# Random Riemannian Geometry in 4 Dimensions

**Theo Sturm**

**Hausdorff Center for Mathematics**
**Institute for Applied Mathematics**

**Universität Bonn**

based on joint work
*Conformally invariant random fields, Liouville Quantum Gravity measures, and random Paneitz operators on Riemannian manifolds of even dimension*
*Arxiv 2105.13925*
with Lorenzo Dello Schiavo, Ronan Herry, and Eva Kopfer

# Random Riemannian Geometry and Conformal Invariance

Let a compact Riemannian manifold $(M, g)$ of dimension $n$ be given and denote its conformal class by

$$(M, [g]) := \{(M, g') : \ g' = e^{2\varphi} g, \ \varphi \in \mathcal{C}(M)\}.$$

Consider probability measures $\mathbf{P}_{M,g}$ on "fields" (continuous functions, distributions) on $M$ such that

- $\mathbf{P}_{M,g'} = \mathbf{P}_{M,g}$ if $g' = e^{2\varphi} g$ for some $\varphi \in \mathcal{C}(M)$
- $h \overset{(d)}{=} h' \circ \Phi$ if $\Phi : M \to M'$ is an isometry and $h$ and $h'$ are distributed according to $\mathbf{P}_{M,g}$ and $\mathbf{P}_{M',g'}$, resp.

Throughout the sequel, fix $M$.

# Random Riemannian Geometry and Conformal Invariance

Typically, $\mathbf{P}_g$ is a Gaussian field, informally given as

$$d\mathbf{P}_g(h) = \frac{1}{Z_g} \exp\Big(-\frac{1}{2}\mathfrak{e}_g(h,h)\Big)\, dh$$

with (non-existing) uniform distribution $dh$ on $\mathcal{C}(M)$, normalizing constant $Z_g$, and bilinear form $\mathfrak{e}_g$.

Rigorous definition (on spaces of distributions rather than continuous functions) via Bochner–Minlos Theorem

$$\int e^{i\langle u,h\rangle}\, d\mathbf{P}_g(h) = \exp\Big(-\frac{1}{2}\mathfrak{k}_g(u,u)\Big)$$

where $\mathfrak{k}_g(u,u)^{1/2} := \sup_h \frac{\langle u,h\rangle}{\mathfrak{e}_g(h,h)^{1/2}}$ norm dual to $\mathfrak{e}_g$.

# Random Riemannian Geometry and Conformal Invariance

Typically, $\mathbf{P}_g$ is a Gaussian field, informally given as

$$d\mathbf{P}_g(h) = \frac{1}{Z_g} \exp\Big( -\frac{1}{2}\mathfrak{e}_g(h,h)\Big)\, dh$$

with (non-existing) uniform distribution $dh$ on $\mathcal{C}(M)$, normalizing constant $Z_g$, and bilinear form $\mathfrak{e}_g$.
Rigorous definition (on spaces of distributions rather than continuous functions) via Bochner–Minlos Theorem

$$\int e^{i\langle u,h\rangle}\, d\mathbf{P}_g(h) = \exp\Big( -\frac{1}{2}\mathfrak{k}_g(u,u)\Big)$$

where $\mathfrak{k}_g(u,u)^{1/2} := \sup_h \frac{\langle u,h\rangle}{\mathfrak{e}_g(h,h)^{1/2}}$ norm dual to $\mathfrak{e}_g$.

*Conformal Invariance Requirement*

$$\mathfrak{e}_g(u,u) = \mathfrak{e}_{e^{2\varphi}g}(u,u) \qquad \forall \varphi,\ \forall u.$$

In case $n = 2$, celebrated property of the *Dirichlet energy*

$$\mathcal{E}_g(u,u) := \int_M \big|\nabla_g u\big|^2\, d\,\mathrm{vol}_g\,.$$

# Random Riemannian Geometry and Conformal Invariance

**In case $n = 2$:**

Gaussian Free Field [Sheffield, Miller, ...],

$$d\mathbf{P}_g(h) = \frac{1}{Z_g}\ \exp\Big(-\frac{1}{2}\mathfrak{e}_g(h,h)\Big)\, dh$$

with conformally invariant Dirichlet energy

$$\mathfrak{e}_g(u,u) = \int_M \big|\nabla_g u\big|^2\, d\,\mathrm{vol}_g = \mathfrak{e}_{e^{2\varphi}g}(u,u) \qquad \forall\varphi,\ \forall u.$$

Liouville Quantum Gravity: random measure

$$e^{\gamma h(x) - \frac{\gamma^2}{2}\mathbf{E}h(x)^2}\, d\,\mathrm{vol}(x)$$

rigorously defined as weak limit of RHS with $h$ replaced by regular approximations $(h_\ell)_{\ell\in\mathbb{N}}$

Links to Schramm–Loewner evolution [Lawler/Schramm/Werner, ...], convergence to Brownian map: universal scaling limit of planar random graphs [LeGall, Miermond]

# Random Riemannian Geometry and Conformal Invariance

Gaussian fields $d\mathbf{P}_g(h) = \frac{1}{Z_g} \exp\Big(-\frac{1}{2}\mathfrak{e}_g(h,h)\Big)\, dh$ with conformally invariant energy

$$\mathfrak{e}_g(u,u) = \mathfrak{e}_{e^{2\varphi}g}(u,u) \qquad \forall\varphi,\ \forall u.$$

In $n \neq 2$, Dirichlet energy no longer conformally invariant:

$$\mathcal{E}_{e^{2\varphi}g}(u,u) = \int_M \big|\nabla_g u\big|^2 e^{(n-2)\varphi}\, d\,\mathrm{vol}_g\,.$$

In $n = 4$, more promising: bi-Laplacian energy

$$\tilde{\mathfrak{e}}_g(u,u) := \int_M \big(\Delta_g u\big)^2\, d\,\mathrm{vol}_g\,.$$

Still not conformally invariant but close to:

$$\tilde{\mathfrak{e}}_{e^{2\varphi}g}(u,u) := \int_M \big(\Delta_g u + 2\nabla_g\varphi\,\nabla_g u\big)^2\, d\,\mathrm{vol}_g = \tilde{\mathfrak{e}}_g(u,u) + \text{ low order terms.}$$

## Conformally invariant energy

$$\mathfrak{e}_g(u,u) = c\int_M \big(-\Delta_g u\big)^{n/2}\, d\,\mathrm{vol}_g + \text{low order terms}$$

Paneitz ($n = 4$), Graham/Jenne/Mason/Sparling (even $n$)

# Paneitz Energy on 4-Manifolds

From now on: $(M, g)$ is 4-dimensional smooth, compact, connected Riemannian manifold without boundary

**Def. Paneitz Energy,** bilinear form on $L^2(M, \mathrm{vol}_g)$ with domain $H^2(M)$

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M \left[(\Delta_g u)^2 - 2\,\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + \frac{2}{3}\,\mathrm{scal}_g \cdot |\nabla_g u|^2\right] d\,\mathrm{vol}_g$$

- for every 4-dimensional Einstein manifold with $\mathrm{Ric}_g = k\,g$, $k \in \mathbb{R}$,

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M \left[(\Delta_g u)^2 + \frac{2}{3}k\,|\nabla_g u|^2\right] d\,\mathrm{vol}_g$$

- for the 4-sphere $M = \mathbb{S}^4$

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M \left[(\Delta_g u)^2 + 2|\nabla_g u|^2\right] d\,\mathrm{vol}_g$$

- for the 4-torus $M = \mathbb{T}^4$

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M (\Delta_g u)^2\, d\,\mathrm{vol}_g$$

# Paneitz Energy on 4-Manifolds

**Theorem**

*The Paneitz energy*

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M \left[(\Delta_g u)^2 - 2\,\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + \frac{2}{3}\,\mathrm{scal}_g \cdot |\nabla_g u|^2\right] d\,\mathrm{vol}_g$$

*is conformally invariant:* $\forall \varphi \in \mathcal{C}^\infty(M)$, $\forall u \in H^2(M)$

$$\mathfrak{e}_g(u,u) = \mathfrak{e}_{e^{2\varphi}g}(u,u).$$

# Paneitz Energy on 4-Manifolds

**Definition**

The 4-manifold $(M, g)$ is called admissible if $\mathfrak{e}_g > 0$ on $\mathring{H}^2(M)$.

Admissibility is a conformal invariance.

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M \left[(\Delta_g u)^2 - 2\,\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + \frac{2}{3}\,\mathrm{scal}_g \cdot |\nabla_g u|^2\right] d\,\mathrm{vol}_g$$

Large classes of 4-manifolds are admissible

- all compact Einstein 4-manifolds with $\mathrm{Ric} \geq 0$ are admissible.
- all compact hyperbolic 4-manifolds with spectral gap $\lambda_1 > 2$ are admissible.

If $M_1, M_2$ are compact hyperbolic Riemannian surfaces with $\lambda_1(M_1) \leq \frac{2}{3}$, then $M = M_1 \times M_2$ is not admissible.

# Paneitz Energy on 4-Manifolds

**Definition**

The 4-manifold $(M,g)$ is called admissible if $\mathfrak{e}_g > 0$ on $\mathring{H}^2(M)$.

Admissibility is a conformal invariance.

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M \left[(\Delta_g u)^2 - 2\,\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + \frac{2}{3}\,\mathrm{scal}_g \cdot |\nabla_g u|^2\right] d\,\mathrm{vol}_g$$

Large classes of 4-manifolds are admissible

- all compact Einstein 4-manifolds with $\mathrm{Ric} \geq 0$ are admissible.
- all compact hyperbolic 4-manifolds with spectral gap $\lambda_1 > 2$ are admissible.

If $M_1, M_2$ are compact hyperbolic Riemannian surfaces with $\lambda_1(M_1) \leq \frac{2}{3}$, then $M = M_1 \times M_2$ is not admissible.

Integrable functions (or distributions) $u$ on $M$ will be called grounded if $\int_M u\,d\,\mathrm{vol}_g = 0$ (or $\langle u, \mathbf{1}\rangle = \mathbf{0}$, resp.).

Grounded Sobolev spaces $\mathring{H}^s(M,g) = (-\Delta_g)^{-s/2}\mathring{L}^2(M,\mathrm{vol}_g)$ for $s \in \mathbb{R}$,
usual Sobolev spaces $H^s(M,g) = (1-\Delta)^{-s/2}L^2(M,\mathrm{vol}_g) = \mathring{H}^s(M,g) \oplus \mathbb{R}\cdot\mathbf{1}$

Laplacian $-\Delta : H^s \to \mathring{H}^{s-2}$; grounded Green operator $\mathring{\mathrm{G}}_g : \mathring{H}^s \to \mathring{H}^{s+2}$

# Paneitz Energy and Co-Biharmonic Green Kernel

Assume $(M, g)$ is admissible. Then Paneitz operator

$$\mathrm{p}_g = \frac{1}{8\pi^2}\left[\Delta_g^2 + \mathrm{div}\left(2\mathrm{Ric}_g - \frac{2}{3}\mathrm{scal}_g\right)\nabla\right]$$

is a self-adjoint positive operator on $L^2(M, \mathrm{vol}_g)$ with domain $H^4(M)$.

Let $(\psi_j)_{j\in\mathbb{N}_0}$ be complete ONB of $L^2(M, \mathrm{vol}_g)$ of eigenfunctions for $\mathrm{p}_g$ with eigenvalues $(\nu_j)_{j\in\mathbb{N}_0}$. Define operator $\mathrm{k}_g$ on $H^{-4}(M)$ , inverse to $\mathrm{p}_g$ on $\mathring{L}^2$, by

$$\mathrm{k}_g : \; u \mapsto \mathrm{k}_g u := \sum_{j\in\mathbb{N}} \frac{1}{\nu_j} \langle u, \psi_j\rangle \psi_j,$$

and associated bilinear form with domain $H^{-2}(M)$ by

$$\mathfrak{k}_g(u, v) := \langle u, \mathrm{k}_g v\rangle_{L^2} = \sum_{j\in\mathbb{N}} \frac{1}{\nu_j} \langle u, \psi_j\rangle \langle v, \psi_j\rangle.$$

# Paneitz Energy and Co-Biharmonic Green Kernel

Assume $(M, g)$ is admissible.

**Theorem**

$\mathrm{k}_g$ *is an integral operator with an integral kernel* $k_g$ *which is grounded, symmetric, and satisfies*

$$\left| k_g(x,y) + \log d_g(x,y) \right| \le C_0.$$

**Theorem**

*Assume that* $g' := e^{2\varphi} g$ *for some* $\varphi \in \mathcal{C}^\infty(M)$*. Then the co-biharmonic Green kernel* $k_{g'}$ *for the metric* $g'$ *is given by*

$$k_{g'}(x,y) = k_g(x,y) - \frac{1}{2}\bar{\phi}(x) - \frac{1}{2}\bar{\phi}(y)$$

*with* $\bar{\phi} \in \mathcal{C}^\infty(M)$ *defined by*

$$\bar{\phi} := \frac{2}{\mathrm{vol}_{g'}(M)} \int k_g(.,z)\, d\mathrm{vol}_{g'}(z) - \frac{1}{\mathrm{vol}_{g'}(M)^2} \iint k_g(z,w)\, d\mathrm{vol}_{g'}(z)\, d\mathrm{vol}_{g'}(w)$$

# Co-Biharmonic Gaussian Field

**Definition**

A co-biharmonic Gaussian field on $(M, g)$ is a linear family $\big\{\langle h, u\rangle\big\}_{u\in H^{-2}}$ of centered Gaussian random variables (defined on some probability space) with

$$\mathbf{E}\big[\langle h, u\rangle\,\langle h, v\rangle\big] = \mathfrak{k}_g(u, v) \qquad \forall u, v \in H^{-2}(M).$$

Let a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ be given and an i.i.d. sequence $(\xi_j)_{j\in\mathbb{N}}$ of $\mathcal{N}(0, 1)$ random variables. Furthermore, let $(\psi_j)_{j\in\mathbb{N}_0}$ and $(\nu_j)_{j\in\mathbb{N}_0}$ denote the sequences of eigenfunctions and eigenvalues for $\mathfrak{p}_g$ (counted with multiplicities).

**Theorem**

*A co-biharmonic field is given by*

$$h := \sum_{j\in\mathbb{N}} \nu_j^{-1/2}\,\xi_j\,\psi_j.$$

# Co-Biharmonic Gaussian Field

**Theorem**

*A co-biharmonic field is given by*

$$h := \sum_{j\in\mathbb{N}} \xi_j \cdot \sqrt{\mathsf{k}_g}\, \psi_j = \sum_{j\in\mathbb{N}} \nu_j^{-1/2}\, \xi_j\, \psi_j .$$

*More precisely,*

1. *For each $\ell \in \mathbb{N}$, a centered Gaussian random variable $h_\ell$ with values in $\mathcal{C}^\infty(M)$ is given by*

$$h_\ell := \sum_{j=1}^{\ell} \nu_j^{-1/2}\, \xi_j\, \psi_j .$$

2. *The convergence $h_\ell \to h$ holds in $L^2(\mathbf{P}) \times H^{-\epsilon}(M)$ for every $\epsilon > 0$. In particular, for a.e. $\omega$ and every $\epsilon > 0$,*

$$h^\omega \in H^{-\epsilon}(M),$$

3. *For every $u \in H^{-2}(M)$, the family $(\langle u, h_\ell\rangle)_{\ell\in\mathbb{N}}$ is a centered $L^2(\mathbf{P})$-bounded martingale and*

$$\langle u, h_\ell\rangle \to \langle u, h\rangle \quad \text{in } L^2(\mathbf{P}) \text{ as } \ell \to \infty .$$

# Co-Biharmonic Gaussian Field

Core arguments of the proof:

$$\mathbf{E}\Big[\langle u, h_\ell\rangle^2\Big] = \sum_{j=1}^{\ell} \frac{1}{\nu_j}\big\langle u, \psi_j\big\rangle^2 \;\to\; \sum_{j=1}^{\infty} \frac{1}{\nu_j}\big\langle u, \psi_j\big\rangle^2 = \langle u, ku\rangle = \mathfrak{k}(u,u).$$

Thus $(\langle u, h_\ell\rangle)_{\ell\in\mathbb{N}}$ is an $L^2$-bounded martingale. Convergence follows readily.

For $\epsilon > 0$, since $h = \sum_j \xi_j \cdot \sqrt{\mathsf{k}}\psi_j$ and since $\mathring{G}^{(s)}$ for $s > n/2 = 2$ is a bounded function,

$$\mathbf{E}\Big[\|h\|^2_{H^{-\epsilon}}\Big] = \mathbf{E}\Big[\big\|\mathring{\mathsf{G}}^{\epsilon} h\big\|^2_{L^2}\Big] = \sum_j \Big\|\mathring{\mathsf{G}}^{\epsilon}\sqrt{\mathsf{k}}\psi_j\Big\|^2_{L^2} \sim \sum_j \Big\|\mathring{\mathsf{G}}^{1+\epsilon}\psi_j\Big\|^2_{L^2}$$

$$= \int_M \Big\|\mathring{G}^{(1+\epsilon)}(.,z)\Big\|^2 d\,\mathrm{vol}_g(z) = \int_M \mathring{G}^{(2+2\epsilon)}(z,z)\, d\,\mathrm{vol}_g(z) < \infty$$

This proves that $\|h\|_{H^{-\epsilon}} < \infty$ for a.e. $\omega$.
The convergence $h_\ell \to h$ in $H^{-\epsilon}(M)$ follows similarly.

# Co-Biharmonic Gaussian Field

- A co-biharmonic Gaussian field on $(M, g)$ can be regarded as a random variable with values in $\mathring{H}^{-\epsilon}(M)$ for any $\epsilon > 0$.
- Given a grounded white noise $\Xi$ on $(M, g)$, then $h := \sqrt{\mathsf{k}_g}\,\Xi$ is a co-biharmonic Gaussian field on $(M, g)$.
- If $(M, g)$ is Ricci flat (e.g. torus) then $\sqrt{\mathsf{k}_g} = c\,\mathring{\mathsf{G}}_g$ is the grounded Green operator. For general $n$, $\sqrt{\mathsf{k}_g} = c\,\mathring{\mathsf{G}}_g^{n/4}$.

**Theorem**

*Let $h : \Omega \to H^{-\epsilon}(M)$ denote a co-biharmonic Gaussian field for $(M, g)$ and let $g' = e^{2\varphi} g$ with $\varphi \in \mathcal{C}^\infty(M)$. Then*

$$h' := h - \frac{1}{\mathrm{vol}_{g'}(M)} \left\langle h, \mathbf{1} \right\rangle_{H^{-\epsilon}(M,g'), H^{\epsilon}(M,g')}$$

*is a co-biharmonic Gaussian field for $(M, g')$.*

# Liouville Quantum Gravity Measure

Fix an admissible manifold $(M, g)$ and a co-polyharmonic Gaussian field $h : \Omega \to \mathfrak{D}'$. Our naive goal is to study the 'random geometry' $(M, g_h)$ obtained by the random conformal transformation,

$$g_h = e^{2h} g \,,$$

and in particular to study the associated 'random volume measure' given as

$$d\,\mathrm{vol}_{g_h}(x) = e^{nh(x)} d\,\mathrm{vol}_g(x). \tag{1}$$

It easily can be seen that — due to the singular nature of the noise $h$ — all approximating sequences of this measure diverge as long as no additional renormalization is built in.

# Liouville Quantum Gravity Measure

Fix an admissible manifold $(M, g)$ and a co-polyharmonic Gaussian field $h : \Omega \to \mathfrak{D}'$. Our naive goal is to study the 'random geometry' $(M, g_h)$ obtained by the random conformal transformation,

$$g_h = e^{2h} g \,,$$

and in particular to study the associated 'random volume measure' given as

$$d\,\mathrm{vol}_{g_h}(x) = e^{nh(x)} d\,\mathrm{vol}_g(x). \tag{1}$$

It easily can be seen that — due to the singular nature of the noise $h$ — all approximating sequences of this measure diverge as long as no additional renormalization is built in.

A more tractable goal is to study (for suitable $\gamma \in \mathbb{R}$) the random measure $\mu^h$ formally given as

$$d\mu^h(x) = e^{\gamma h(x) - \frac{\gamma^2}{2}\mathbf{E}[h(x)^2]} d\,\mathrm{vol}_g(x). \tag{2}$$

Since $h$ is not a function but only a distribution, both (1) and (2) are ill-defined.

# Liouville Quantum Gravity Measure

Fix an admissible manifold $(M, g)$ and a co-polyharmonic Gaussian field $h : \Omega \to \mathfrak{D}'$. Our naive goal is to study the 'random geometry' $(M, g_h)$ obtained by the random conformal transformation,

$$g_h = e^{2h} g \,,$$

and in particular to study the associated 'random volume measure' given as

$$d\,\mathrm{vol}_{g_h}(x) = e^{nh(x)} d\,\mathrm{vol}_g(x). \tag{1}$$

It easily can be seen that — due to the singular nature of the noise $h$ — all approximating sequences of this measure diverge as long as no additional renormalization is built in.

A more tractable goal is to study (for suitable $\gamma \in \mathbb{R}$) the random measure $\mu^h$ formally given as

$$d\mu^h(x) = e^{\gamma h(x) - \frac{\gamma^2}{2}\mathbf{E}[h(x)^2]} d\,\mathrm{vol}_g(x). \tag{2}$$

Since $h$ is not a function but only a distribution, both (1) and (2) are ill-defined.

However, replacing $h$ by finite-dimensional noise approximations $h_\ell$ as before, leads to a sequence $(\mu^{h_\ell})$ of random measures on $M$ which as $\ell \to \infty$, almost surely, converges to a random measure $\mu^h$.

# Liouville Quantum Gravity Measure

For $\ell \in \mathbb{N}$ define a random measure $\mu_\ell = \rho_\ell \operatorname{vol}_g$ on $M$ with density

$$\rho_\ell(x) := \exp\left(\gamma h_\ell(x) - \frac{\gamma^2}{2} k_\ell(x,x)\right)$$

where as before $h_\ell := \sum_{j=1}^{\ell} \nu_j^{-1/2} \xi_j \psi_j$ and $k_\ell(x,x) := \mathbf{E}\left[h_\ell^2(x)\right] = \sum_{j=1}^{\ell} \nu_j^{-1} \psi_j^2(x)$.

**Theorem**

*If $|\gamma| < \sqrt{2n}$, then there exists a random measure $\mu$ on $M$ with $\mu_\ell \to \mu$. More precisely, for every $u \in \mathcal{C}(M)$,*

$$\int_M u \, d\mu_\ell \longrightarrow \int_M u \, d\mu \quad \text{in } L^1(\mathbf{P}) \text{ and } \mathbf{P}\text{-a.s. as } \ell \to \infty.$$

The random measure $\mu := \lim_{\ell \to \infty} \mu_\ell$ is called *Liouville Quantum Gravity measure*.

# Liouville Quantum Gravity Measure

Choose symmetric Markov kernels $q_\ell$ on $M$ with $q_\ell(x,.)\,\mathrm{vol}_g \to \delta_x$ as $\ell \to \infty$, put $h_\ell = \langle h, q_\ell(x,.)\rangle$, $k_\ell(x,y) = \iint k(x',y')q_\ell(x,x')q_\ell(y,y')dy'dx'$, and $d\mu_\ell(x) = \exp\left(\gamma h_\ell(x) - \frac{\gamma^2}{2}k_\ell(x,x)\right) d\,\mathrm{vol}_g(x)$.

## Theorem

*If $|\gamma| < \sqrt{n}$, then for every $u \in \mathcal{C}_b(M)$,*

$$\left(Y_\ell\right)_{\ell\in\mathbb{N}} := \left(\int_M u\, d\mu_\ell\right)_{\ell\in\mathbb{N}} \quad \textit{is } L^2\textit{-bounded martingale}$$

Proof: Assume $0 \le u \le 1$. Then

$$\begin{aligned}\sup_\ell \mathbf{E}\left[{Y_\ell}^2\right] &= \sup_\ell \mathbf{E}\Big[\iint e^{\gamma h_\ell(x)+\gamma h_\ell(y) - \frac{\gamma^2}{2}k_\ell(x,x) - \frac{\gamma^2}{2}k_\ell(y,y)} \cdot u(x)u(y)\,dx\,dy\Big]\\ &= \sup_\ell \iint e^{\gamma^2 k_\ell(x,y)} \cdot u(x)u(y)\,dx\,dy\Big]\\ &\le \iint e^{\gamma^2 k(x,y)}\,dx\,dy\\ &= \iint \frac{1}{d(x,y)^{\gamma^2}}\,dx\,dy + \mathcal{O}(1)\end{aligned}$$

due to the log divergence of $k$. The latter integral is finite if and only if $\gamma^2 < n$.

# Liouville Quantum Gravity Measure

A key property of the Liouville Quantum Gravity measure is its quasi-invariance under conformal transformations.

**Theorem**

*Let $\mu$ be the Liouville Quantum Gravity measure for $(M, g)$, and $\mu'$ be the Liouville Quantum Gravity measure for $(M, g')$ where $g' = e^{2\varphi} g$ for some $\varphi \in \mathcal{C}^\infty(M)$. Then*

$$\mu' \stackrel{(\mathrm{d})}{=} e^{-\gamma\xi + \frac{\gamma^2}{2}\bar{\varphi} + n\varphi} \mu$$

*where $\xi := \frac{1}{v'}\langle h, e^{n\varphi}\rangle$ and $\bar{\varphi} := \frac{2}{v'}\,\mathrm{k}_g(e^{n\varphi}) - \frac{1}{v'^2}\,\mathfrak{k}_g(e^{n\varphi}, e^{n\varphi})$ with $v' := \mathrm{vol}_{g'}(M)$.*

# Liouville Quantum Gravity Measure

A key property of the Liouville Quantum Gravity measure is its quasi-invariance under conformal transformations.

**Theorem**

*Let $\mu$ be the Liouville Quantum Gravity measure for $(M, g)$, and $\mu'$ be the Liouville Quantum Gravity measure for $(M, g')$ where $g' = e^{2\varphi} g$ for some $\varphi \in \mathcal{C}^\infty(M)$. Then*

$$\mu' \overset{(\mathrm{d})}{=} e^{-\gamma\xi + \frac{\gamma^2}{2}\bar{\varphi} + n\varphi} \mu$$

*where* $\xi := \frac{1}{v'}\langle h, e^{n\varphi}\rangle$ *and* $\bar{\varphi} := \frac{2}{v'}\mathrm{k}_g(e^{n\varphi}) - \frac{1}{v'^2}\mathfrak{k}_g(e^{n\varphi}, e^{n\varphi})$ *with* $v' := \mathrm{vol}_{g'}(M)$.

All the previous results extend verbatim to Riemannian manifolds of even dimension $n$, now with a conformally invariant energy functional $\mathfrak{e}_g$ generated by the co-polyharmonic operator

$$\mathrm{p}_g = a_n (-\Delta_g)^{n/2} + \text{low order terms}$$

introduced by Graham, Jenne, Mason, Sparling (rather than the Paneitz operator). Its inverse has an integral kernel $k_g$ with the two key properties

- conformal invariance (modulo additive corrections)
- log divergence.

# Liouville Brownian Motion, Random Paneitz Operator

If $\gamma < \sqrt{2}$ then a.s. the LQG measure $\mu$ does not charge sets of vanishing $H^1$-capacity

$\longrightarrow$ Dirichlet form $\int_M |\nabla u|^2 d\,\mathrm{vol}_g$ on $L^2(M, \mu)$
$\longrightarrow$ Liouville Brownian motion (random time change of BM)

If $\gamma < \sqrt{2n}$ then a.s. the LQG measure $\mu$ does not charge sets of vanishing $H^{n/2}$-capacity

$\longrightarrow$ energy form $\int u((-\Delta)^{n/2} + l.o.t.)u\, d\,\mathrm{vol}_g$ on $L^2(M, \mu)$
$\longrightarrow$ random Paneitz operators, conformally invariant

# Discrete Approximations

Now let $M$ be the continuous torus $\mathbb{T}^n \cong [0,1)^n$ and consider its discrete approximations $\mathbb{T}^n_L \cong \{0, \frac{1}{L}, \ldots, \frac{L-1}{L}\}^n$ for $L \in \mathbb{N}$.

**Polyharmonic Gaussian Field on the discrete torus $\mathbb{T}^n_L$**

centered Gaussian field $(h_L(v))_{v \in \mathbb{T}^n_L}$ with covariance function

$$k_L(u,v) = \frac{1}{a_n} \mathring{G}_L^{n/2}(u,v) = \frac{1}{a_n} \sum_{z \in \mathbb{Z}^n_L \setminus \{0\}} \frac{1}{\lambda_{L,z}^{n/2}} \cdot \cos\Big(2\pi\, z \cdot (v-u)\Big)$$

where $\lambda_{L,z} = 4L^2 \sum_{k=1}^n \sin^2\left(\pi z_k / L\right)$ and $\mathbb{Z}^n_L = \{z \in \mathbb{Z}^n : 0 < \|z\|_\infty < L/2\}$.

Given iid standard normals $(\xi_z)_{z \in \mathbb{Z}^n_L}$ and Fourier basis functions $\varphi_z(x) = \frac{1}{\sqrt{2}} \cos(2\pi x z)$ and $\varphi_{-z}(x) = \frac{1}{\sqrt{2}} \sin(2\pi x z)$, a Polyharmonic Gaussian Field is given as

$$h_L = \frac{1}{\sqrt{a_n}} \sum_{z \in \mathbb{Z}^n_L \setminus \{0\}} \frac{1}{\lambda_{L,z}^{n/4}} \cdot \xi_z\, \varphi_z.$$

Given white noise on $\mathbb{T}^n_L$, i.e. iid centered Gaussian variables $(\Xi(v))_{v \in \mathbb{T}^n_L}$ with variance $L^{n/2}$, then

$$h_L = \frac{1}{\sqrt{a_n}} \mathring{G}_L^{n/4} \Xi.$$

# Discrete Approximations

The law of the "ungrounded" Polyharmonic Gaussian Field is given explicitly as

$$c_n \exp\left(-\frac{a_n}{2N}\left\|(-\Delta_L)^{n/4}h\right\|^2\right) d\mathcal{L}^N(h)$$

on $\mathbb{R}^N \cong \mathbb{R}^{\mathbb{T}_L^n}$ where $N = L^n$.

- Convergence of fields $h_L \to h$ as $L \to \infty$: tested against $f \in \bigcup_{s>n/2} H^s(\mathbb{T}^n)$
- Convergence of Fourier extension of $h_L$ to $h$: in each $H^{-\epsilon}(\mathbb{T}^n)$ and also tested against $f \in H^{-n/2}(\mathbb{T}^n)$

**Theorem (Convergence of Liouville Quantum Gravity Measures )**

*Assume $\gamma < \sqrt{n/e}$ and $L = 3^\ell$.*

$$\mu_L \to \mu \quad \textit{weakly on } \mathbb{T}^n \textit{ in } \mathbf{P}\textit{-probabilty.}$$

Convergence of "reduced LQG measures" on $\mathbb{T}_L^n$ for all $\gamma < \sqrt{2n}$.

# Polyakov-Liouville Measure and Conformal Field Theory

**Aim:** rigorous meaning to the Polyakov–Liouville measure $\boldsymbol{\nu}^*_g$, informally given as

$$\frac{1}{Z_g}\exp\Big(-S_g(h)\Big)\,dh$$

with (non-existing) uniform distribution $dh$ on the set of fields and action

$$S_g(h) := \int_M \Big(\frac{1}{2}\left|\sqrt{\mathsf{p}_g}\,h\right|^2 + \Theta\, Q_g h + \frac{\Theta^*}{\mathrm{vol}_g(M)} h + m e^{\gamma h}\Big)\, d\,\mathrm{vol}_g\,. \qquad (3)$$

Here $\mathsf{p}_g$ is the co-polyharmonic operator, $Q_g$ denotes Branson's curvature, and $m, \Theta, \Theta^*, \gamma$ are parameters.

In the case $n = 2$, $Q_g$ is the usual curvature and $\int \left|\sqrt{\mathsf{p}_g}\,h\right|^2 d\,\mathrm{vol}_g$ is the Dirichlet energy $\int |\nabla h|^2 d\,\mathrm{vol}_g$.

# Polyakov-Liouville Measure and Conformal Field Theory

**Aim:** rigorous meaning to the Polyakov–Liouville measure $\boldsymbol{\nu}_g^*$, informally given as

$$\frac{1}{Z_g}\exp\Big(-S_g(h)\Big)\,dh$$

with (non-existing) uniform distribution $dh$ on the set of fields and action

$$S_g(h) := \int_M \left(\frac{1}{2}\left|\sqrt{\mathsf{p}_g}\,h\right|^2 + \Theta\,Q_g h + \frac{\Theta^*}{\mathrm{vol}_g(M)} h + m e^{\gamma h}\right) d\,\mathrm{vol}_g\,. \qquad (3)$$

Here $\mathsf{p}_g$ is the co-polyharmonic operator, $Q_g$ denotes Branson's curvature, and $m, \Theta, \Theta^*, \gamma$ are parameters.

In the case $n = 2$, $Q_g$ is the usual curvature and $\int \left|\sqrt{\mathsf{p}_g}\,h\right|^2 d\,\mathrm{vol}_g$ is the Dirichlet energy $\int |\nabla h|^2 d\,\mathrm{vol}_g$. With the Polyakov–Liouville action, this ansatz for the measure $\boldsymbol{\nu}_g^*(dh) = \frac{1}{Z_g^*} e^{-S_g(h)} dh$ reflects the coupling of the gravitational field with a matter field. It can be regarded as quantization of the the classical Einstein–Hilbert action $S_g^{EH}(h) = \frac{1}{2\kappa}\int_M \left(R_g - 2\Lambda\right) dx$ or, more precisely, of its coupling with a matter field

$$S_g^{EH}(h) = \int_M \left[\frac{1}{2\kappa}\left(R_g - 2\Lambda\right) + \mathcal{L}_M\right] dx.$$

# Polyakov-Liouville Measure and Conformal Field Theory

In the case $n = 4$, $Q_g = -\frac{1}{6}\Delta_g \mathrm{scal}_g - \frac{1}{2}|\mathrm{Ric}_g|^2 + \frac{1}{6}\mathrm{scal}_g^2$. In general, total $Q$-curvature is conformally invariant, and if $g' = e^{2\varphi} g$ then

$$e^{n\varphi} Q_{g'} = Q_g + \frac{1}{a_n} \mathsf{p}_g \varphi.$$

Informal ansatz

$$\boldsymbol{\nu}_g^*(dh) = \frac{1}{Z_g^*} \exp\left( - \int_M \left( \frac{1}{2} \left| \sqrt{\mathsf{p}_g}\, h \right|^2 + \Theta\, Q_g h + \frac{\Theta^*}{\mathrm{vol}_g(M)} h + m e^{\gamma h} \right) d\,\mathrm{vol}_g \right) dh$$

**Rigorous**

$$d\boldsymbol{\nu}_g^*(h) := \exp\Big( - \Theta \langle h, Q_g \rangle - \Theta^* \langle h \rangle_g - m\, \mu^h(M) \Big)\, d\widehat{\boldsymbol{\nu}}_g(h)$$

with $d\widehat{\boldsymbol{\nu}}_g$ = law of ungrounded co-polyharmonic Gaussian field = image of $d\boldsymbol{\nu}_g(h) \otimes d\mathcal{L}^1(t)$ under map $(h, t) \mapsto h + t$, informally characterized as

$$d\widehat{\boldsymbol{\nu}}_g(h) = \frac{1}{Z_g} \exp\left( - \frac{1}{2} \mathfrak{e}_g(h, h) \right) dh,$$

and $\mu^h$ denotes the Liouville Quantum Gravity measure on the $n$-manifold $M$.

# Polyakov-Liouville Measure and Conformal Field Theory

$$d\boldsymbol{\nu}^*_g(h) := \exp\Big( -\Theta\langle h, Q_g\rangle - \Theta^*\langle h\rangle_g - m\,\mu^h(M)\Big)\, d\widehat{\boldsymbol{\nu}}_g(h)$$

with $d\widehat{\boldsymbol{\nu}}_g$ = law of ungrounded co-polyharmonic Gaussian field and $\mu^h$ = Liouville Quantum Gravity measure.

**Theorem**

*Assume that $0 < \gamma < \sqrt{2n}$ and $\Theta\, Q(M) + \Theta^* < 0$. Then $\boldsymbol{\nu}^*_g$ is a finite measure.*

**Theorem**

*If $\Theta = a_n \frac{n}{\gamma}$, and $\Theta^* = \gamma$, then $\boldsymbol{\nu}^*_g$ is conformally quasi-invariant modulo shift:*

$$\boldsymbol{\nu}^*_{e^{2\varphi}g} = Z(g,\varphi)\cdot T_*\boldsymbol{\nu}^*_g \qquad \forall\varphi \tag{4}$$

*with explicitly given conformal anomaly $Z(g,\varphi)$.*

For $n = 2$: David, Kupiainen, Rhodes, Vargas '16 for surfaces of genus 0, David, Rhodes, Vargas '16 for surfaces of genus 1, and Garban, Rhodes, Vargas '19 for surfaces of higher genus.