Moduli of Annuli in Random Conformal Geometry

Xin Sun

University of Pennsylvania

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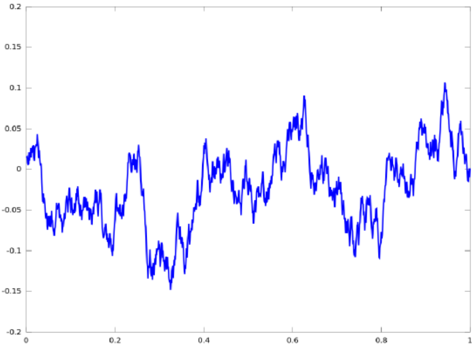
Joint work with Guillaume Remy and Morris Ang

How to sample a random path?

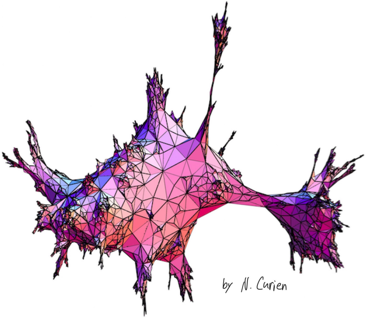


Brownian Bridge

$e_n = \frac{\sqrt{2}}{n\pi}\sin(n\pi t)$. $\{\alpha_n\}$: independent standard Gaussians.

$B = \sum_{n=1}^{\infty} \alpha_n e_n$, (convergence in the uniform topology).

How to sample a random surface?



M_n: uniformly sampled triangulation/quadrangulation.

d_n: graph distance.

A_n: counting measure on the vertex set.

Le Gall (2011), Miermont (2011)

(M_n, d_n, A_n) after proper scaling converge to a random metric measure space in Gromov-Hausdorff-Prokhorov topology.

Brownian sphere: the limiting random sphere.

Polyakov (1981), *Quantum geometry of bosonic strings*

"In my opinion at the present time we have to develop an art of handling sums over random surfaces.".

$\mathcal{S}$: a topological surface, e.g. sphere, disk, annulus.

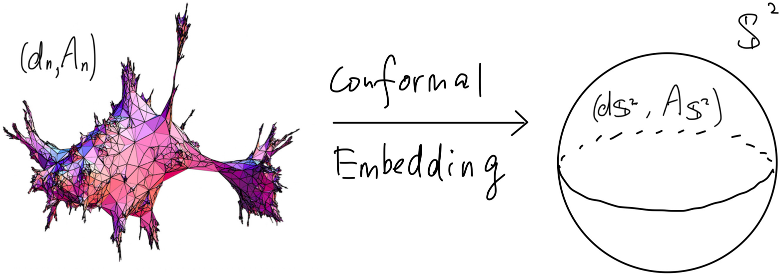
Quantum gravity on $\mathcal{S}$ = random geometry on $\mathcal{S}$.

$(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$ are conformally equivalent
if $\exists\ \psi : S_1 \to S_2$ and a function φ on S_2 s.t. $\psi_* \mathbf{g}_1 = e^{\varphi} \mathbf{g}_2$.
ψ: conformal embedding. φ: conformal factor.

From random geometry to random function

A random geometry on $\mathcal{S}$, conditioned on being, conformally equivalent to a fixed $(S, \mathbf{g})$, can be written as $(S, e^{\varphi}\mathbf{g})$ for some random conformal factor φ.

Conformal Embedding of Brownian Sphere



Polyakov's idea in modern math language

Conformal embedding of Brownian sphere = $\sqrt{8/3}$-LQG on $\mathbb{S}^2$.

- Liouville quantum gravity on the sphere and disk.
- Mathematical status of Polyakov's idea.
- The case of annulus.
- Relation to 2D statistical physics

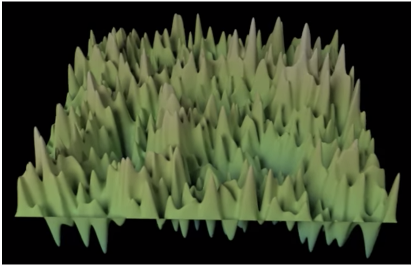
Gaussian Free Field on a Riemannian Manifold

$\{e_n\}_{n\geq 1}$: non-constant eigenfunctions of the Δ on $(S, \mathbf{g})$ normalized by e.g. $\int |\nabla e_n|^2 \mathrm{d}v_{\mathbf{g}} = 2\pi$ and $\int e_n dv_{\mathbf{g}} = 0$.

Gaussian free field (GFF) on $(S, \mathbf{g})$

$$h := \sum_{n=1}^{\infty} \alpha_n e_n, \qquad \{\alpha_n\} \text{ i.i.d. standard Gaussians.}$$

- Convergence holds almost surely in $H^{-1}(S, \mathbf{g})$.
- $\mathbb{E}[h(x)h(y)] = -\log|x-y| + \text{smooth}$.



$h(z)$ is not well defined.

$h_\varepsilon(z)$: average of h over the circle $\{w : |w-z| = \varepsilon\}$.

Simulation of h_ε by H. Jackson.

Random Geometry of γ-LQG

$\gamma \in (0,2)$ h: a variant of GFF on a planar domain D

γ-LQG area

$A_h := e^{\gamma h} \mathrm{d}^2 z := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon} \mathrm{d}^2 z.$

Example of Gaussian multiplicative chaos

Kahane (1985), Duplantier-Sheffield & Rhodes-Vargas, around 2010

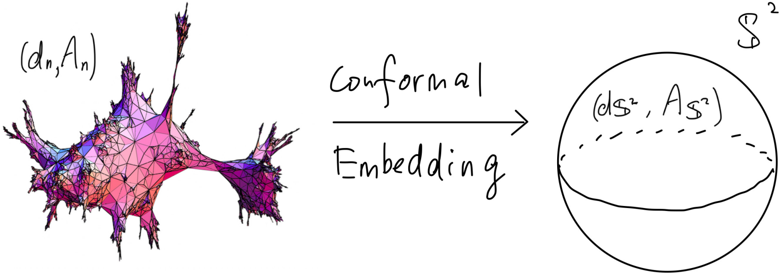
γ-LQG boundary length

$L_h := e^{\frac{\gamma}{2} h} \mathrm{d}^2 z$ on ∂D. (Gaussian multiplicative chaos)

γ-LQG metric

$d_h := e^{\xi_\gamma h}(\mathrm{d}x^2 + \mathrm{d}y^2)$. (more difficult but done)

Dubedat-Ding-Dunlap-Falconet & Gwynne-Miller (2019)

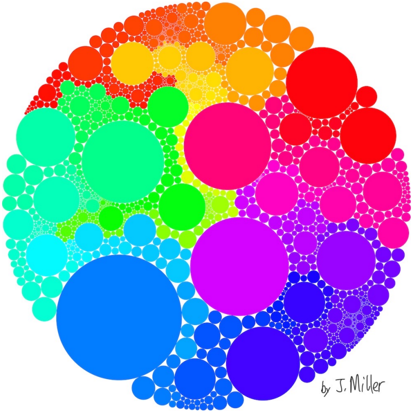


Miller-Sheffield (2015)

There exists a variant of GFF φ on $\mathbb{S}^2$ such that
(d_φ, A_φ) is isometric to Brownian sphere.

If uniform triangulations under conformal embedding converge,
the scaling limit has to be (d_φ, A_φ).

A Similar Story for Random Disk



Brownian disk under conformal embedding = $\sqrt{8/3}$-LQG on $\mathbb{D}$.

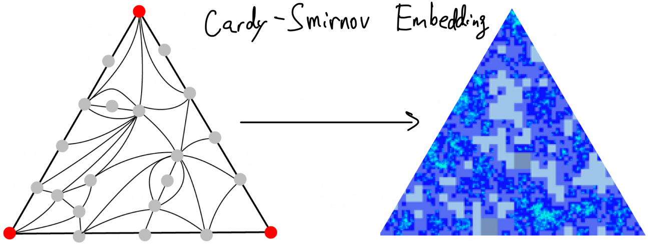
Brownian Disk:
Bettinelli-Miermont (2015)

Circle Packing:
a discrete conformal embedding

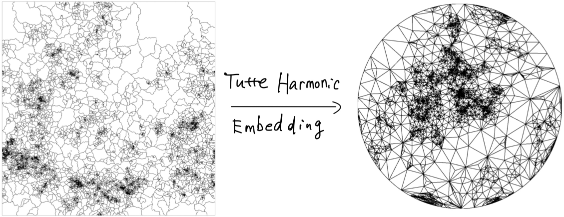
Miller-Sheffield (2015)

There exists a variant of GFF φ on $\mathbb{D}$ such that
$(d_\varphi, A_\varphi, L_\varphi)$ is isometric to the Brownian disk.
If uniform triangulations under conformal embedding converge,
the scaling limit has to be $(d_\varphi, A_\varphi, L_\varphi)$.

Holden-S. (2019)



Gwynne-Miller-Sheffield (2018)



Poisson-Voronoi tessellation of Brownian disk under Tutte embedding

What is the law of φ?

Polyakov (1981)

The conformal factor φ is governed by the Liouville field theory, the 2D quantum field theory defined by Liouville action.

Liouville field theory is a conformal field theory, and is a locally trivial but globally nontrivial perturbation of Gaussian free field.

Polyakov's idea in model math language

The law of φ for Brownian sphere/disk embedded to $(S, \mathbf{g})$ is given by the Liouville field on $(S, \mathbf{g})$ with $\gamma = \sqrt{8/3}$.

2D quantum gravity coupled with a conformal matter

Random surface of the sphere/disk topology sampled from the uniform measure weighted by $\det(\Delta_{S,\mathbf{g}})^{-\mathfrak{c}/2}$

$$\implies \gamma\text{-LQG with } \mathfrak{c} = 25 - 6(\tfrac{\gamma}{2} + \tfrac{2}{\gamma})^2.$$

$\gamma = \sqrt{8/3}, \mathfrak{c} = 0$ $\qquad$ $\gamma \in (0,2), \mathfrak{c} < 1$

Liouville (Conformal) Field Theory

David, Kupiainen, Rhodes, Vargas, Huang, Guillarmou, Remy

Liouville CFT has been constructed rigorously
by making sense of the defining path integral.

Before the Liouville CFT approach, φ for Brownian sphere/disk were constructed by Sheffield via a more geometrical approach.

Aru-Huang-S., Circlé, Ang-Holden-S.

Two approaches agree.

Integrability of 2D CFT — Belavin-Polyakov-Zamolodchikov '84

2D CFT —> local conformal symmetry —> Virasoro algebra
—> exact solution of partition functions/correlation functions.

(Rigorous) Integrability of Liouville CFT

Bulk: Kupiainen-Rhodes-Vargas, Guillarmou-KRV.
Boundary: Remy, Remy-Zhu, Ang-Remy-S., Wu

Random Annulus

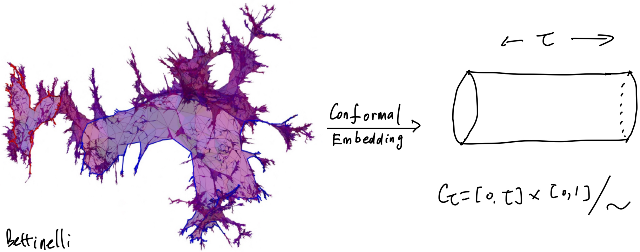
a_n, b_n: even integers with $\lim_{n\to\infty} \frac{a_n}{3n^2} = a$, $\lim_{n\to\infty} \frac{b_n}{3n^2} = b$.
$\mathcal{Q}_n$: set of annular quadrangulations with bdy lengths a_n, b_n.

Sample Q_n from $\mathcal{Q}_n$ with probability $\propto 12^{-\#\text{vertices}}$.
View Q_n as compact metric space: graph distance scaled by $\frac{1}{n}$.

Definition: Brownian annulus with boundary lengths a, b

$\lim_{n\to\infty} Q_n$ in the Gromov-Hausdorff-Prokhorov topology.

Existence follows from work of Betinelli-Miermont.



Question: What's the law of τ?

The Law of the Modulus of the Brownian Annulus

$\mathrm{BA}(a,b)^{\#}$: law of the Brownian annulus with bdy lengths a, b.
Dedekind eta function: $\eta(i\tau) = e^{-\frac{\pi\tau}{12}} \prod_{k=1}^{\infty}(1 - e^{-2\pi k\tau})$
ρ_τ: density function for the positive random variable X_τ s.t.
$\mathbb{E}[X_\tau^{it}] = \frac{2\pi t e^{-2\pi\tau t^2/3}}{3\sinh(2\pi t/3)}$.

Ang-Remy-S. '22

$$\mathrm{BA}(a,b)^{\#}[\tau \in I] = \int_I \eta(i2\tau)\rho_\tau\left(\frac{b}{a}\right)\, \mathrm{d}\tau, \qquad \forall I \subset (0,\infty).$$

$\mathrm{BA} = \iint_0^\infty \frac{1}{\sqrt{ab(a+b)}}\mathrm{BA}(a,b)^{\#}$. (free boundary)
LF_τ: pushforward of $\mathbb{P}_\tau \times \mathrm{d}x$ under $(h,x) \mapsto \varphi = h + x$.
$\mathbb{P}_\tau$: law of GFF on $\mathcal{C}_\tau$. $\mathrm{d}x$: Lebesgue measure on $\mathbb{R}$

Ang-Remy-S. '22

$$\mathrm{BA} = \int_0^\infty (\sqrt{2})^{-1}\eta(2i\tau)\mathrm{LF}_\tau(\mathrm{d}\varphi)\mathrm{d}\tau.$$

Polyakov '81, David '88, Distler-Kawai '89

2D QG coupled with conformal matter can be decomposed into

$$\underbrace{\mathcal{Z}_{\text{matter}}(\tau)}_{\text{matter CFT}} \times \underbrace{\mathcal{Z}_{\text{GFF}}(\tau)\text{LF}_\tau(d\phi)}_{\text{Liouville CFT}} \times \underbrace{\mathcal{Z}_{\text{ghost}}(\tau)}_{\text{ghost CFT}}\, d\tau.$$

$\mathcal{Z}_{\text{GFF}}(\tau) := \frac{1}{\sqrt{2}\eta(2i\tau)}$. $\mathcal{Z}_{\text{ghost}}(\tau) := \eta(2i\tau)^2$.

$\mathcal{Z}_{\text{GFF}}(\tau)\mathcal{Z}_{\text{ghost}}(\tau) = \eta(2i\tau)/\sqrt{2}$. Brownian annulus: $\mathcal{Z}_{\text{matter}} \equiv 1$.

Origin of 26 in String Theory — Polyakov '81

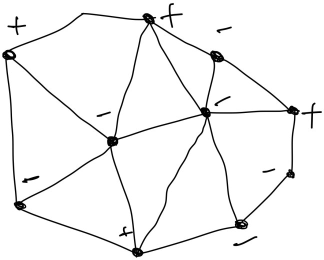
$$\mathcal{Z}_{\text{matter}}(\mathcal{C}_\tau, \mathbf{g}) = \mathcal{Z}_{\text{matter}}(\tau)\det(\Delta_{\mathbf{g}})^{-\mathfrak{c}/2}$$

$$\mathcal{Z}_{\text{Liouville}}(\mathcal{C}_\tau, \mathbf{g}) = \mathcal{Z}_{\text{Liouville}}(\tau)\det(\Delta_{\mathbf{g}})^{-c_L/2}, \quad c_L = 1 + 6(\tfrac{\gamma}{2} + \tfrac{2}{\gamma})^2.$$

$$\mathcal{Z}_{\text{ghost}}(\mathcal{C}_\tau, \mathbf{g}) = \mathcal{Z}_{\text{ghost}}(\tau)\det(\Delta_{\mathbf{g}})^{-(-26)/2}.$$

$$\mathfrak{c} + c_L + (-26) = 0 \implies \mathfrak{c} = 25 - 6(\tfrac{\gamma}{2} + \tfrac{2}{\gamma})^2.$$

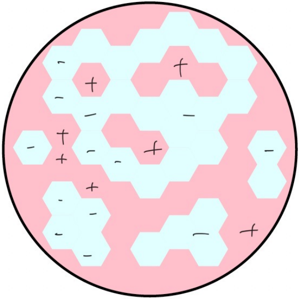
Ghost field: non-physical, come from gauge fixing.



Ising Model on a graph.

Hamiltonian: $H(\sigma) = \sum_{i \sim j} \sigma_i \sigma_j$.

Partition function:
$Z(T) = \sum_\sigma e^{-H(\sigma)/T}$.



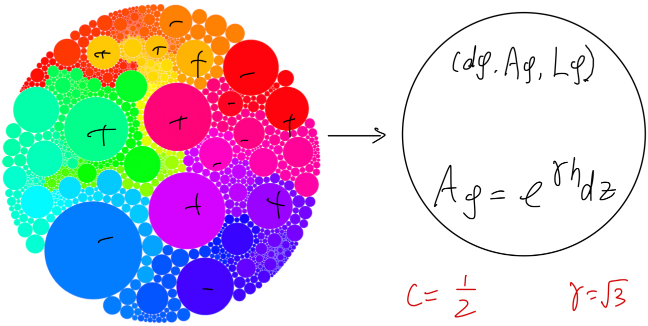
Ising Model on 2D lattice.

T_c: critical temperature
for phase transition.

$Z(T_c) \sim (\det \Delta)^{-1/4}$
$\mathfrak{c}_{\text{Ising}} = 1/2$.

Physics: 2D QG coupled with conformal matter

Surface from uniform measure weighted by $\det(\Delta)^{-\mathfrak{c}/2}$ converge to γ-LQG with $\mathfrak{c} = 25 - 6(\frac{\gamma}{2} + \frac{2}{\gamma})^2$.



Math: Scaling Limit Conjecture

Random triangulation weighted by Ising partition function under conformal embedding converge to $\sqrt{3}$-LQG.

CFT and 2D Statical Physics

Many 2D statistical physics model (e.g. Ising) enjoys conformal symmetry at their criticality.

Correlation function governed by a CFT.
Partition function $\sim (\det \Delta)^{-\mathfrak{c}/2}$ with $\mathfrak{c} < 1$.
$\mathfrak{c}$: central charge of the corresponding CFT.

BPZ, Cardy et. al. See book of Di Francesco-Mathieu-Sénécha.

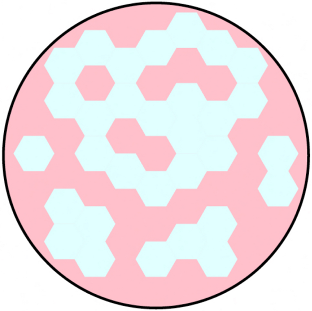
Scaling Limit Conjecture: 2D QG+conformal matter

Random planar map weighted by statistical physics models under conformal embedding converge to

$$\underbrace{\mathcal{Z}_{\text{matter}}(\tau)}_{\text{matter CFT}} \times \underbrace{\mathcal{Z}_{\text{GFF}}(\tau)\text{LF}_\tau(d\phi)}_{\text{Liouville CFT}} \times \underbrace{\mathcal{Z}_{\text{ghost}}(\tau)}_{\text{ghost CFT}}\ d\tau.$$

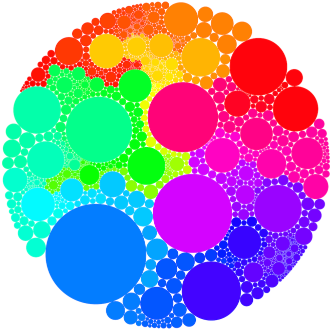
$\mathfrak{c} = 25 - 6(\frac{\gamma}{2} + \frac{2}{\gamma})^2$.

Knizhnik-Polyakov-Zamolodchikov (KPZ) Relation



Question: if we have n^2 vertices on the lattice, what is the size of the boundary connecting cluster?

Answer: $\sim n^{91/48}$



On weighted random triangulation, the answer is $n^{\text{some quantum exponent}}$.

$91/48 = \mathrm{KPZ}(\text{quantum exponent}).$

Physics Verification of the Scaling Limit Conjecture

In physics, scaling exponent can be computed by e.g.

- Algebraic CFT principle (BPZ, Cardy)
- Coulomb gas method (Nienhuis, di Francesco-Saleur-Zuber)
- Transfer matrix/quantum group (Andrews, Baxter, and Forrester)

KPZ '88, David '88, Distler-Kawai '89

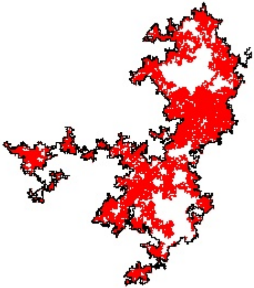
Derived KPZ relation from matter-Liouville-ghost decomposition of 2D quantum gravity coupled with conformal matter.

Douglas, Gross, Kazakov, Kostov, Migdal, Shenker ... $\sim 90s$

Derive KPZ relation for random planar maps weighted by statistical physics models using enumeration techniques for planar maps such as random matrix.

Two approaches give the same exponents.

KPZ relation provides a powerful framework to study fractals.



Mandelbrot (1982) conjectured that the frontier of the planar Brownian motion has fractal dimension $4/3$.

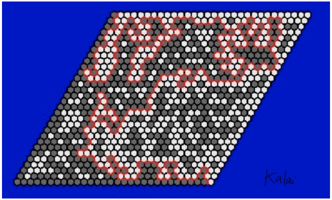
(Image by Schramm)

Duplantier (1998): Physics "Proof" of Mandelbrot's conjecture

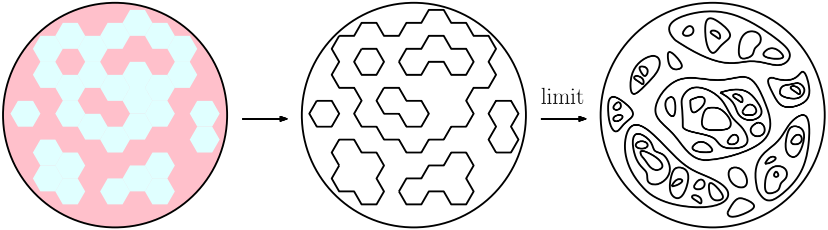
The quantum dimension of the Brownian frontier on the Brownian disk is $1/2$. $KPZ(1/2) = 4/3$. □

Many predictions on scaling dimensions in physics via CFT and/or KPZ are derived by Schramm-Loewner evolution (SLE). e.g. Mandelbrot conjecture: Lawler-Schramm-Werner (2000).

SLE and Conformal Loop Ensemble



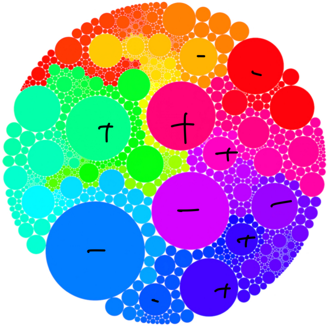
Schramm (1999)
Random interfaces in many 2D statical physics models should converge to SLE_κ with $\kappa > 0$.



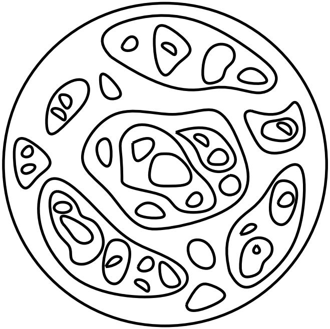
A few scaling limit results, many more conjectures.

Percolation $\to$ SLE_6, Ising model $\to$ CLE_3 (Smirnov et. al.)

$\mathfrak{c} = 25 - 6(\frac{\sqrt{\kappa}}{2} + \frac{2}{\sqrt{\kappa}})^2$ $\kappa = 6, \mathfrak{c} = 0;$ $\kappa = 3, \mathfrak{c} = \frac{1}{2}.$



Triangulation + Ising



CLE$_3$ + $\sqrt{3}$-LQG

Scaling Limit Conjecture: 2D QG + conformal matter

Random planar map + statistical physics models under conformal embedding converge to (SLE$_\kappa$/CLE$_\kappa$) + (γ-LQG).

$\mathfrak{c} = 25 - 6(\frac{\sqrt{\kappa}}{2} + \frac{2}{\sqrt{\kappa}})^2$ $\qquad$ $\mathfrak{c} = 25 - 6(\frac{\gamma}{2} + \frac{2}{\gamma})^2$.

Holden-S. '19: triangulation+ percolation, Cardy-Smirnov embedding

Although the scaling limit conjecture is open in most cases, the limiting object: SLE/CLE + LQG is well understood:

- quantum zipper — Sheffield (2010)
- mating of trees — Duplantier-Miller-Sheffield (2014)

Applications include

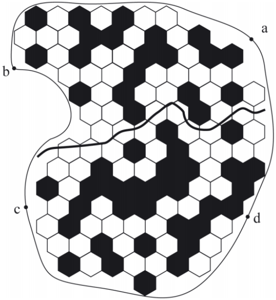
- all scaling limit results on random planar maps $\to$ LQG.
- rigorous KPZ derivation of SLE exponents/dimensions.
  Duplantier-Sheffield, Gwynne-Holden-Miller.

Proof ingredients for — Ang-Remy-S. '22

$$\mathrm{BA} = \int_0^\infty \frac{1}{\sqrt{2}\eta(2i\tau)} \mathrm{LF}_\tau(d\varphi)d\tau.$$

- SLE/CLE + LQG.
- Integrability of random planar maps. (Bernardi-Fusy)
- Integrability of boundary Liouville CFT. (Remy, Wu)

The predictive power of CFT goes beyond exponent/dimension.

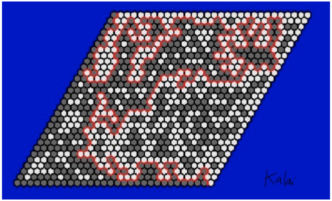


Cardy's formula for percolation

$\frac{2\Gamma(2/3)}{\Gamma(1/3)^2} \times F(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}, z)$

z: cross ratio
F: hypergeometric function.

Proof: Cardy-Smirnov embedding.



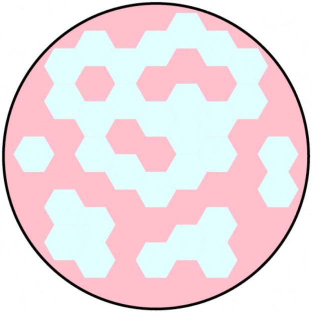
Cardy formula for SLE_6

expresses the probability of an SLE_6 event.

Ito's calculus problem.

Similar results: Watts' formula, Schramm's formula, etc.

Cardy's formula for CLE$_\kappa$ on Annulus



O(n)-loop model (dilute phase)

$$Z(T_c) = \sum_{\text{loop collection}} e^{\frac{-1}{T_c}\text{total length}} n^{\#loops}$$

$g = \frac{4}{\kappa}$, $\mathfrak{c} = 25 - 6(\frac{\sqrt{\kappa}}{2} + \frac{2}{\sqrt{\kappa}})^2$

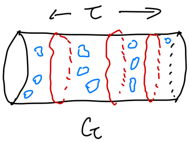
$\chi = (1-g)\pi$ $n = 2\cos\chi \in (0, 2]$

Ising model: $n = 1$

Cardy's formula for # of non-contratible loops of CLE on $\mathcal{C}_\tau$

$\mathbb{E}[(n'/n)^{\mathcal{N}}] = Z(\tau, \kappa, \chi')/Z(\tau, \kappa, \chi),$ $n' = 2\cos\chi'.$

$q = e^{-\pi/\tau}, \tilde{q} = e^{-2\pi\tau}$



$$Z(\tau, \kappa, \chi')$$

$$= q^{\frac{-c}{24}} \prod_{r=1}^{\infty} (1-q^r)^{-1} \sum_{p\in\mathbb{Z}} \frac{\sin(p+1)\chi'}{\sin\chi'} q^{\frac{gp^2}{4} - \frac{(1-g)p}{2}}$$

$$= (\frac{2}{g})^{1/2} \tilde{q}^{-\frac{c}{12}} \prod_{r=1}^{\infty} (1-\tilde{q}^{2r})^{-1} \sum_{m\in\mathbb{Z}} \frac{\sin((\chi'+2m\pi)/g)}{\sin\chi'} \tilde{q}^{\frac{(\chi'+2\pi m)^2}{2\pi^2 g} - \frac{(1-g)^2}{2g}}$$

Cardy (2006)

$Z(\kappa,\tau,\chi')$ is the partition function of the $O(n)$ loop model on $\mathcal{C}_\tau$. via (non-rigorous) Coulomb gas applied to lattice on $\mathcal{C}_\tau$

$Z(\tau,\kappa,\chi')$ $\qquad$ $q = e^{-\pi/\tau}$ and $\tilde{q} = e^{-2\pi\tau}$

$$= \frac{q^{\frac{-c}{24}}}{\prod_{r=1}^{\infty}(1-q^r)} \sum_{p\in\mathbb{Z}} \frac{\sin(p+1)\chi'}{\sin\chi'} q^{\frac{gp^2}{4}-\frac{(1-g)p}{2}}$$

$$= \frac{(\frac{2}{g})^{1/2}\tilde{q}^{-\frac{c}{12}}}{\prod_{r=1}^{\infty}(1-\tilde{q}^{2r})} \sum_{m\in\mathbb{Z}} \frac{\sin((\chi'+2m\pi)/g)}{\sin\chi'} \tilde{q}^{\frac{(\chi'+2\pi m)^2}{2\pi^2 g}-\frac{(1-g)^2}{2g}}.$$

$g = \frac{4}{\kappa}$ $\qquad$ $\mathfrak{c} = 25 - 6(\frac{\sqrt{\kappa}}{2} + \frac{2}{\sqrt{\kappa}})^2,$

$\chi = (1-g)\pi,$ $\qquad$ $n = 2\cos\chi \in (0,2]$

Ang-Remy-S. (2022)

Cardy's formula for CLE on annulus holds.

Hard to prove by Ito's calculus. No canonical bdy point to start.

Proof: Set Matter to be CLE in the decomposition:

$$\underbrace{\mathcal{Z}_{\mathrm{matter}}(\tau)}_{\text{matter CFT}} \times \underbrace{\mathcal{Z}_{\mathrm{GFF}}(\tau)\mathrm{LF}_\tau(d\phi)}_{\text{Liouville CFT}} \times \underbrace{\mathcal{Z}_{\mathrm{ghost}}(\tau)}_{\text{ghost CFT}}\ d\tau.$$

Use the same method for the Brownian annulus. □

Matter can set to be any model with SLE/CLE as scaling limit.

$$Z_{\text{self avoiding loop}}(\tau) = \prod_{r=1}^{\infty}(1-q^r)^{-1}\sum_{k\in\mathbb{Z}} k(-1)^{k-1}q^{\frac{3k^2}{2}-k+\frac{1}{8}}.$$

- Werner (2005): construction of self-avoiding loop; $\tau \to \infty$ asymptotic of $Z(\tau)$.
- Cardy (2006) conjectural formula; SAW = O(0) model.
- Ang-Remy-S. (2022): rigorous proof.

Summary

Random sphere/disk and their relation to LQG are well understood mudulo scaling limit conjectures.

A new difficuty arises for annulus: the random modulus. Ang-Remy-S. (2022) solves the Brownian annulus by establishing the matter-Liouville-ghost decomposition.

CFT preditions for SLE are relatively well understood. Much less has been proved for exact formulae predicted by CFT for CLE. Ang-S. (2021), Ang-Remy-S. (2022)

Outlook

- Arbirary surface: measures on the moduli space.
- Full CFT content of CLE: conformal boostrap.

Bigger Picture: Integrabilty in Conformal Probability

Two concrete goals among many others

1. Brownian surface moduli = ghost partition function.
2. Connectivity and loop statistics of CLE = CFT prediction.

What have been very useful so far

1. coupling of SLE/CLE and finite volume LQG surfaces.
2. Integrability of Liouville CFT.

What need to be done?

1. Continue to develop SLE/LQG coupling and their connection to Liouville CFT. (Ang-S.-Yu, in progress)
2. Better understanding of conformal blocks (Ghosal-Remy-Sun-S. (2020), in progress.)
3. More guidance from physics/geometry/algebra. ghost field; transfer matrix/quantum group, topo recursion, ???