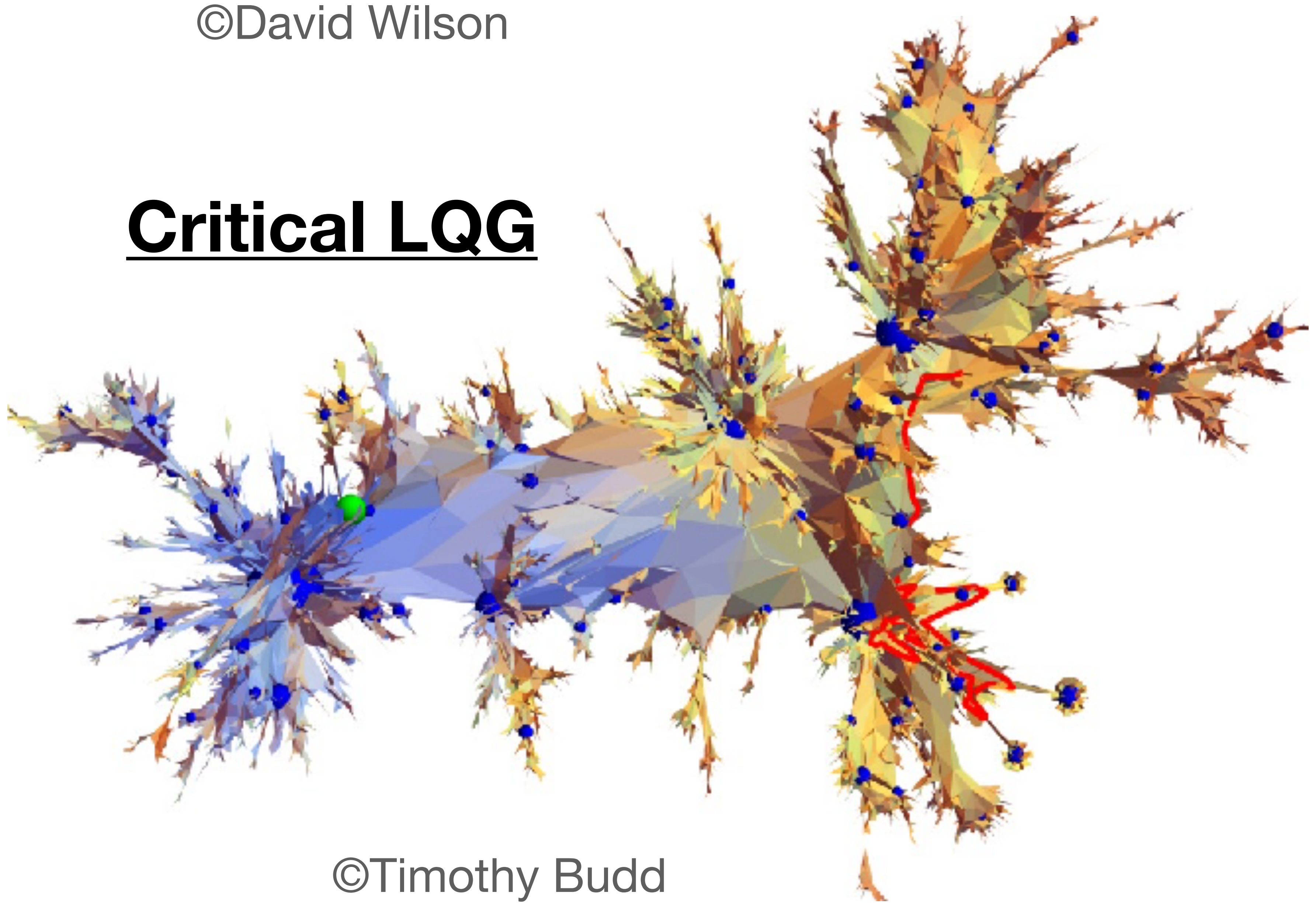
Critical LQG



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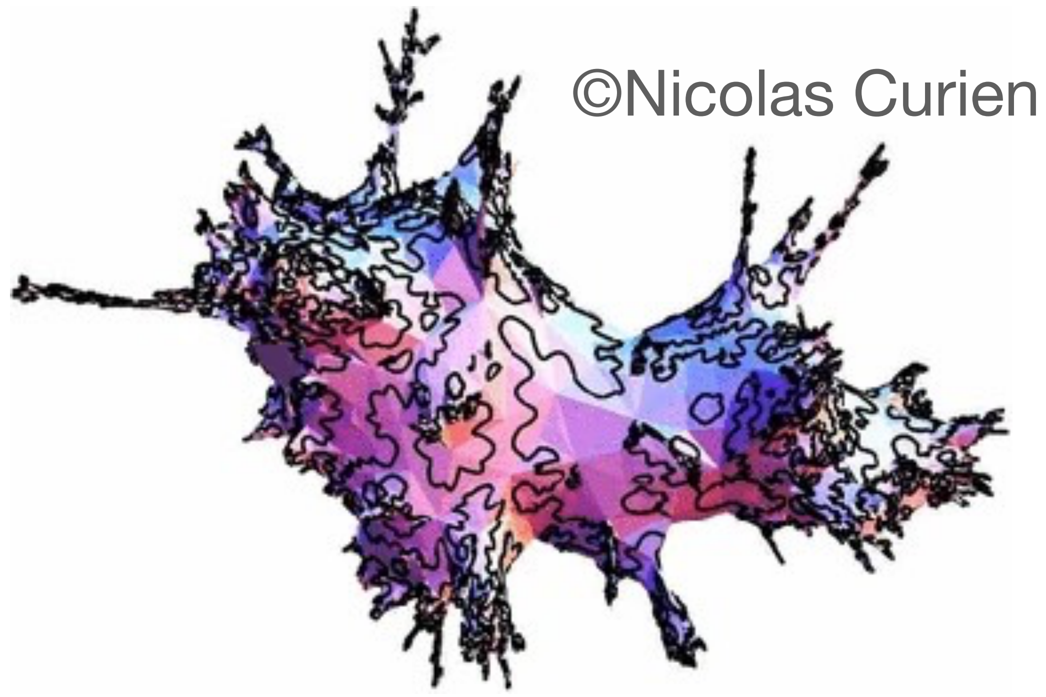
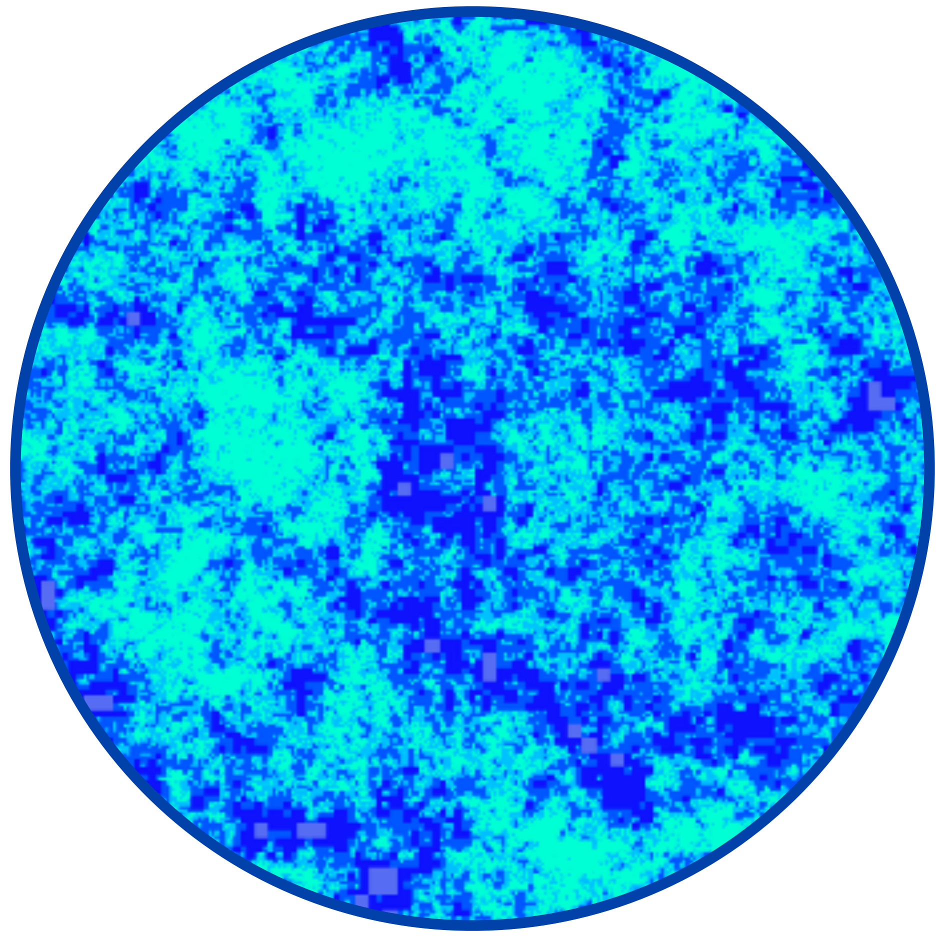
Duplantier-Miller-Sheffield's mating of trees



Encodes **SLE/loop** decorated γ-**quantum surfaces** by a **correlated planar Brownian motion:** $\gamma \in (0,2)$

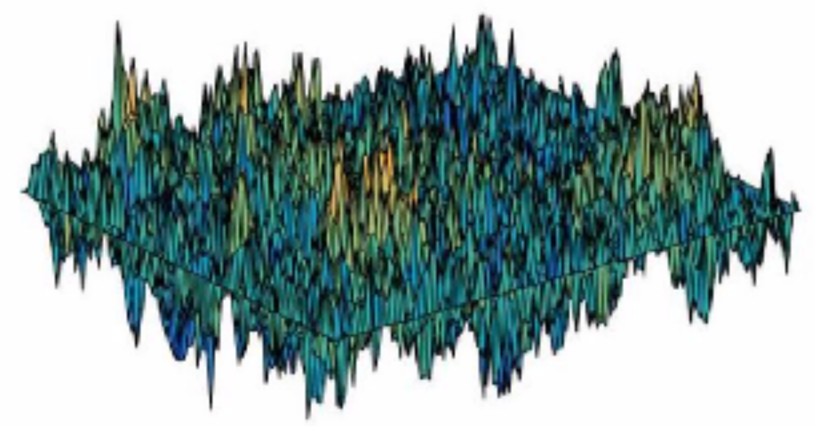
Disc version (D-M-S, Ang-Gwynne)



Unit boundary length γ-LQG disc

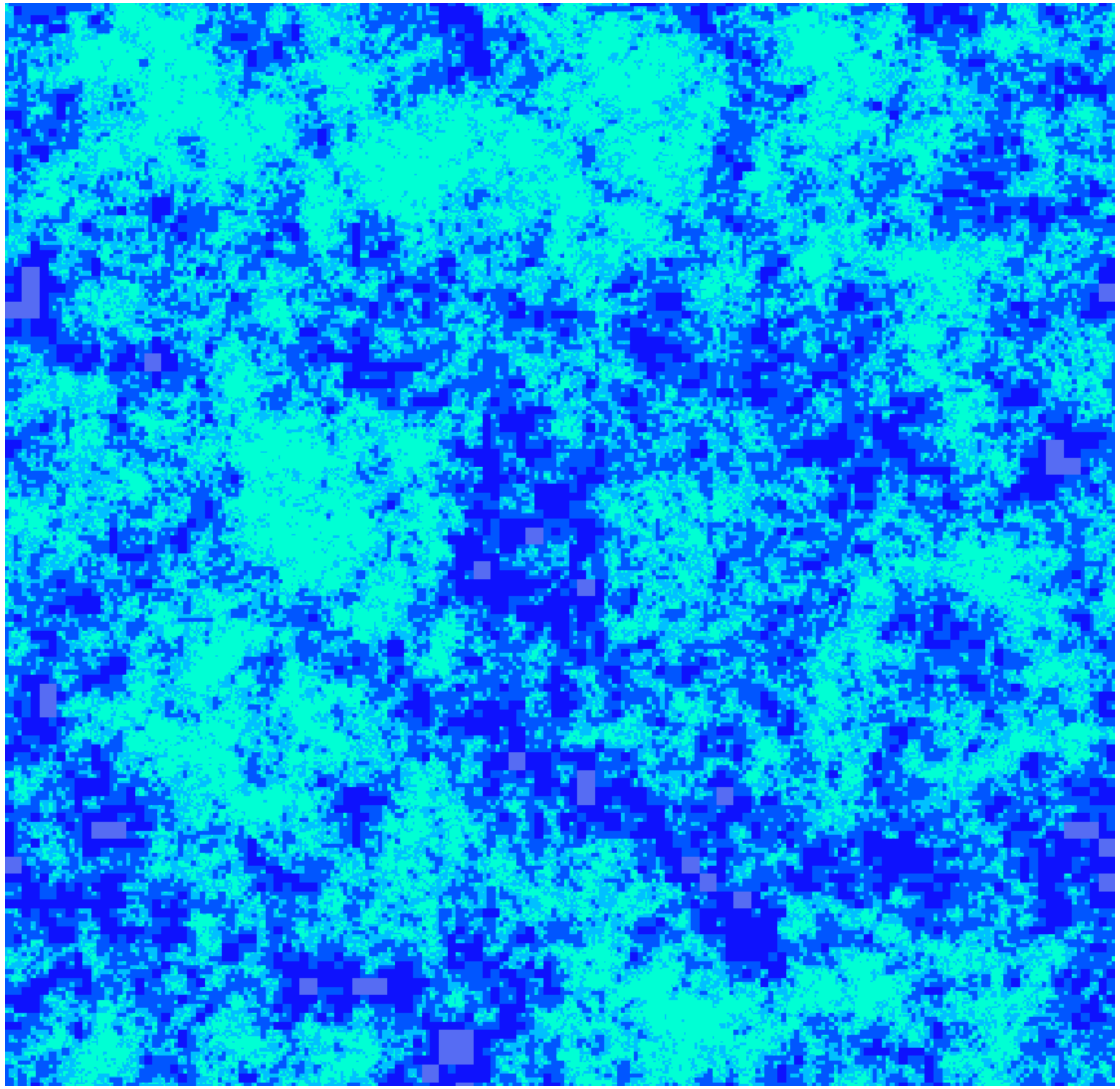
Quantum gravity discs

- **LQG disc** = **random measure** on unit disc of $\mathbb{C}$
- **Gaussian multiplicative chaos measure** of variant of Gaussian free field **(GFF)**

$h =$ 

$$\mu_\gamma(dx) \text{ "=" } \exp(\gamma h(x))\, dx$$

- Can measure **quantum areas** of regions and **quantum lengths** of curves
- **Unit boundary length disc,** boundary of the disc has **quantum boundary length = 1**
- (Conjectured/proven) scaling limit of **random planar map** models (Gwynne, Holden, Miller, Sheffield, Sun…)
- $\gamma \in (0,2) \leftrightarrow$ discrete model, $\gamma = 2$ is **critical**



The approximate density of a Liouville quantum gravity measure
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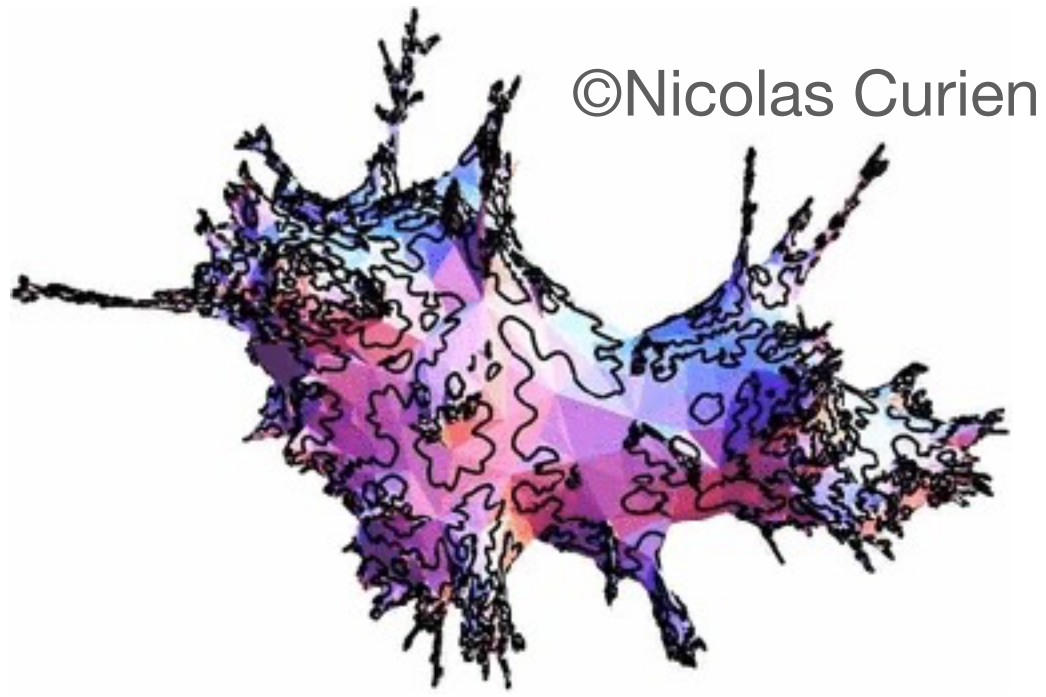
Duplantier-Miller-Sheffield's mating of trees



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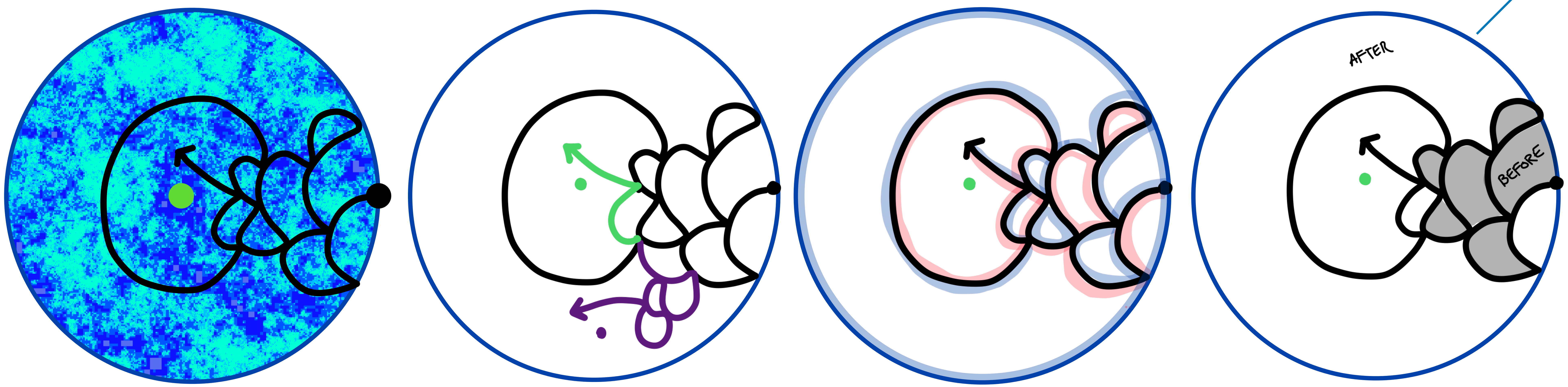
Encodes **SLE/loop** decorated γ-**quantum surfaces** by a **correlated planar Brownian motion:** $\gamma \in (0,2)$

Disc version (D-M-S, Ang-Gwynne)

Ordering on points $\rightsquigarrow$ space-filling SLE_κ



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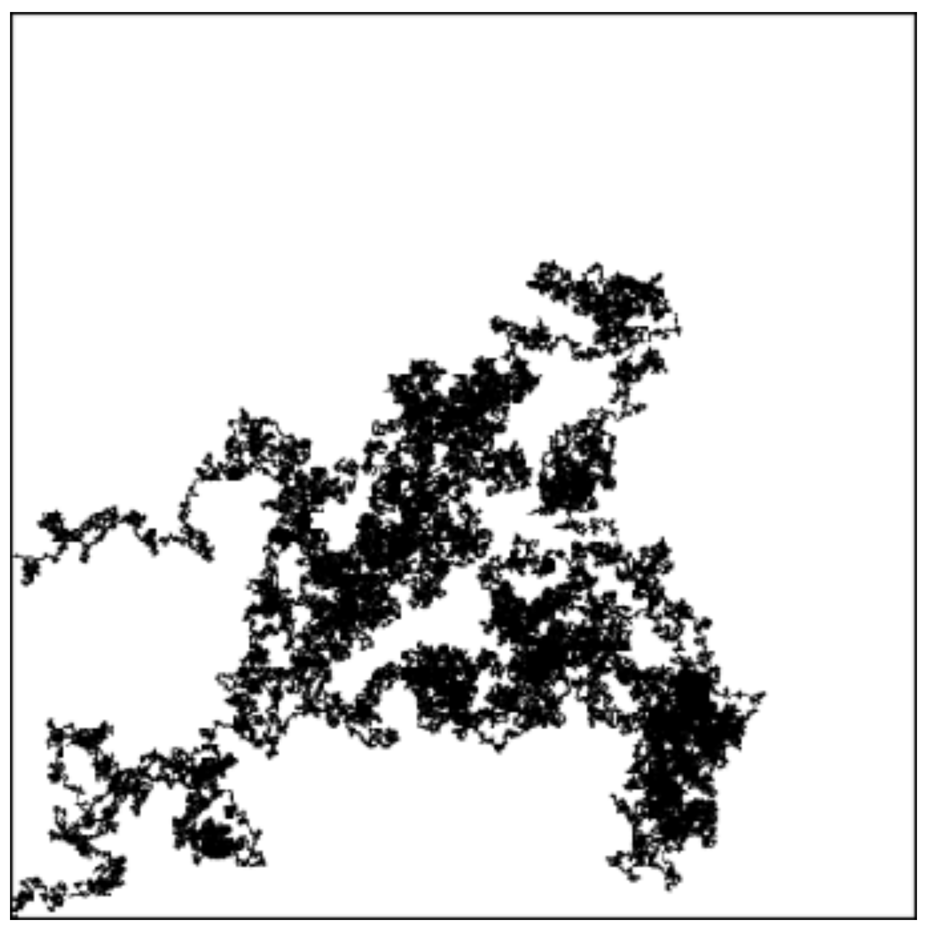
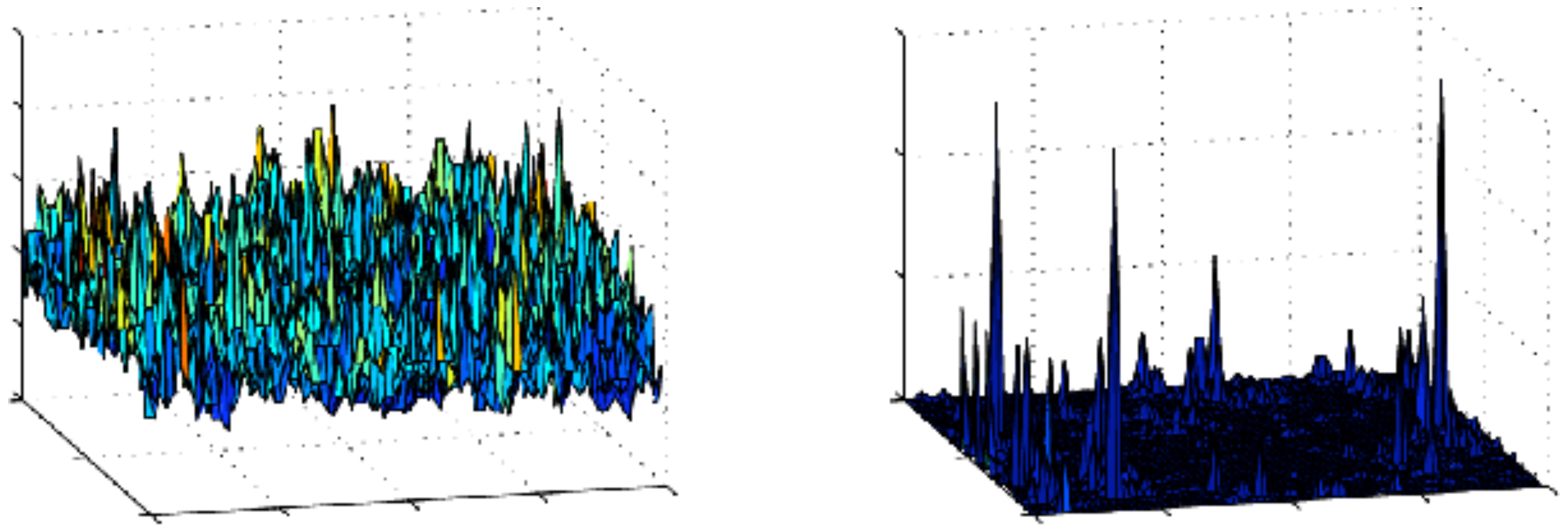
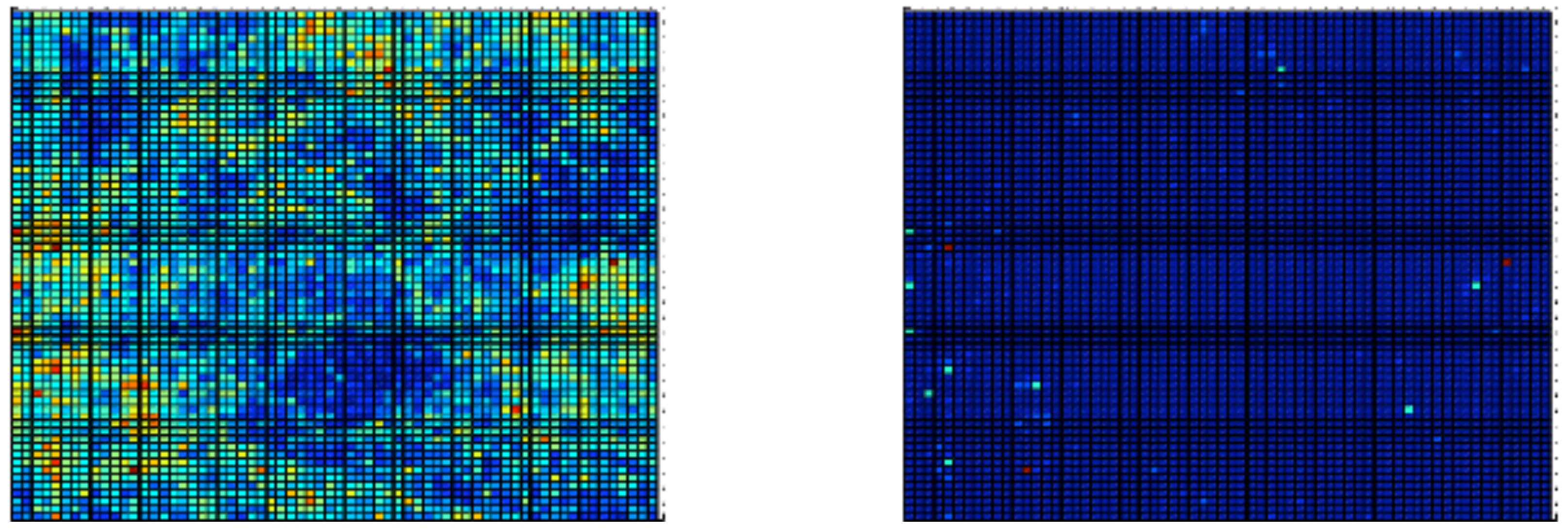
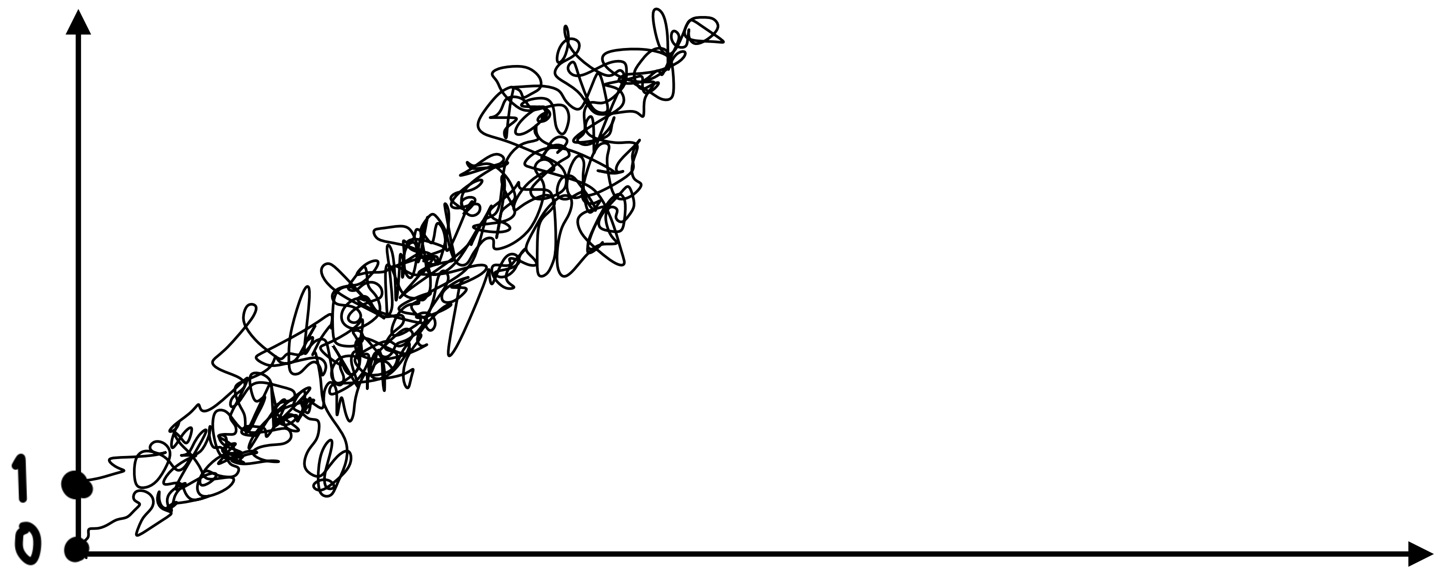


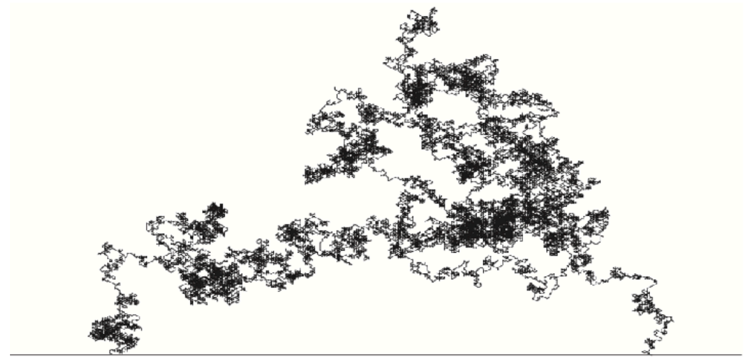
Unit boundary length γ-LQG disc
$\kappa = 16/\gamma^2 > 4$

+ independent branching $\mathrm{SLE}(\kappa; \kappa - 6)$

$\leftrightarrow$ correlated Brownian "cone excursion"

What happens as $\gamma \uparrow 2, \kappa \downarrow 4$?

LQG disc	Branching/space-filling SLE	Correlated Brownian cone excursion
 ©Remi Rhodes/Vincent Vargas  Field has to be huge to give unit boundary length	• Value of $\rho = \kappa - 6$ in the $\mathrm{SLE}(\kappa;\rho)$ approaches -2 • Curves start spinning round the boundary very fast… • Continuous space-filling curve degenerates	Correlation $= -\cos(\pi\gamma^2/4) \to 1$ Variance $= \left((2-\gamma)\sin(\pi\gamma^2/4)\right)^{-1} \to \infty$ 

LQG disc	Branching/space-filling SLE	Correlated Brownian cone excursion
• Take a $(2-\gamma)$-boundary length γ-quantum disc • **Rescale** boundary lengths and areas by $(2-\gamma)^{-1}$ • Field + GMC measures converge to **unit boundary length critical LQG disc**	• Branching $\mathrm{SLE}(\kappa; \kappa-6)$ converges to the **branching uniform CLE_4 exploration** of Werner and Wu • Order converges to a sequence of **independent coin tosses** indexed by the branch points of the exploration tree.	• Take Brownian excursion for new discs with rescaled QA and QBL • **Shear** transformation so components become **independent** • Limit as $\gamma \uparrow 2$ is a **Brownian half-plane excursion** 

Theorem (Aru-Holden-P.-Sun ‘21)

Limits exists simultaneously in law when objects are coupled as in mating of trees correspondence

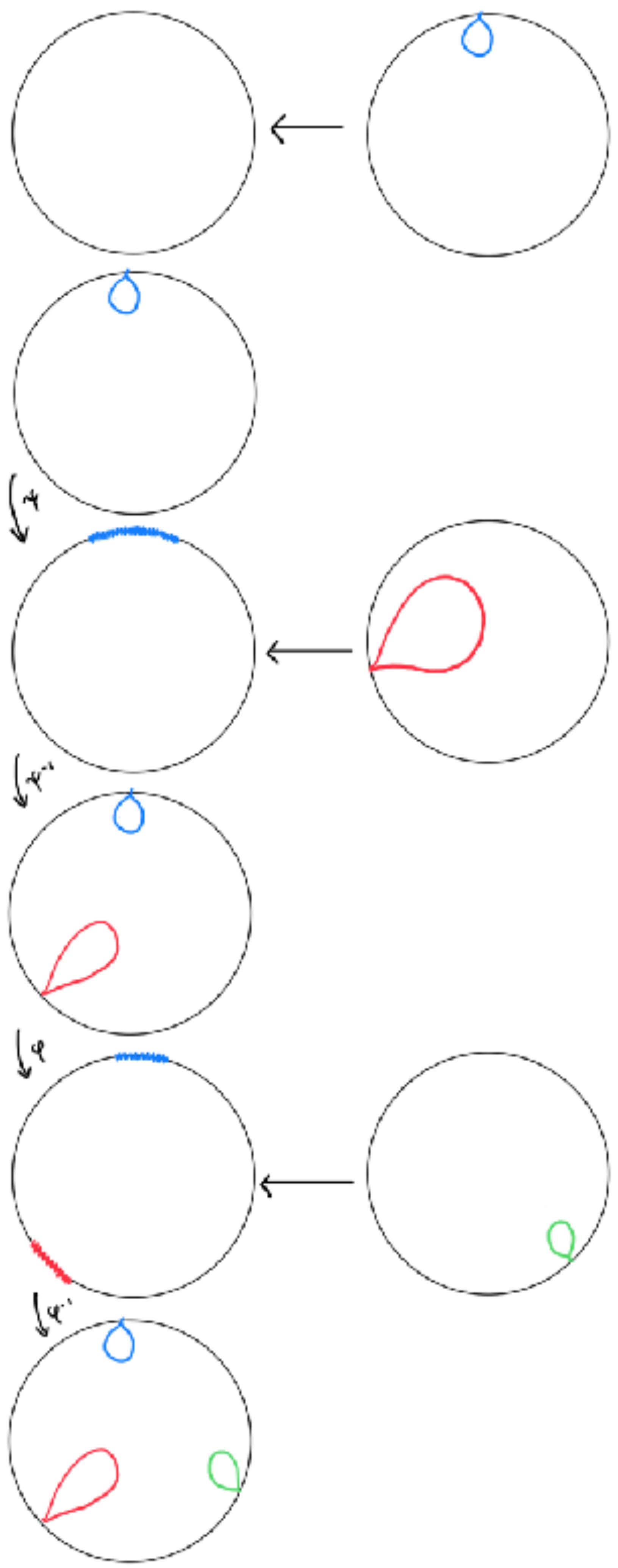
Limit gives coupling between critical LQG disc + branching uniform CLE_4 exploration and Brownian half plane excursion

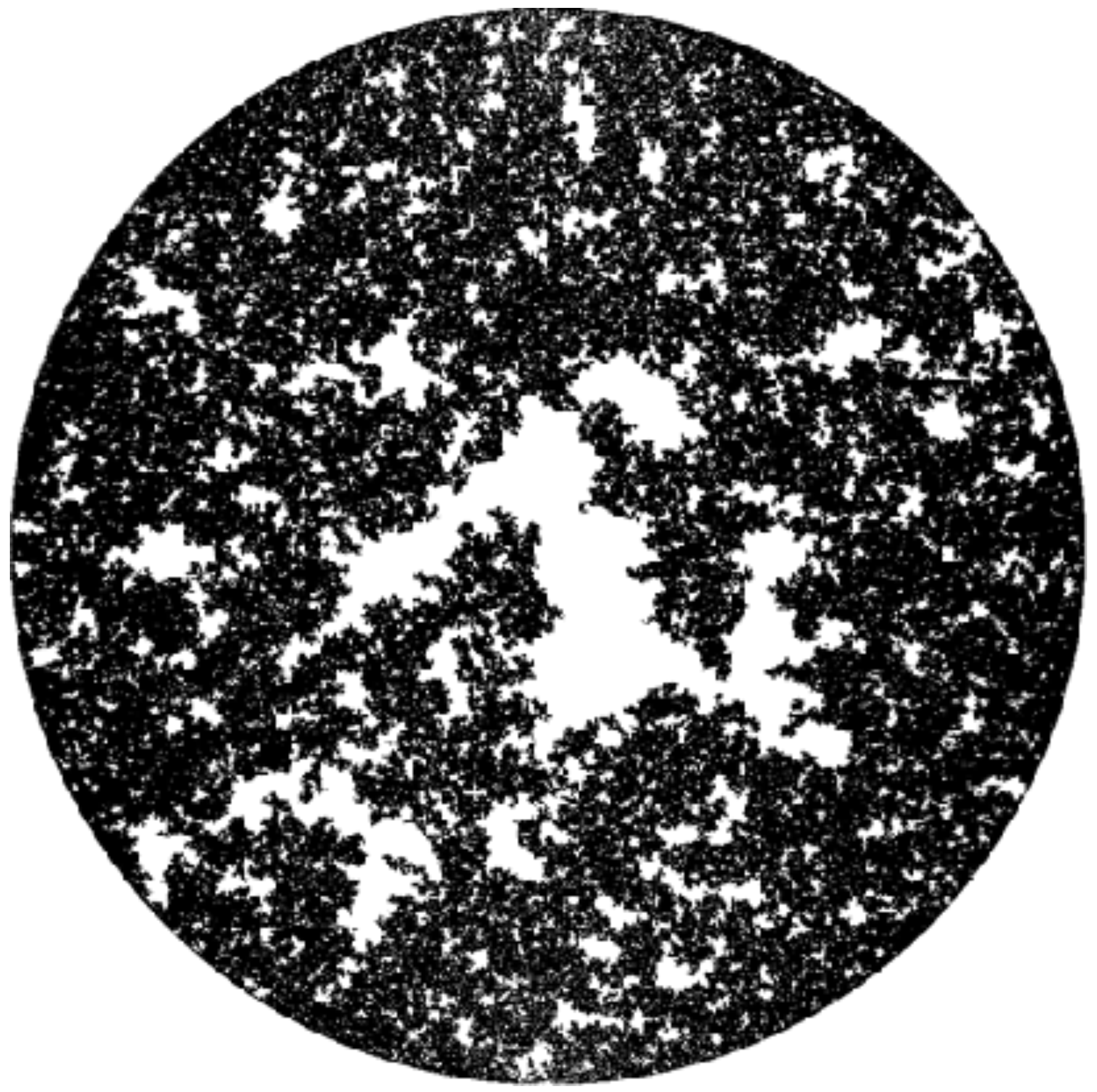
Uniform CLE_4 exploration

Werner & Wu

Uniform Exploration

- Construct each branch from a **PPP of SLE_4 bubbles** rooted **uniformly** only the boundary of the unit disk
- Add them in to the unit disk chronologically





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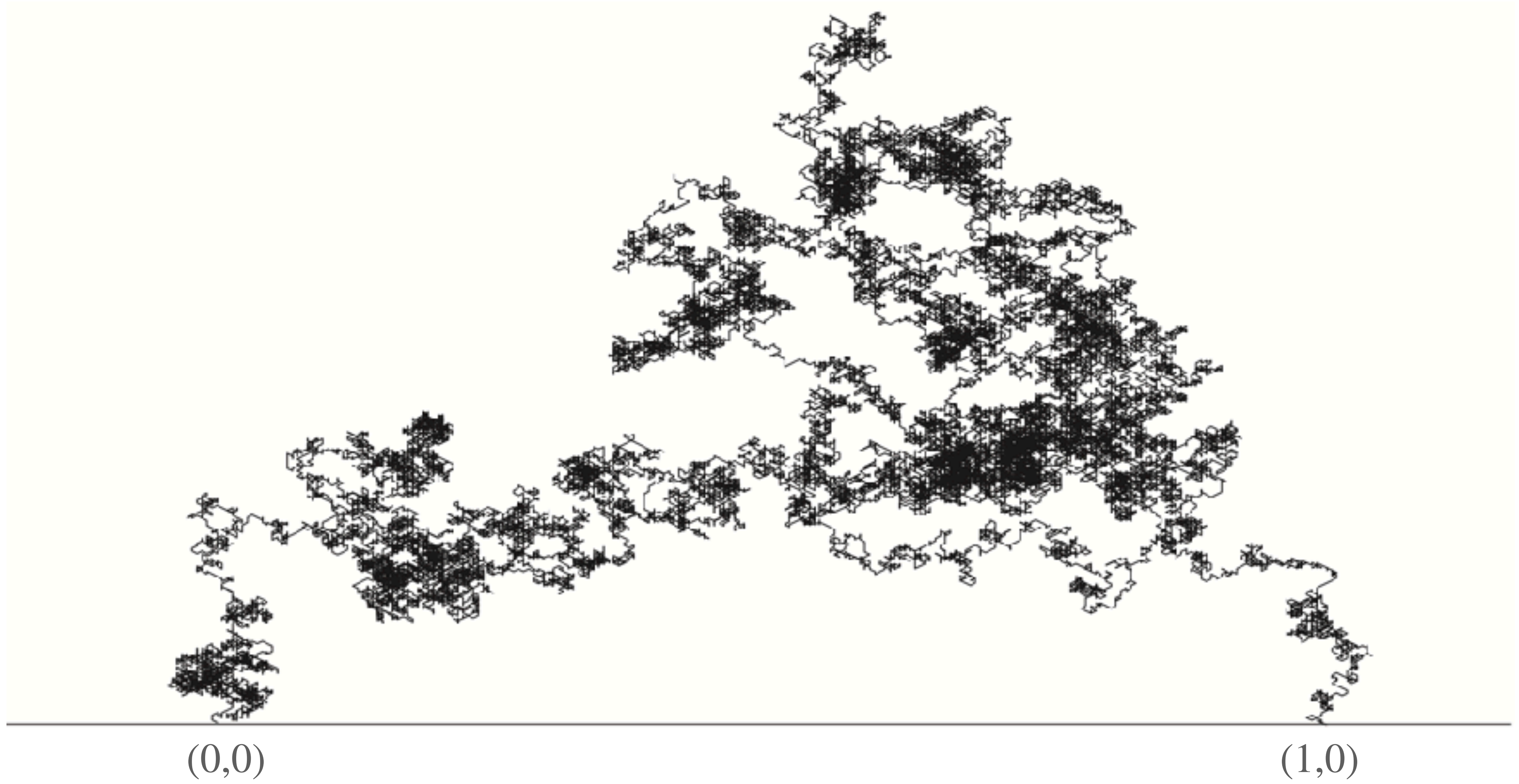
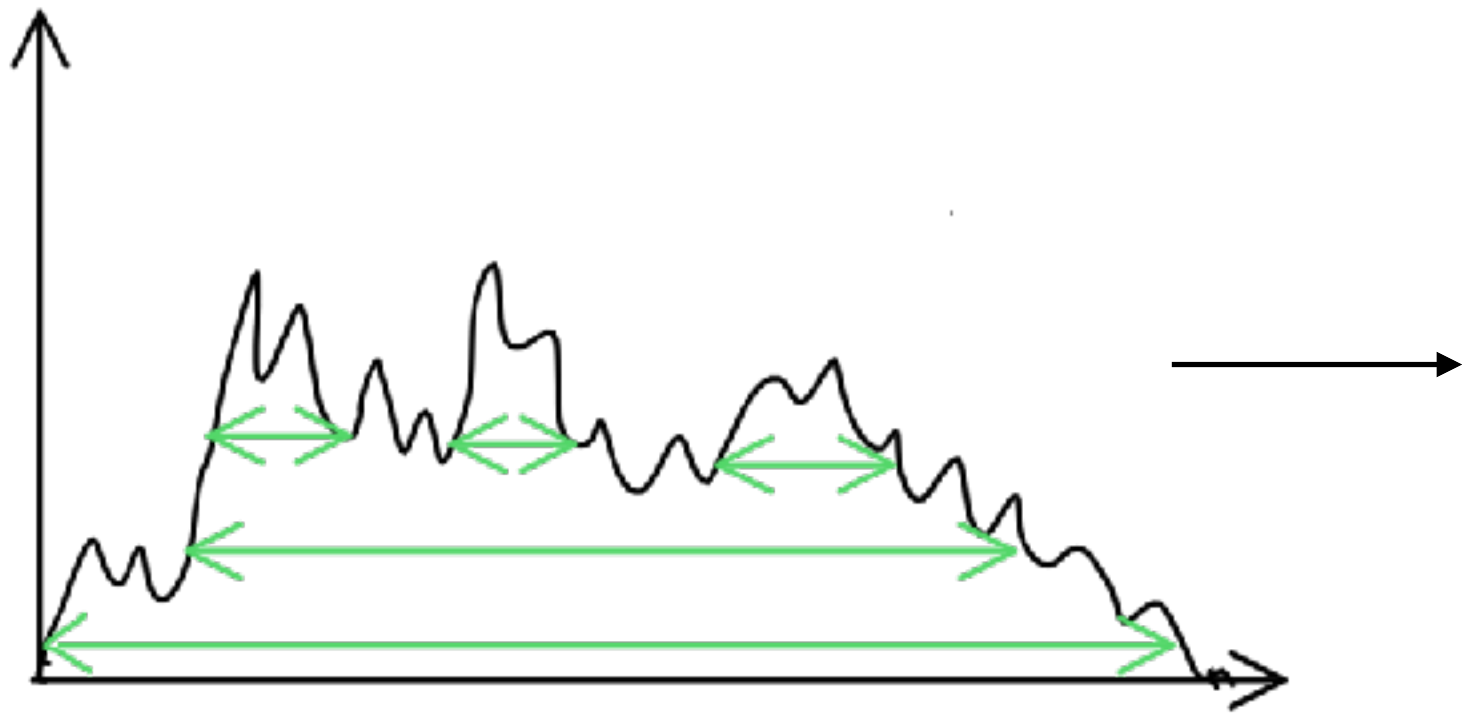
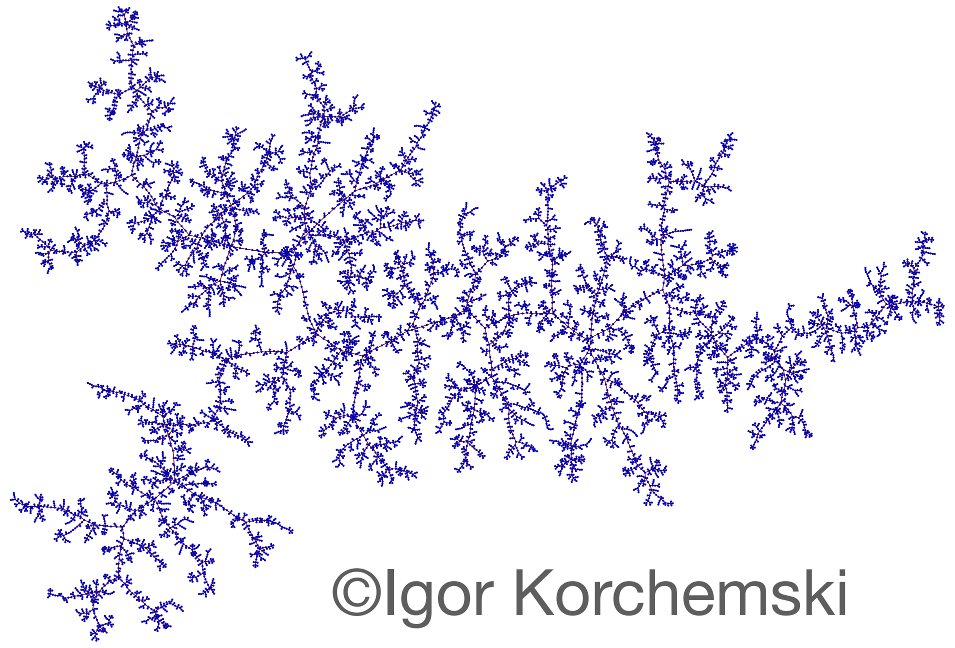
Branching Version

- Target invariance property means can couple towards dense set of target points
- Also decide **order** by flipping a coin when two points are disconnected

Brownian half-plane excursions



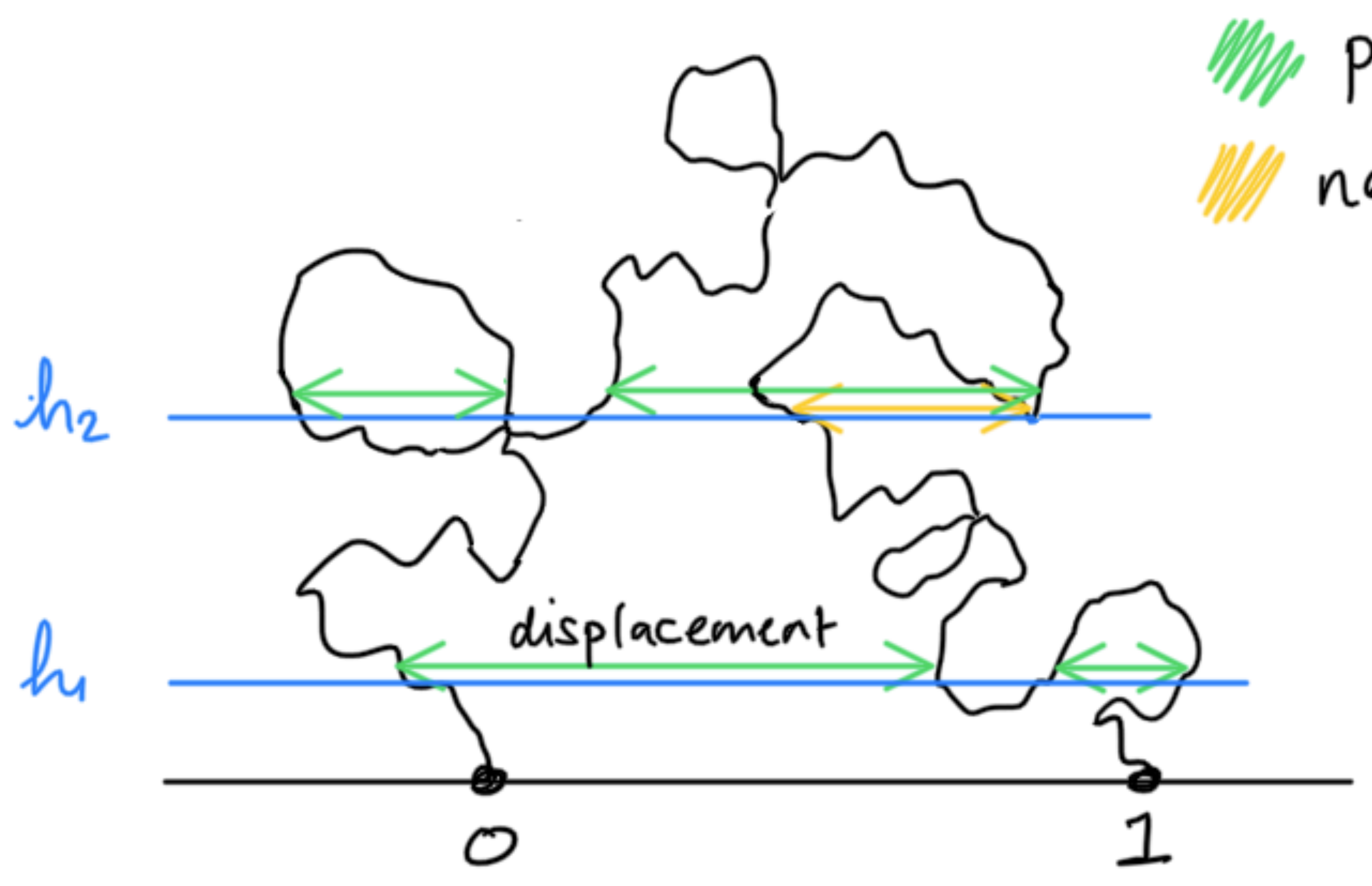




- Planar Brownian motion started at (1,0) and conditioned to exit the upper half plane at the origin
- Given its duration (specified law), the x coordinate is a Brownian bridge and the y coordinate is an independent Brownian excursion from $0 \mapsto 0$
- The y coordinate defines a Brownian CRT (and therefore a **branching** structure)
- In fact, there is an embedded **growth fragmentation**

Brownian half-plane excursions

Growth fragmentations and Cauchy processes (Aïdekon & Da Silva)



- At each height $h \geq 0$ have collection of "displacements" of sub-excursions above h
- These have **signs**
- Built from an **explicit** self-similar Markov process with index 1, starting at 1, and with **positive and negative jumps**
- For each jump of this Markov process, introduce a new cell of mass = signed size of jump. Then start a new copy of the process/minus the process starting from this value.

Correspondence
(= Main Theorem)

There exists a coupling so that the following correspondences hold...

CLE_4 decorated critical LQG disk	Brownian half-plane excursion
Branching structure defined by exploration	Branching structure in the associated CRT
Order defined by independent coin tosses	Order defined by chronology
Boundary lengths of discovered disks	Displacements of sub-excursions above heights
Areas of discovered disk	Durations of sub-excursions above heights
Parity of nesting	Sign of subexcursion
Some notion of “quantum” distance from boundary	Height

... and in this coupling, the Brownian excursion is a deterministic function of the CLE (+exploration) + LQG + coin tosses

Open Questions

- To what extent does the Brownian excursion encode the whole CLE decorated LQG picture?
- Is there a link between our “distance” from the boundary (encoded by height) and the conformally invariant metric on CLE_4 of Sheffield, Wu, Watson.
- Can one obtain any rigorous scaling limit results?

Thanks!