Approximation of Liouville Brownian Motion

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Joint work with Yang Yu
(in progress)

Liouville Brownian motion

• Introduced by Garban-Rohdes-Vargas '16, also by Berestycki '15.

• a time change of two-dimensional Brownian motion via Liouville measure or Gaussian multiplicative chaos.

• canonical diffusion process under Liouville quantum gravity: scaling limit of random walk on mated-CRT planar maps. Berestycki-Gwynne '20, Gwynn-Miller-Sheffield '21

Liouville measure

h: massive Gaussian free field on $\mathbb{R}^2$ with

$$\begin{aligned}\mathrm{Cov}(h(x), h(y)) &= G_{m^2}(x,y) \\ &= \int_0^\infty e^{-m^2 t} \frac{1}{2\pi t} e^{-(x-y)^2/2t} dt.\end{aligned}$$

Liouville measure: for $\gamma \in (0,2)$,

$$\mu_h(dz) = \text{“}e^{\gamma h(z)} dz\text{”} = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz.$$

$h_\varepsilon(z)$ centered Gaussuan with $\mathrm{Var}(h_\varepsilon(z)) \asymp \log(1/\varepsilon)$.

μ_h concentrates on those z where “$h(z) = +\infty$”, in fact on γ-thick points. Thus $\mu_h(dz) \perp dz$.

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X: Brownian motion indepependent of GFF h.

μ_h is a Radon measure not charging on zero capacity sets.

$$\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz \longleftrightarrow A_t^\varepsilon = \int_0^t \varepsilon^{\gamma/2} e^{\gamma h_\varepsilon(X_s)} ds.$$

Garban-Rohdes-Vargas '16:

(i) $\lim_{\varepsilon \to 0} A_t^\varepsilon = A_t$ is a strictly increasing continuous additive functional of X having Revuz measure μ_h.

(ii) Liouville Brownian motion $Y_t := X_{\tau_t}$ is μ_h-symmetric, where $\tau_t = \inf\{r > 0 : A_r > t\}$.

(iii) $\{X_{\tau_t^\varepsilon}; t \geq 0\} \Longrightarrow \{X_t; t \geq 0\}$.

$$\mu(dz) = c\,dx \longleftrightarrow A_t = ct \longleftrightarrow \tau_t = t/c.$$

$X_{\tau_t} = X_{t/c}$: speeds up if $c < 1$, slows down if $c > 1$.

X_{τ_t} speeds up at places where μ has small masses and slows down where μ has large masses.

Time change

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Conformal change of metrics

(M^n, g): $g(x) = \sum_{i,j} g_{ij}(x) dx^i \otimes dx^j$.

$$\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{i,j} \partial_i (\sqrt{g} g^{ij} \partial_j).$$

Volume elemenet $m_g(dx) = \sqrt{\det g}\, dx$.

For $g_w = e^w g$, $\sqrt{\det g_w}\, g_w^{ij} = e^{(n/2-1)w} \sqrt{\det g}\, g^{ij}$.

When $n = 2$, $\Delta_{g_w} = e^{-w} \Delta_g$ and

$$\begin{aligned}(-\Delta_g u, v)_{m_g} &= \int_M \nabla u(x) \cdot (\sqrt{\det g(x)} g^{ij}(x)) \nabla v(x) dx \\ &= (-\Delta_{g_\varphi} u, v)_{m_{g_w}}.\end{aligned}$$

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In $n = 2$, the Dirichlet form is invariant under conformal change of metrics $g \to e^w g$, while $m_{g_w} = e^w m_g$.

Silverstein '73, '74, Fitzsimmons '90: Brownian motion on (M, g_w) is a time change of Brownian motion on (M, g).

Liouville quantum gravity: "$e^{\gamma h(z)} dx \otimes dy$".
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Approximation

Question: Can one construct/approximate Liouville Brownian motion directly from μ_h?

More generally, suppose X is a Brownian motion on $\mathbb{R}^d$ and μ is a Radon measure with full support that does not charge on zero capacity sets.

• μ uniquely determines a positive continuous additive functional A_t^μ of X as $f(x)dx \longleftrightarrow \int_0^t f(X_s)ds$.

• Time changed Brownian motion $Y_t = X_{\tau_t}$, where $\tau_t = \inf\{r > 0 : A_r^\mu > t\}$.

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Dirichlet form characterization

Transition semigroup: $P_t f(x) = \mathbb{E}_x[f(Y_t)] = \mathbb{E}_x[f(X_{\tau_t})]$.

$$\int_{\mathbb{R}^d} g(x) P_t f(x) \mu(dx) = \int_{\mathbb{R}^d} f(x) P_t g(x) \mu(dx).$$

Dirichlet form $(\mathcal{E}, \mathcal{F})$ of Y on $L^2(\mathbb{R}^d; \mu)$:

$$\begin{aligned} \mathcal{F} &:= \inf \left\{ u \in L^2(\mu) : \sup_{t>0} \frac{1}{t} (u - P_t u, u)_{L^2(\mu)} < \infty \right\}, \\ \mathcal{E}(u, v) &:= \lim_{t \to 0} \frac{1}{t} (u - P_t u, u)_{L^2(\mu)}, \quad u, v \in \mathcal{F}. \end{aligned}$$

Dirichlet form characterization

When $\mu = dx$ (i.e. $Y = X$ is BM), $\mathcal{F} = W^{1,2}(\mathbb{R}^d)$ and

$$\mathcal{E}(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x) dx.$$

For general μ, as a special case of Silverstein '73, '74, Fitzsimmons '90,

$$\begin{aligned} \mathcal{F} &= \left\{ u \in W^{1,2}_{loc}(\mathbb{R}^d) \cap L^2(\mu) : \nabla u \in L^2(\mathbb{R}^d; dx) \right\}, \\ \mathcal{E}(u, v) &= \frac{1}{2} \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x) dx, \quad u, v \in \mathcal{F}. \end{aligned}$$

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Brownian motion can be approximated as follows.
Let $r > 0$.

(i) When at x, stay for an exponentially distributed time with parameter r^{-2} (mean holding time is r^2).
(ii) Jump to point x_1 chosen randomly from $B(x, r)$.
(iii) Repeat this procedure to get a process $X_t^{(r)}$.

Donsker '51: $\{X_t^{(r)}; t \geq 0\} \Longrightarrow \{B_t; t \geq 0\}$ as $r \to 0$.

Random walk approximation

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RW approximation of time changed BM:

Two possible natural schemes.

Scheme 1:

(i) When at x, stay for an exponentially distributed time with parameter $\lambda(x) = \frac{|B(x,r)|}{r^2\, \mu(B(x,r))}$ (mean holding time is $1/\lambda(x)$).

(ii) Jump to point x_1 chosen randomly from $B(x, r)$.

(iii) Repeat this procedure to get a process $X_t^{(r)}$.

$X^{(r)}$ is symmetric with respect to $\mu_r(dx) = \frac{\mu(B(x,r))}{|B(x,r)|} dx$,

$$\mathcal{E}^{(r)}(u,u) = \frac{1}{2r^2|B(0,r)|} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 \mathbb{1}_{|x-y|<r} dxdy$$

Second possible scheme

(i) When at x, stay for an exponentially distributed time with parameter r^{-2} (mean holding time is r^2).

(ii) Jump to point x_1 chosen randomly from $B(x, r_x)$, where $\mu(B(x, r_x)) = |B(x, r)|$.

(iii) Repeat this procedure to get a process $X_t^{(r)}$.

$X^{(r)}$ is typically not symmetric. Need a symmetrizing procedure.

Main result I

We work under scheme 1.

Theorem (C.-Yu 22+)

Suppose $\mu \ll dx$ *has full support on* $\mathbb{R}^d$. *For any* $\phi \geq 0$ *in* $C_c(\mathbb{R}^d)$, $\{X_t^{(2^{-n})}; t \geq 0\}$ *with initial distribution* $\phi(x)\mu_{2^{-n}}dx$ *converges weakly in* $\mathbb{D}([0,\infty);\mathbb{R}^d)$ *in Skorohod topology to the time-changed Brownian motion by* μ *with initial distribution* $\phi(x)\mu(dx)$.

Main result II

Theorem (C.-Yu 22+)

For general Radon measure μ with full support on $\mathbb{R}^d$ that does not charge on zero capacity sets and $r > 0$, let $\mu_r(dx) = \frac{\mu(B(x,r))}{|B(x,r)|}dx$ and $X^{(r)}$ time-changed Brownian motion by $\mu^{(r)}$. For any $\phi \geq 0$ in $C_c(\mathbb{R}^d)$, $\{X^{(r)}; t \geq 0\}$ with initial distribution $\phi(x)\mu_r(dx)$ converges weakly in $\mathbb{D}([0,\infty); \mathbb{R}^d)$ in Skorohod topology to the time-changed Brownian motion by μ with initial distribution $\phi(x)\mu(dx)$ as $r \to 0$.

We can also take $\mu_r = p_r * \mu$, where p_r is the heat kernel of BM.

Remark

Suppose that μ satisfies locally that

$$\mu(B(x,r)) \le c\, r^{d-2+\varepsilon} \quad \text{for all } r \le 1.$$

Then the weak convergence holds for every starting point $x \in \mathbb{R}^d$.

γ-Liouville measure μ_h satisfies the above condition with $d = 2$ and any $\varepsilon < 2(1-\gamma/2)^2$.

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- Tightness:

In pseudo-path topology (weak convergence topology), then strengthen it to Skorohod topology afterwards.

- Convergence of finite dimensional distributions.

Mosco convergence of Dirichlet forms

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Weak convergence

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Weak convergence topology

A rcll path $\gamma_n : [0, \infty) \to \mathbb{R}^d \cup \{\partial\}$ is said to convergent to γ in pseudo-path topology on $\mathbb{D}([0,\infty), \mathbb{R}^d_\partial)$ if $\int_0^\infty e^{-t}(|\gamma_n(t) - \gamma(t)| \wedge 1)dt \to 0$.

$\Omega := \mathbb{D}([0,\infty), \mathbb{R}^d_\partial)$, probability measures $\mathbb{P}_n$, $\mathbb{P}$ on Ω.

Coordinate processes: $X^n_t(\omega) = \omega(t)$, $X_t(\omega) = \omega(t)$.

Weak convergence topology

Meyer-Zheng '84:

(i) If $\mathbb{P}_n \Longrightarrow \mathbb{P}$ in pseudo-path topology, there is a Lebesgue null set $\Lambda \subset [0, \infty)$ so that the finite dimensional distributions of X^n converges to that of X outside Λ.

(ii) $\{\mathbb{P}_n\}$ is tight on $\mathbb{D}([0, T], \mathbb{R}^d_\partial)$ if and only if for any relatively compact disjoint open sets E and F, $\sup_n \mathbb{E}^{\mathbb{P}_n}[N_T^{E,F}] < \infty$, where $N_T^{E,F}(\omega)$ is the number of crossings from E to F by a path ω over $[0, T]$.

C.-Fitzsimmons-Song 01': For an m-symmetric strong Markov process X and disjoint relatively compact open sets E and F,

$$\mathbb{E}_m\left[N_T^{E,F}\right] \leq 2T \inf\left\{\mathcal{E}(u,u):\ u\in\mathcal{F}: u=0 \text{ on } E, u=1 \text{ on } F\right\}.$$

Tightness

For $f \in C_c^2(\mathbb{R}^d)$,

$$
\begin{aligned}
& \lim_{r\to 0} \mathcal{E}^{(r)}(f,f) \\
= & \lim_{r\to 0} \frac{1}{2r^2|B(0,r)|} \iint_{\mathbb{R}^d\times\mathbb{R}^d} (f(x)-f(y))^2 \mathbb{1}_{|x-y|<r} dxdy \\
= & \frac{1}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2 dx < \infty.
\end{aligned}
$$

Thus for any $\phi \geq 0$ in $C_c(\mathbb{R}^d)$, the laws $\{\mathbb{P}^{(r)}_{\phi\mu_r}; r>0\}$ are tight on $\mathbb{D}([0,\infty),\mathbb{R}^d)$ w.r.t. the pseudo-path topology.

Convergence in finite dimensional distributions

Mosco '94: $(\mathcal{E}^n, \mathcal{F}^n)$ and $(\mathcal{E}, \mathcal{F})$ DFs on $L^2(\mathcal{X}; m)$. Then P_t^n strongly converges to P_t in $L^2(\mathcal{X}; m)$ for every $t > 0$ if and only if $(\mathcal{E}^n, \mathcal{F}^n)$ is Mosco-convergent to $(\mathcal{E}, \mathcal{F})$.

Set $\mathcal{E}(u, u) = \infty$ for $u \notin \mathcal{F}$.

Definition

$(\mathcal{E}^n, \mathcal{F}^n)$ is Mosco-convergent to $(\mathcal{E}, \mathcal{F})$ if

(i) $u_n \to u$ *weakly in* $L^2 \Longrightarrow \mathcal{E}(u, u) \leq \liminf_n \mathcal{E}^n(u, u)$.

(ii) *For any* $u \in \mathcal{F}$, $\exists u_n \to u$ *strongly in* L^2 *with* $\mathcal{E}(u, u) \geq \limsup_n \mathcal{E}^n(u_n, u_n)$.

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(i) *$u_n \to u$ weakly in $L^2 \Longrightarrow \mathcal{E}(u, u) \leq \liminf_n \mathcal{E}^n(u, u)$.*

(ii) *For any $u \in \mathcal{F}$, $\exists u_n \to u$ strongly in L^2 with $\mathcal{E}(u, u) \geq \limsup_n \mathcal{E}^n(u_n, u_n)$.*

Kuwae-Shioya '03 extended Mosco's result to cases with varying spaces and reference measures. Let $r_n = 2^{-n}$. Specialized to our setting, their result says that Mosco's result continues to holds if

(i) $u_n \in L^2(\mu_{r_n})$ is said to converge weakly to $u \in L^2(\mu)$ if $\int_{\mathbb{R}^d} u_n(x)\phi(x)\mu_{r_n}(dx) \to \int_{\mathbb{R}^d} u(x)\phi(x)\mu(dx)$ for every $\phi \in C_c(\mathbb{R}^d)$.

(ii) $u_n \in L^2(\mu_{r_n})$ is said to converge strongly to $u \in L^2(\mu)$ if there are $v_n \in C_c(\mathbb{R}^d)$ so that $v_n \to u$ in $L^2(\mu)$ and $\|v_n - u_n\|_{L^2(\mu_{rn})} \to 0$.

Sketch of the proof

Condition (ii) of Mosco convergence is easy to establish. For (i), we have

(a) $u_n \in L^2(\mu_{r_n})$ converge weakly to $u \in L^2(\mu)$ if and only if $v_n(x) := \frac{1}{|B(x,r_n)|}\int_{B(x,r_n)} u_n(y)dy$ converges weakly to u in $L^2(\mu)$.

(b) $\mathcal{E}^{(r_n)}(v_n, v_n) \leq \mathcal{E}^{(r_n)}(u_n, u_n)$.

(c) Suffices to consider $\lim_n \mathcal{E}^{(r_n)}(v_n, v_n) < \infty$.

(d) $\mathcal{E}^{(r)}(f, f) \leq \mathcal{E}^{(r/2)}(f, f)$.

Sketch of the proof for (i)

By Banach-Saks' theorem, taking a subsequence if needed, $w_n = (v_1 + \ldots + v_n)/n$ is $\mathcal{E}^{(r_k)}$-Cauchy for every $k \geq 1$ and $w_n \to u$ in $L^2(\mu)$.
When $\mu \ll dx$, $w_n \to u$ in $\mathcal{E}^{(r_k)}$ for every $k \geq 1$. So

$$\begin{aligned} \mathcal{E}^{(r_k)}(u,u) &\leq \lim_n \mathcal{E}^{(r_k)}(w_n, w_n) \leq \liminf_n \mathcal{E}^{(r_k)}(v_n, v_n) \\ &\leq \liminf_n \mathcal{E}^{(r_n)}(v_n, v_n) \leq \liminf_n \mathcal{E}^{(r_n)}(u_n, u_n). \end{aligned}$$

Taking $k \to \infty$, we get $u \in W^{1.2}_{loc}(\mathbb{R}^d)$ with

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \leq \liminf_n \mathcal{E}^{(r_n)}(u_n, u_n).$$

Sketch of Proof

So $X^{(2^{-n})}$ converges weakly in pseudo-path topology to the time-changed Brownian motion Y.

They in fact converge in Skorohod topology through a localization argument and a result of Aldous as Y is a continuous local martingale.

Theorem 2 can also be established via Mosco convergence.

Key: μ does not charge on zero capacity sets. Forms $\mathcal{E}$ are the same but with different speed measures.

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Pointwise starting point

When μ locally satisfies

$$\mu(B(x,r)) \leq c\, r^{d-2+\varepsilon} \quad \text{for all } r \leq 1,$$

μ does not charge on zero capacity sets. We can obtain locally uniform exit time estimates for $X^{(r)}$ from balls. This together with Hölder regularity of harmonic functions of $X^{(r)}$ enable us to strengthen weak convergence to pointwise starting point.

Thank you!