Scattering of harmonic functions and forms in quasicircles

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The Analysis and Geometry of Random Spaces

The bigger picture

General programme: understand conformal invariants associated to nested surfaces and its relation to the global geometry of moduli spaces of nested surfaces.

This “nesting” appears in:

- classical geometric function theory (image of a conformal map f in its codomain)
- conformal field theory (sewing)
- quasiconformal Teichmüller theory (univalent function model, conformal welding $=$sewing).

What's in this talk

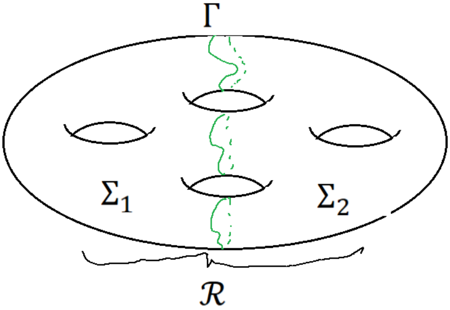
How do harmonic objects interact with irregular ($=$ quasicircles) seams?

1. "overfare" of harmonic forms and functions
2. Faber and Grunsky operators
3. comparison integral operators of Schiffer
4. scattering matrix

Overfare

Geometric sketch of overfare

Setup: A compact Riemann surface $\mathcal{R}$, separated into Σ_1 and Σ_2 by a complex of quasicircles $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$.



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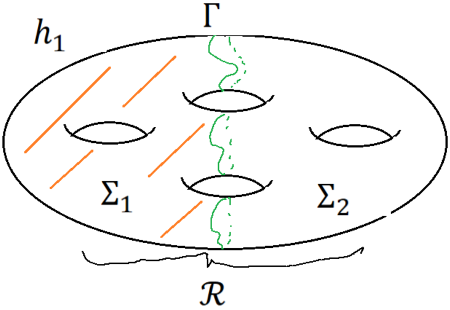

h_1 harmonic on Σ_1

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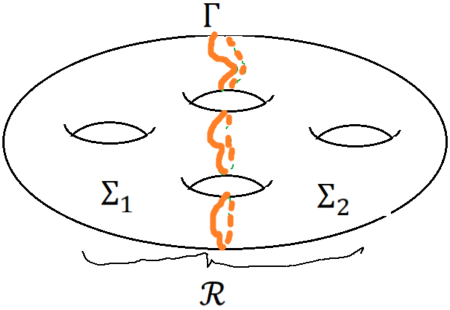

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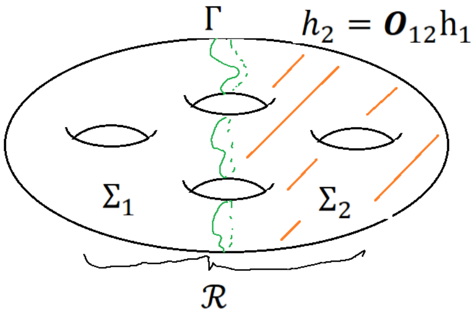


h_1 harmonic on $\Sigma_1 \mapsto h_2 = \mathbf{O}_{12} h_1$ harmonic on Σ_2.

Bergman spaces

Hodge star operator: One-form locally $a(z)dx + b(z)dy$ ($z = x + iy$)

$$*\,(a(z)dx + b(z)dy) = a(z)dy - b(z)dx.$$

$$(\alpha, \beta)_\Sigma = \iint_\Sigma \alpha \wedge *\overline{\beta}, \quad \|\alpha\|^2 = (\alpha, \alpha).$$

Bergman spaces of one-forms:

$$\begin{aligned} \mathcal{A}_{\text{harm}}(\Sigma) &= \{\alpha \ \text{ harm one form} : \|\alpha\| < \infty\} = \mathcal{A}(\Sigma) \oplus \overline{\mathcal{A}(\Sigma)} \\ \mathcal{A}(\Sigma) &= \{\alpha \in \mathcal{A}_{\text{harm}}(\Sigma) : \alpha \ \text{ holomorphic}\} \\ \overline{\mathcal{A}(\Sigma)} &= \{\alpha \in \mathcal{A}_{\text{harm}}(\Sigma) : \alpha \ \text{ antiholomorphic}\} \end{aligned}$$

pullback: $f^*(h(z)dz) = h(f(z))f'(z)dz$, $f^*\overline{h(z)}d\bar{z} = \overline{h(f(z))f'(z)}d\bar{z}$.

Dirichlet spaces

Dirichlet spaces of functions:

$$\mathcal{D}_{\mathrm{harm}}(\Sigma) = \{h : \Sigma \to \mathbb{C} : dh \in \mathcal{A}_{\mathrm{harm}}(\Sigma)\}$$
$$\mathcal{D}(\Sigma) = \{h : \Sigma \to \mathbb{C} : dh \in \mathcal{A}(\Sigma)\}$$
$$\overline{\mathcal{D}(\Sigma)} = \{\overline{h} : \Sigma \to \mathbb{C} : d\overline{h} \in \overline{\mathcal{A}(\Sigma)}\}.$$

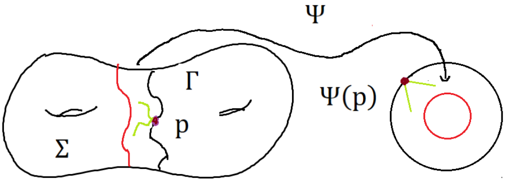
$$(h, g)_\Sigma = (dh, dg)_\Sigma$$

- the norm is now a semi-norm
- In a planar domain, these reduce to the usual Bergman and Dirichlet spaces.

pullback: $f^* h = h \circ f$.

Conformally non-tangential limits

Let Γ be a Jordan curve with a biholomorphism ψ of a collar of Γ in Σ.

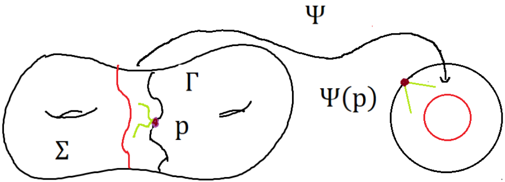


Definition (CNT limits)

The CNT limit of a function $h : \Sigma \to \mathbb{C}$ at p is ζ if $h \circ \psi^{-1}$ has a non-tangential limit of ζ at $\psi(p)$

- This is independent of the choice of ψ.
- This is the same as treating Γ as the abstract border of the Riemann surface Σ.

Null sets



Definition

We say that a set $I \subset \Gamma$ is null if it is a Borel set and $\psi(I)$ has logarithmic capacity zero in $\mathbb{S}^1$.

- This is independent of the choice of ψ, not independent of the side!

Theorem (S Staubach, 2020)

A function $h \in \mathcal{D}_{\mathrm{harm}}(\Sigma)$ has CNT boundary values except possibly on a null set.

- follows from Beurling's classical theorem, and a bit of easy surgery.

Bounded overfare theorem

Let $\mathcal{R}$ be a compact Riemann surface, separated by a complex of quasicircles Γ into Σ_1 and Σ_2.

Theorem (S & Staubach, 2021)

$I \subset \Gamma$ is null with respect to Σ_1 if and only if it is null with respect to Σ_2. Given any $h_1 \in \mathcal{D}_{\mathrm{harm}}(\Sigma_1)$, there is a unique $h_2 \in \mathcal{D}_{\mathrm{harm}}(\Sigma_2)$ whose boundary values agree with those of h_1 except possibly on a null set.

Theorem (S & Staubach, 2021)

Assume that Σ_1 is connected. The overfare operator

$$\mathbf{O}_{12} : \mathcal{D}_{\mathrm{harm}}(\Sigma_1) \rightarrow: \mathcal{D}_{\mathrm{harm}}(\Sigma_2)$$

is bounded with respect to the Dirichlet semi-norm.

Earlier versions

- in the plane, S & Staubach 2017 (note: existence of a bounded overfare characterizes quasicircles!)
- Riemann surfaces with a single curve Γ S & Staubach 2020.

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Nomenclature problems:
In 2017: "~~harmonic reflection~~"
For a while after: "~~transmission~~"
Finally settled on "overfare".

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Results in the sphere

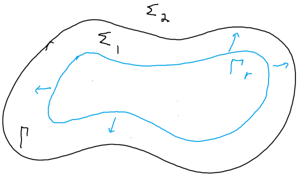
Faber and Grunsky operators

Cauchy operator

- $\mathcal{R}$ = Riemann sphere, Γ a Jordan curve separating into Σ_1, Σ_2
- Let $\Gamma_r = f(|z| = r)$ where $f : \mathbb{D} \to \Sigma_1$ is conformal

Define for $h \in \mathcal{D}_{\mathrm{harm}}(\Sigma_1)$

$$\mathbf{J}_{1k} h = \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{\Gamma_r} \frac{h(w)}{w - z}\, dw \quad z \in \Sigma_k.$$



$\mathbf{J}_{1k} : \mathcal{D}_{\mathrm{harm}}(\Sigma_1) \to \mathcal{D}(\Sigma_k)$ is bounded.

Faber operator

The **Faber operator** (e.g. Suetin 95) is

$$\mathbf{I}_f = \mathbf{J}_{12}(f^{-1})^* : \overline{\mathcal{D}_0(\mathbb{D})} \to \mathcal{D}_\infty(\Sigma_2)$$

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Faber polynomials: $\Phi_n(z) := \mathbf{I}_F(\bar{z}^n) \in \mathbb{C}_\infty[1/z]$.

If Γ and H are nice enough, the **Faber series** converges to $\mathbf{I}_F H$.

$$\mathbf{I}_F(\sum_{n=1}^{\infty} H_n \bar{z}^n) = \sum_{n=1}^{\infty} H_n \Phi_n$$

where you can fill in the meanings of the blue words in various ways.

Faber isomorphism

Theorem ("iff" Shen 09, "if" Çavuş 1996)

For a domain Ω_-

$$\ell^2 \ni \lambda_n \mapsto \sum \lambda_n/\sqrt{n}\,\Phi_n$$

is an isomorphism onto $\mathcal{D}_\infty(\Sigma_2)$ *if and only if its boundary is a quasicircle.*

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Let Γ *be a rectifiable Jordan curve. Then the Faber operator is an isomorphism if and only if* Γ *is a rectifiable quasicircle.*

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Grunsky operator

Given $f : \mathbb{D} \to \Sigma_1$ conformal, the Grunsky operator is the unique bounded extension of

$$\mathbf{P}_0(\mathbb{D}) f^* \mathbf{O}_{21} \mathbf{I}_f : \overline{\mathbb{C}_0[z]} \to \mathcal{D}_0(\mathbb{D})$$

to $\overline{\mathcal{D}_0(\mathbb{D})}$ where $\mathbf{P}_0(\mathbb{D})$ is projection

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Letting $\hat{\mathbf{Gr}}_f = d \mathbf{Gr}_f d^{-1} : \overline{\mathcal{A}(\mathbb{D})} \to \mathcal{A}(\mathbb{D})$
This is equivalent to the integral operator (Bergman and Schiffer, 1950)

$$[\hat{\mathbf{Gr}}_f \overline{h}](z) = \frac{1}{\pi} \iint_{\mathbb{D}} \left(\frac{f'(\zeta) f'(z)}{(f(\zeta) - f(z))^2} - \frac{1}{(\zeta - z)^2} \right) \overline{h(\zeta)} dA_\zeta.$$

The Ubiquitous Quasicircle

Let Γ be a Jordan curve. TFAE:

1. Γ is a quasicircle.
2. $\mathbf{J}_{12} : \overline{\mathcal{D}_0(\Sigma_1)} \to \mathcal{D}_\infty(\Sigma_2)$ is an isomorphism. (Napalkov and Yulmukhametov, 01).
3. The Faber operator is an isomorphism (Shen 09 & Çavuş 96).
4. There exists a bounded transmission (S & Staubach 17).
5. $\|\mathbf{Gr}_f\| < 1$ (Pommerenke 75, Kühnau 71).

Survey paper: S & Staubach, 2021.

Grunsky literature sample

- The integral operator form of the Grunsky “matrix” (Bergman and Schiffer 50)
- Grunsky operator properties and their relation to regularity of the domain/mapping function (Gavin Jones 98, Shen 07)
- KYNS period map is given by Grunsky matrix (Takhtajan and Teo 06)
- Extension to L^p (Baranov and Hedenmalm 08)

What is the Grunsky operator?

Theorem (Radnell, S & Staubach 17, 20, S & Staubach 19)

If Γ is a quasicircle then $\overline{\mathbf{P}_0(\mathbb{D})}\, f^*\, \mathbf{O}_{21} = \mathbf{I}_f^{-1}$. *So*

$$\begin{aligned} f^*\mathbf{O}_{21}\mathcal{D}_\infty(\Sigma_2) &= f^*\mathbf{O}_{21}\mathbf{I}_f\overline{\mathcal{D}_0(\mathbb{D})} \\ &= \text{Graph}\,\mathbf{Gr}_f. \quad \textit{(mod constants)} \end{aligned}$$

Proof.

$$\begin{aligned} f^*\mathbf{O}_{21}\mathcal{D}_\infty(\Sigma_2) &= \left(\overline{\mathbf{P}_0(\mathbb{D})} + \mathbf{P}_0(\mathbb{D})\right) f^*\mathbf{O}_{21}\mathbf{I}_f\overline{\mathcal{D}_0(\mathbb{D})} \\ &= (\mathbf{Id} + \mathbf{Gr}_f)\,\overline{\mathcal{D}_0(\mathbb{D})} \end{aligned}$$

□

- $\{\bar{z}^n + \mathbf{Gr}_f(\bar{z}^n)\}$ is a basis for $f^*\mathcal{D}_\infty(\Sigma_2)$.

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Grunsky in pictures

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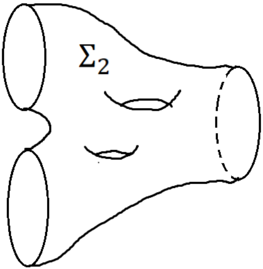

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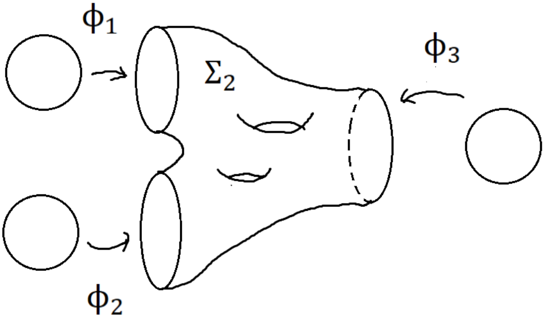

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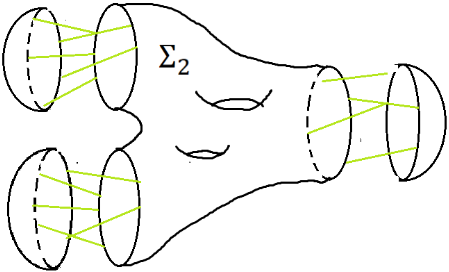

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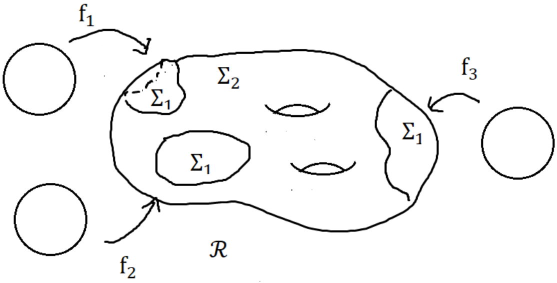

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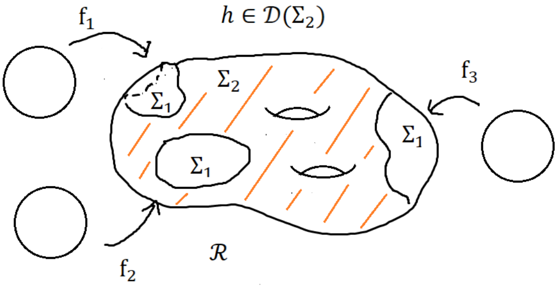

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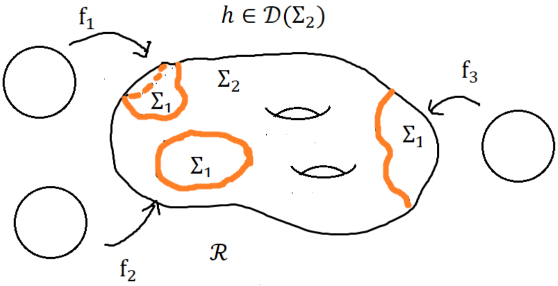

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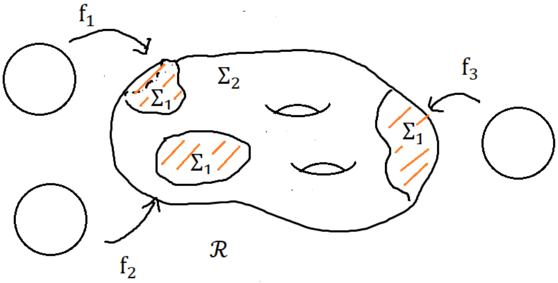

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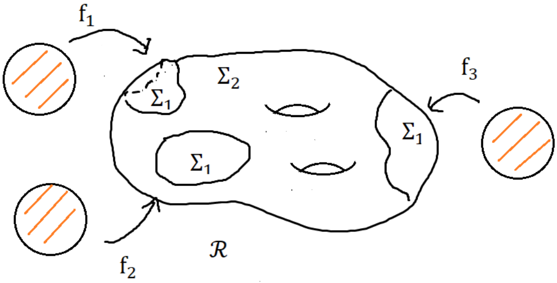

$$f^* \mathbf{O}_{21}\, h \in \mathcal{D}_{harm}(\mathbb{D}) \oplus \mathcal{D}_{harm}(\mathbb{D}) \oplus \mathcal{D}_{harm}(\mathbb{D})$$

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Grunsky in general

- several quasicircles in the sphere Radnell, S & Staubach 20

- general Faber isomorphism and Grunsky operator for genus $g > 0$, many quasicircles, Shirazi 20

Big part of the picture: period mapping

• A chain of papers (Kirillov-Yuri'ev 86, Nag-Sullivan 95, Takhtajan and Teo 06) shows: the Grunsky operator in the plane is an analogue for the Teichmüller space of the disk of the period mapping for compact surfaces.

Classically this period map lies in the Siegel disk

$$\{Z \; n \times n \text{ matrices} : Z^t = Z, \mathrm{Id} - \overline{Z}Z > 0\}$$

associated to the period mapping.

The period matrices generate a "polarization" (a certain Lagrangian decomposition of a symplectic space).

The map

$$\mathrm{Teich}(\mathbb{D}) = \mathsf{QS}(\mathbb{S}^1)/\mathsf{M\ddot{o}b}(\mathbb{S}^1) \to \mathbf{Gr}_f$$

is the “KYNS period map”, which has analogues with the classical period map of compact surfaces.

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One of our motivations: find a way to unify the KYNS period map with the period map of compact surfaces.

Schiffer comparison operators

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Green's function

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Definition

Green's function on R (normalized at w_0) is

1. $g(w; z, q)$ is harmonic in w on $R \backslash \{z, q\}$;
2. for a local coordinate ϕ on an open set U containing z, $g(w; z, q) + \log |\phi(w) - \phi(z)|$ is harmonic for $w \in U$;
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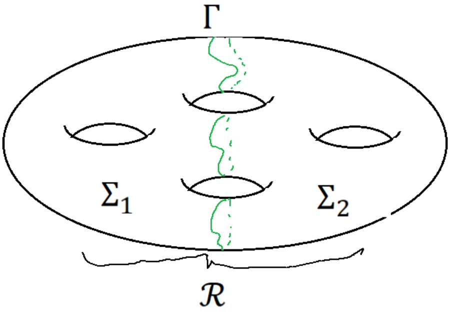
Condition (4) is not essential; we consider only derivatives of g.

The picture from now on



- Γ consists of quasicircles (unless specified to be a general Jordan curve)
- Σ_2 is connected
- Σ_1 might not be connected.

The Schiffer comparison operators: kernels

Bergman kernel:

$$K(z,w) = -\frac{1}{\pi i}\partial_z \overline{\partial}_w g(w;z,q)$$

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Important point:
- kernels depend on entire surface $\mathcal{R}$
- .... but we will integrate only over Σ_1 or Σ_2

Schiffer comparison operators

$$\begin{aligned}\mathbf{T}(\Sigma_j; \Sigma_1 \sqcup \Sigma_2) : \overline{\mathcal{A}(\Sigma_j)} &\to \mathcal{A}(\Sigma_1 \sqcup \Sigma_2) \\ \overline{\alpha(w)} &\mapsto \iint_{\Sigma_j} L(z,w) \wedge_w \overline{\alpha(w)}.\end{aligned}$$

as a principal value integral. Also denote

$$\begin{aligned}\mathbf{T}_{jk} : \overline{\mathcal{A}(\Sigma_j)} &\to \mathcal{A}(\Sigma_k) \\ \overline{\alpha} &\mapsto \mathbf{T}(\Sigma_j, \Sigma_1 \sqcup \Sigma_2)\overline{\alpha}\Big|_{\Sigma_k}\end{aligned}$$

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and

$$\begin{aligned}\mathbf{S}_j : \mathcal{A}(\Sigma_j) &\to \mathcal{A}(\mathcal{R})\\ \alpha(w) &\mapsto \iint_{\Sigma_j} K(z,w)\wedge_w \alpha(w).\end{aligned}$$

Example

Let $\mathcal{R} = \overline{\mathbb{C}}$, Σ_1 is a simply-connected domain in the plane, Σ_2 its complement in $\overline{\mathbb{C}}$.

$$g(w, \infty; z, q) = \log \frac{|w - z|}{|w - q|}.$$

Then

$$\mathbf{T}_{1k}\,\overline{h(w)}d\bar{w} = -\frac{1}{\pi}\iint_{\Sigma_1} \frac{\overline{h(w)}}{(w-z)^2} dA_w \quad z \in \Sigma_k$$

and $K = 0$ so $\mathbf{S}_1 = 0$.

A bit of history

- These operators arise in geometric function theory and are related to the theory of conformal maps, Plemelj-Sokhotski jump formula, Grunsky inequalities, Fredholm eigenvalues etc (e.g. Bergman and Schiffer 1950; Schiffer 1981)
- Multiply-connected case, e.g. Schiffer 1957, Schiffer Springer 1965
- Case when outer domain is not the sphere, Schiffer 1950
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A bit of history

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Some isomorphisms

Theorem (Napalkov and Yulmukhametov, 01)

Let $\mathcal{R}$ be the Riemann sphere, Σ_1, Σ_2 simply connected, Γ a Jordan curve. Then $\mathbf{T}_{12} : \overline{\mathcal{A}(\Sigma_1)} \to \mathcal{A}(\Sigma_2)$ is an isomorphism if and only if Γ is a quasicircle.

remark:

- the Faber operator on forms is just $\hat{\mathbf{I}}_f = \mathbf{T}_{12}(f^{-1})^*$
- the Grunsky operator on forms is $\hat{\mathbf{Gr}}_f = f^* \mathbf{T}_{11}(f^{-1})*$.

Isomorphisms continued

Let $\mathbf{R}_k : \mathcal{A}(\mathcal{R}) \to \mathcal{A}(\Sigma_k)$ denote restriction.

Let $\mathcal{A}_e(\Sigma)$ denote exact one-forms in the Bergman space.

Theorem (S& Staubach 20, Shirazi 20, S & Staubach, 2021)

If Σ_2 is connected, then

$$\mathbf{T}_{12} : \left[\overline{\mathbf{R}}_1 \overline{\mathcal{A}(\mathcal{R})}\right]^{\perp} \to \mathcal{A}_e(\Sigma_2)$$

is an isomorphism.
Furthermore $\|\mathbf{T}_{11}\| < 1$.

A Faber isomorphism for one-forms

Let $\mathcal{A}^{se}(\Sigma_2)$ be those L^2 harmonic one-forms with zero periods on the boundary curves.

Theorem

If Σ_2 is connected, and Σ_1 consists of n disjoint simply connected domains bounded by quasicircles, then

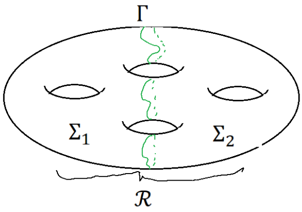
$$\begin{aligned}\Theta : \overline{\mathcal{A}(\Sigma_1)} \oplus \mathcal{A}(\mathcal{R}) &\to \mathcal{A}^{\mathrm{se}}(\Sigma_2) \\ (\overline{\gamma}, \tau) &\mapsto -\mathbf{T}_{1,2}\overline{\gamma} + \mathbf{R}_2\tau\end{aligned}$$

is a bounded isomorphism.

Remark Using this, one can give a general Grunsky operator unifying the Kirillov-Yuriev-Nag-Sullivan period mapping with the classical period mapping for compact surfaces [S Staubach 21].

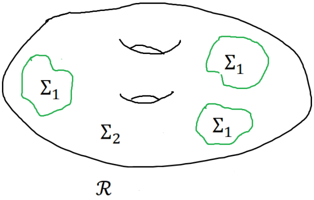
Index theorems for conformally invariant operators

S & Staubach 2021: If Σ_1, Σ_2 connected, of genus g_1 and g_2,



then $\mathrm{index}(\mathbf{T}_{12}) = g_1 - g_2$.

If Σ_2, connected of genus g, Σ_1 is n disjoint simply connected regions,



then $\mathrm{index}(\mathbf{T}_{12}) = 1 - n + g$.

Scattering

Overfare of one-forms in quasicircles

Exact overfare of forms

Overfare of exact harmonic one-forms is defined by

$$\mathbf{O}^e_{21} := d\mathbf{O}_{21}d^{-1}.$$

The boundary values of a harmonic one-form **do not** uniquely specify the one-form. We use "catalyzing forms" in $\mathcal{A}_{harm}(\mathcal{R})$ to specify the cohomology.

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Definition

Let Σ_2 be connected, Let $\mu_2 \in \mathcal{A}_{\mathrm{harm}}(\Sigma_2)$.
Assume $\zeta \in \mathcal{A}_{\mathrm{harm}}(\mathcal{R})$ is such that $\mu_2 - \zeta$ is exact.
Then the overfare of μ_2 with respect to the catalyzing form ζ is

$$\mu_1 := \mathbf{O}_{21}^{e}(\mu_2 - \mathbf{R}_2\zeta) + \mathbf{R}_1\zeta.$$

Remark: μ_1 and μ_2 have the same boundary values in a conformally invariant $H^{-1/2}$ space.

The scattering matrix, genus $g \neq 0$

Given $\mu_2 = \alpha_2 + \overline{\beta}_2 \in \mathcal{A}_{\text{harm}}(\Sigma_2)$.
Let $\zeta = \xi + \overline{\eta} \in \mathcal{A}_{\text{harm}}(\mathcal{R})$ be a catalyzing form.
Let $\mu_1 = \alpha_1 + \overline{\beta}_1 \in \mathcal{A}$ be the overfare of μ_2 with respect to this form.

Theorem (S & Staubach, 2021)

Assume Σ_2 is connected, and genus of $\mathcal{R}$ is non-zero, we have

$$\begin{pmatrix} \overline{\beta}_1 \\ \overline{\beta}_2 \\ \xi \end{pmatrix} = \begin{pmatrix} -\overline{\mathbf{T}}_{1,1} & -\overline{\mathbf{T}}_{2,1} & \overline{\mathbf{R}}_1 \\ -\overline{\mathbf{T}}_{1,2} & -\overline{\mathbf{T}}_{2,2} & \overline{\mathbf{R}}_2 \\ \mathbf{S}_1 & \mathbf{S}_2 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \overline{\eta} \end{pmatrix}. \qquad (1)$$

and this scattering matrix is unitary.

The scattering matrix, genus $g = 0$

Theorem (S & Staubach, 2021)

Assuming Σ_2 is connected, and $\mathcal{R}$ is the sphere we have

$$\begin{pmatrix} \overline{\beta}_1 \\ \overline{\beta}_2 \end{pmatrix} = \begin{pmatrix} -\overline{\mathbf{T}}_{1,1} & -\overline{\mathbf{T}}_{2,1} \\ -\overline{\mathbf{T}}_{1,2} & -\overline{\mathbf{T}}_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}. \tag{2}$$

and the matrix is unitary.

Remark The unitarity amounts to adjoint identities, which are reformulations of norm identities of Bergman and Schiffer 50. The resulting fact that $\|\mathbf{T}_{11}\| < 1$ is Bergman-Schiffer's proof of the Grunsky inequalities.

Thanks!

Thanks!