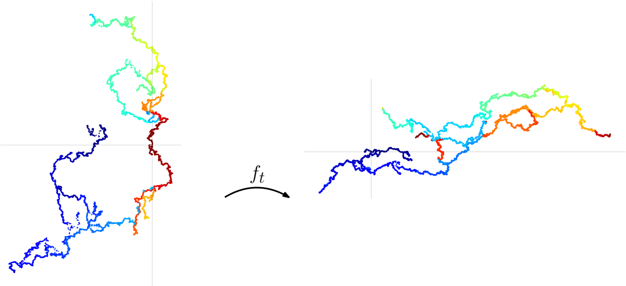
Loewner evolution driven by complex Brownian motion

Ewain Gwynne
(based on joint work with Josh Pfeffer, simulations by Minjae Park)

University of Chicago



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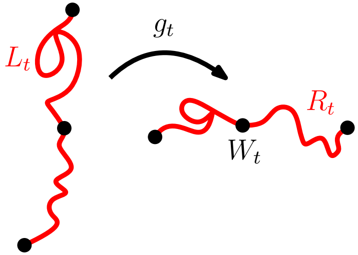
- Can we extend SLE/LQG relationships to complex parameter values?

Outline

1. Loewner evolution with complex driving function
2. Loewner evolution driven by complex Brownian motion
3. Liouville quantum gravity with complex parameters
4. Open problems

Loewner evolution with complex driving function



- (chordal) Loewner equation with continuous driving function $W : [0, \infty) \to \mathbb{C}$:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \; g_0(z) = z,$$

for each $z \in \mathbb{C}$.

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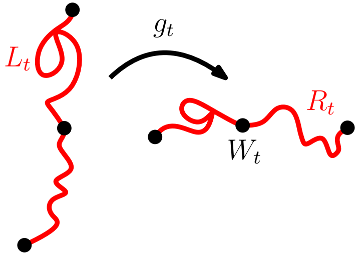

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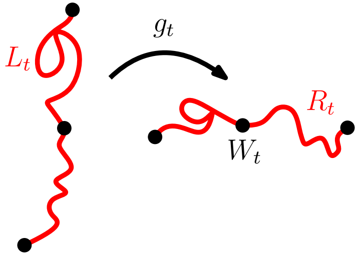

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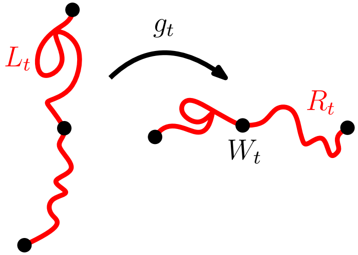

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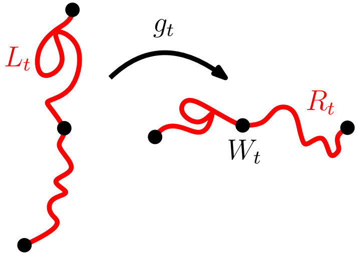

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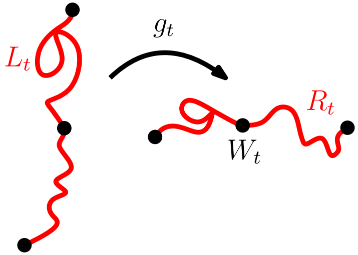

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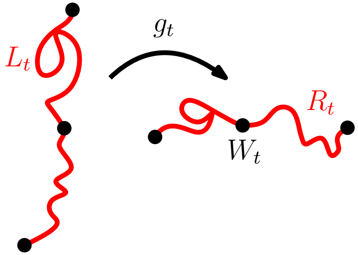

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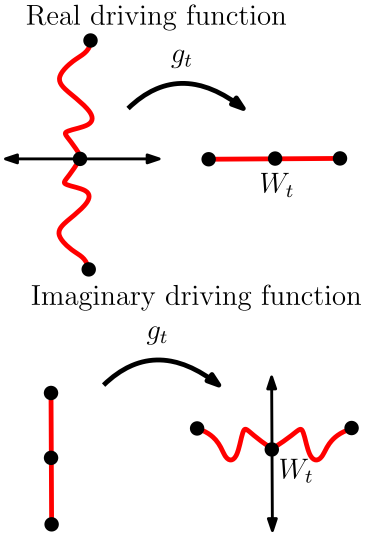
- Previously studied by Rohde-Schramm (unpublished), Tran (2017), Lind-Utley (2021).

Relationship to real Loewner evolution



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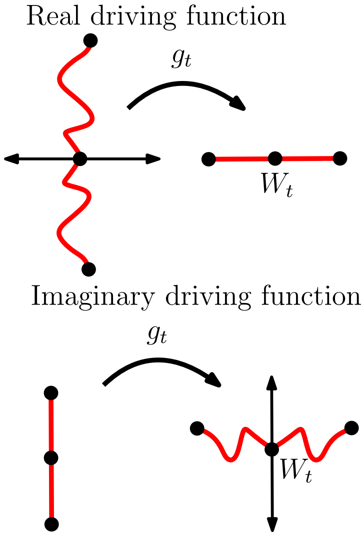

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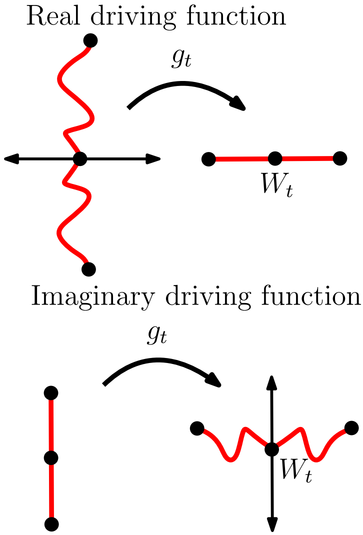

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- "Interpolation between forward and reverse Loewner evolution".

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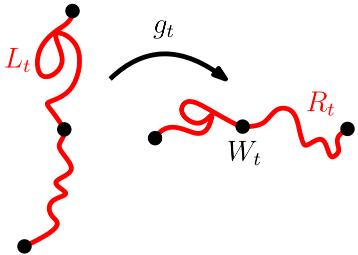
- $c = 0$ corresponds to independent real and imaginary parts (most solvable case).

Forward/reverse duality



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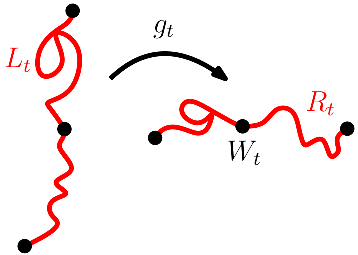

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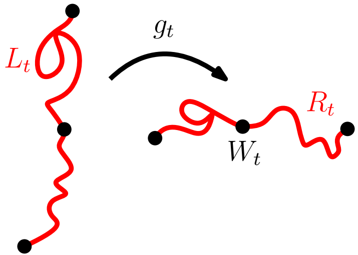

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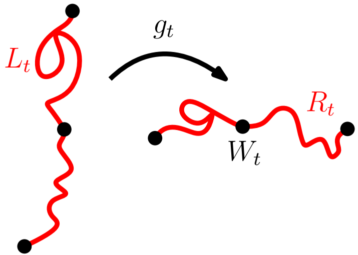

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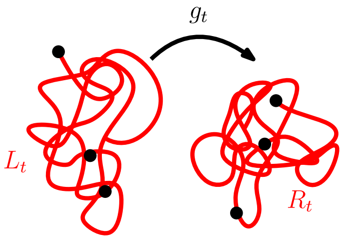
- So, it suffices to consider left hulls.

Differences with real SLE



Theorem (Gwynne-Pfeffer)

Assume that Σ is such that $a, b \neq 0$. For each $z \in \mathbb{C}$, a.s. z is disconnected from ∞ by $(L_t)_{t\geq 0}$ at a time strictly before the smallest t for which $z \in L_t$.

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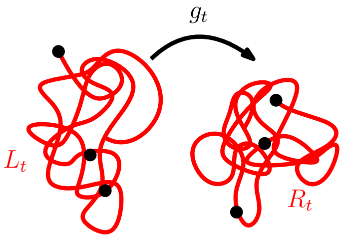

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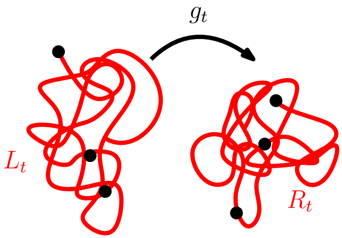

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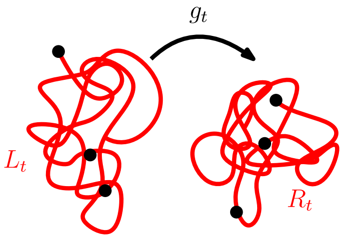

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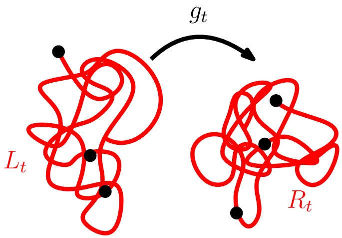

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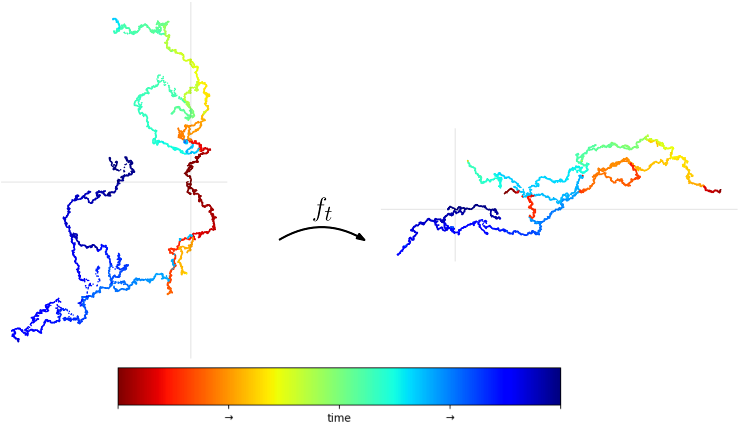
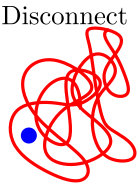
- Not well-posed for SLE_Σ.
- We expect that there is no "reasonable" way to associate a curve with SLE_Σ.

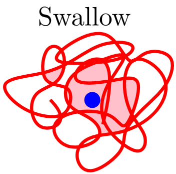
Simulation for $a = 2, b = 2, c = 0$

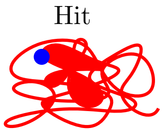


Simulation by M. Park.

Phases







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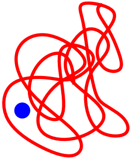
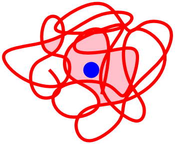


Swallow



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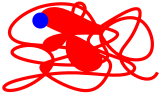

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  - **Disconnected** if $T_z = \infty$ but z is disconnected from ∞ by L_t for some t (does not occur for real SLE).

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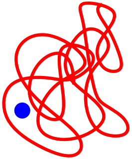

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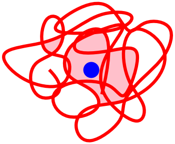

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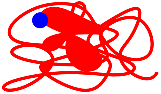

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Phases

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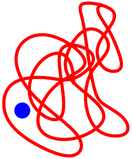
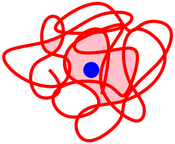


Swallow



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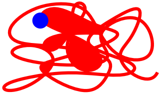

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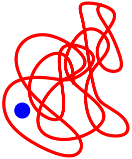

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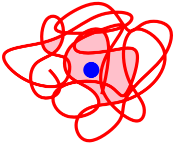

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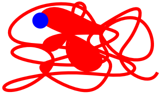

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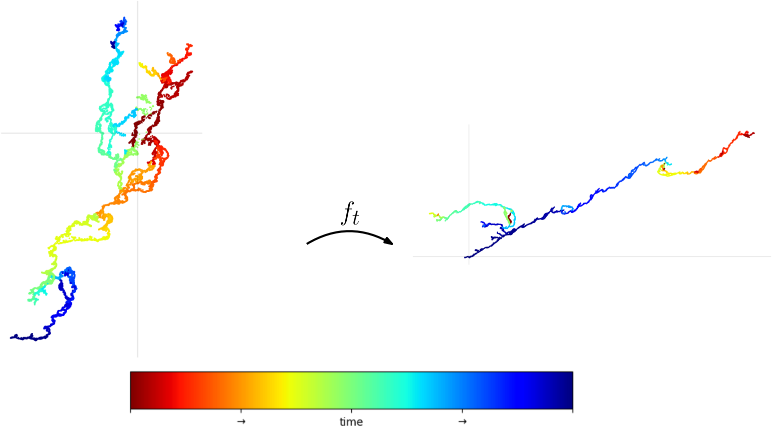
Theorem (Gwynne-Pfeffer)

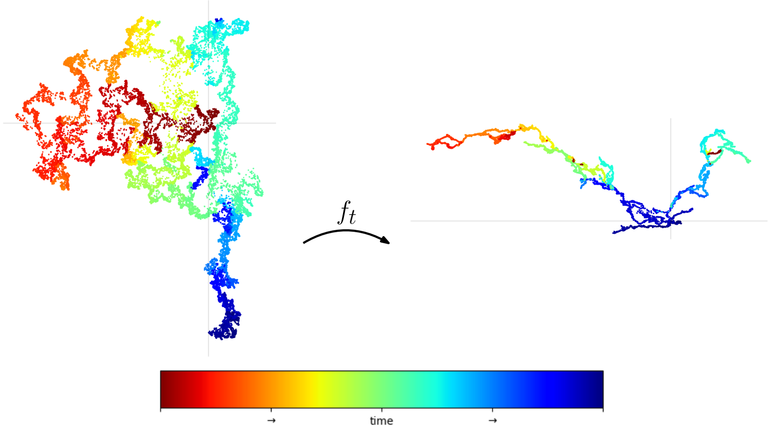
For each Σ, exactly one of the following holds.

- *For each $z \in \mathbb{C}$, a.s. z is disconnected (thin phase).*
- *For each $z \in \mathbb{C}$, a.s. z is swallowed.*
- *For each $z \in \mathbb{C}$, a.s. z is hit.*

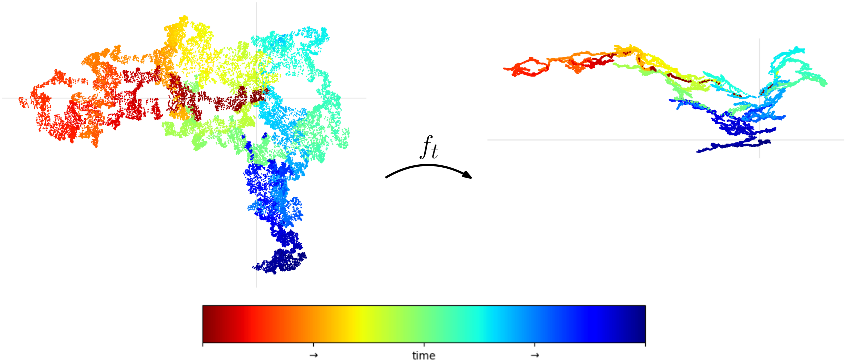
Simulation for $a = 4, b = 2, c = \sqrt{8}$ (thin phase)



Simulation for $a = 7, b = 2, c = 0$ (swallowing phase)



Simulation for $a = 16, b = 3, c = 0$ (hitting phase)



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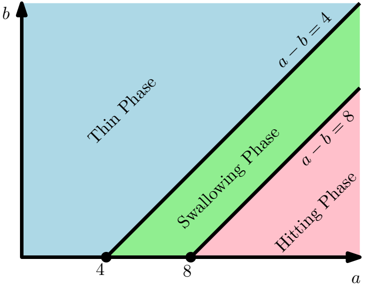

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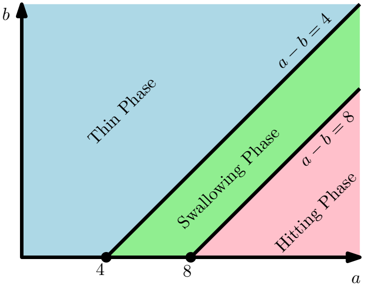

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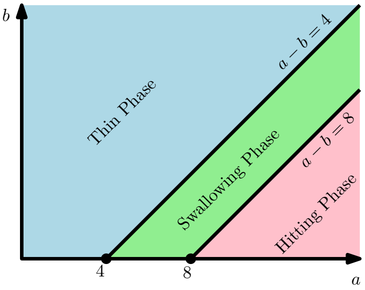

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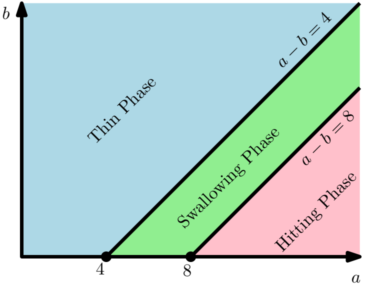

- For $c = 0$ (real and imaginary parts are independent), the phase boundaries are lines.
- Extends phases of SLE$_a$ for $a > 0$.
- When $c \neq 0$, only have numerical approximations of phase boundaries.

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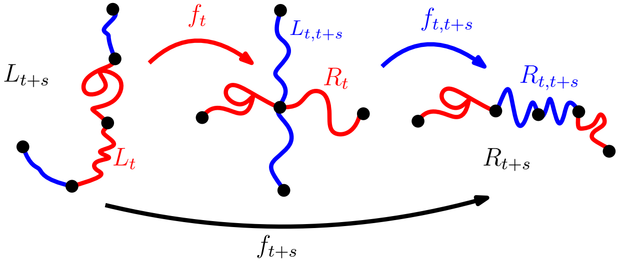

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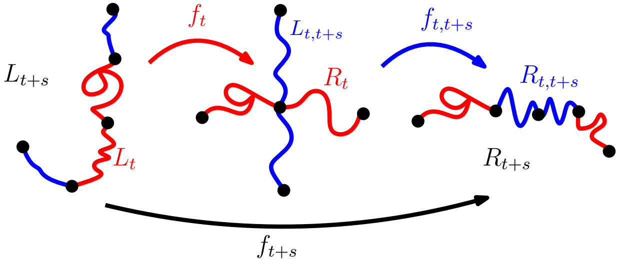

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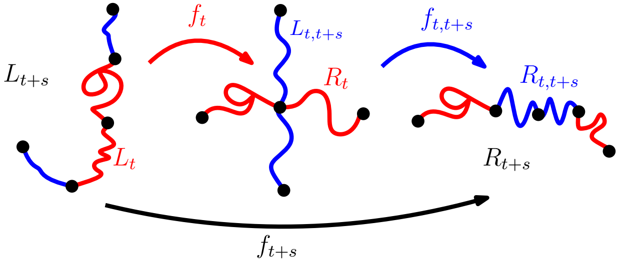

- Proofs based on stochastic calculus + Markov property + complex analysis estimates.
- Very different from proofs for SLE$_\kappa$, since no reference domain or tip.

Outline

1. Loewner evolution with complex driving function
2. Loewner evolution driven by complex Brownian motion
3. Liouville quantum gravity with complex parameters
4. Open problems

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- Euclidean topology, but very different geometry.

Complex Gaussian multiplicative chaos

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$$\lim_{\epsilon \to 0} \epsilon^{(\alpha+i\beta)^2/2} e^{(\alpha+i\beta)h_\epsilon(z)}\, d^2z \quad \text{or} \quad \lim_{\epsilon \to 0} \epsilon^{\alpha^2/2-\beta^2/2} e^{\alpha h_\epsilon(z)+i\beta \widetilde{h}_\epsilon(z)}\, d^2z.$$

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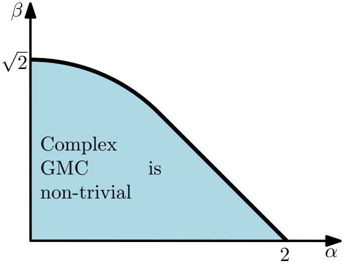

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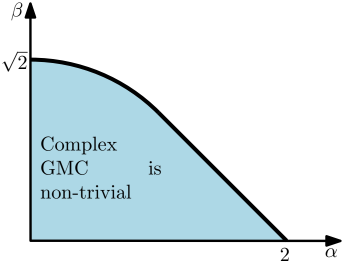

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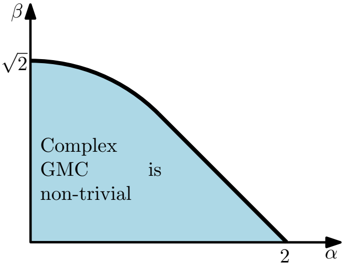

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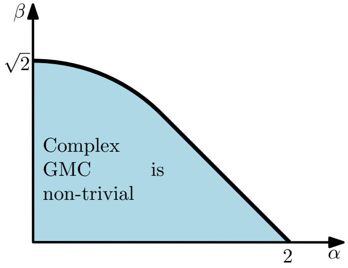

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- Complex distribution (not a measure).
- For other values of α,β, can re-scale differently to get a white noise.

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- Supercritical metric satisfies LQG coordinate change with $Q = Q(\xi) \in (0,2)$.

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Supercritical LQG metric

- For $\gamma \in (0,2]$, we have $\gamma/d_\gamma \leq 2/d_2 \approx 0.41$.
- For $\xi > 2/d_2$, can still define

$$D_h^\epsilon(z,w) = \inf_{P:z\to w} \int_0^1 e^{\xi h_\epsilon(P(t))} |P'(t)|\, dt,$$

  where the inf is over piecewise C^1 paths from z to w.
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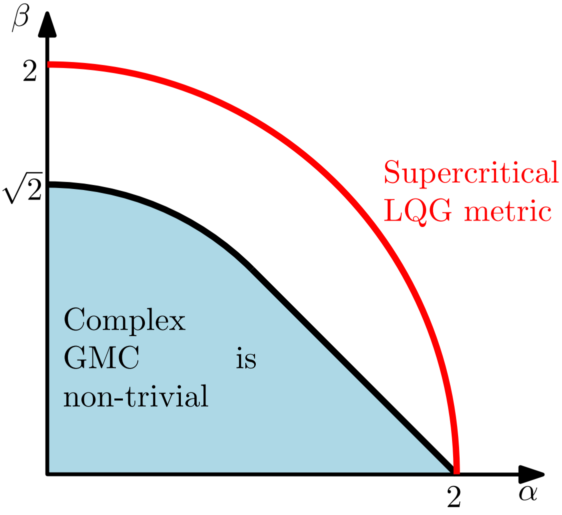
- "γ is complex, but γ/d_γ is real".

Parameter range for “complex LQG”



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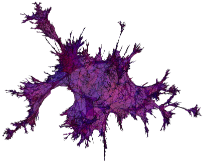

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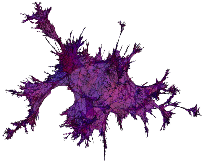

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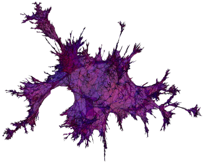

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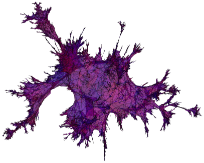

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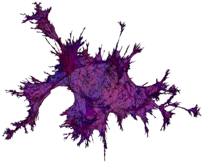

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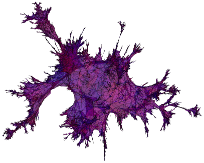

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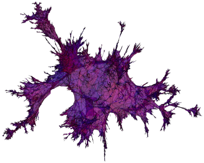

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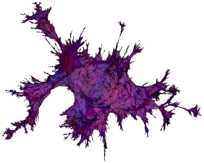

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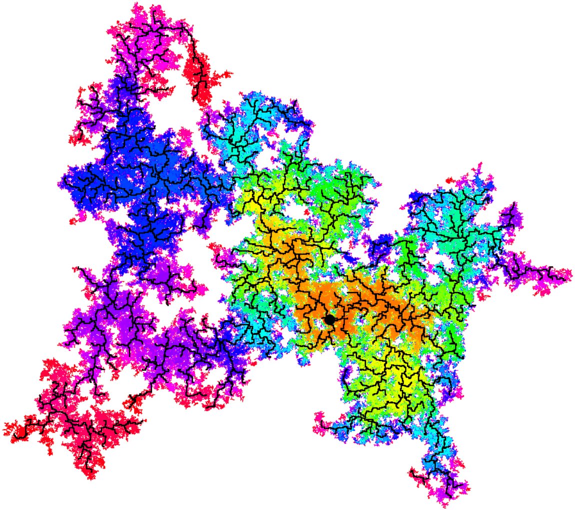
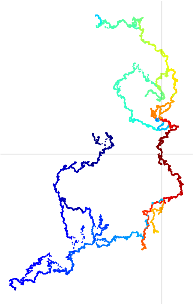
Simulation for $\xi = 1.6$



Outline

Open problems



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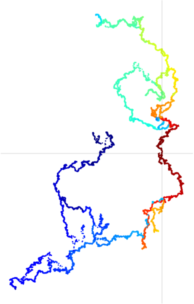

- Relationship to "LQG with complex parameter values":

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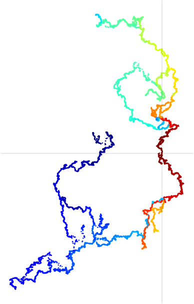

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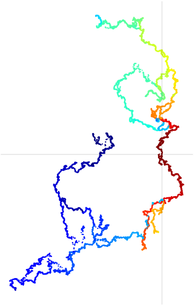

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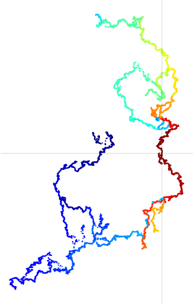

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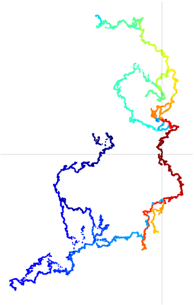

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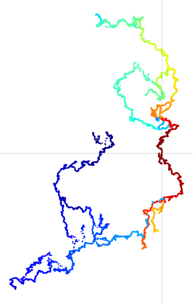

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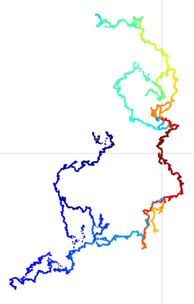

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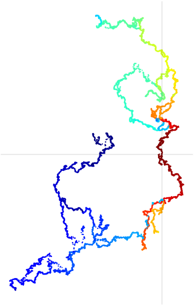

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- Can we describe the outer boundary of SLE_Σ? (Outer boundary of SLE_κ is $\mathrm{SLE}_{16/\kappa}$).