Stability of Regularized Hastings-Levitov Aggregation in the Subcritical Regime

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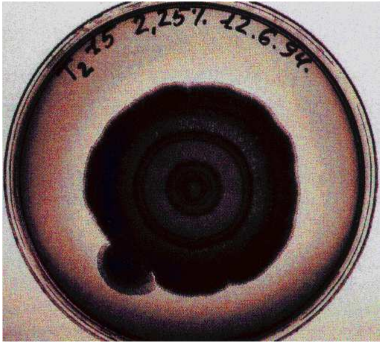
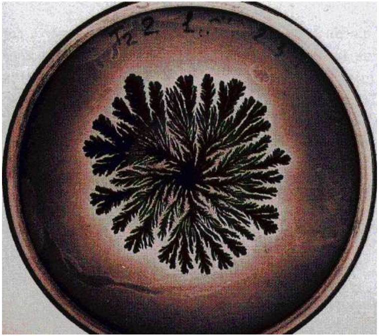
Bacterial growth in increasingly stressed conditions





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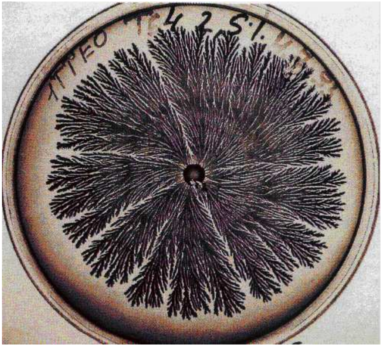

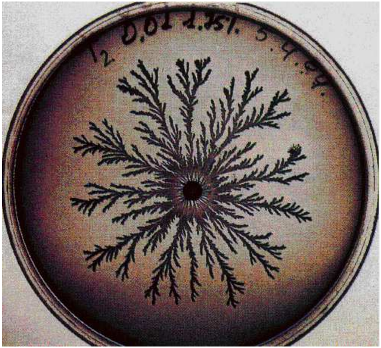

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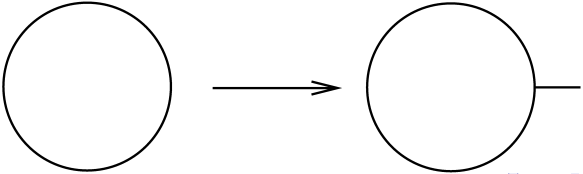
Conformal mapping representation of single particle

Let D_0 denote the exterior unit disk in the complex plane $\mathbb{C}$ and P denote a particle.

There exists a unique conformal mapping $F : D_0 \to D_0 \setminus P$ that fixes ∞ in the sense that

$$F(z) = e^c z + O(1) \quad \text{as} \quad |z| \to \infty,$$

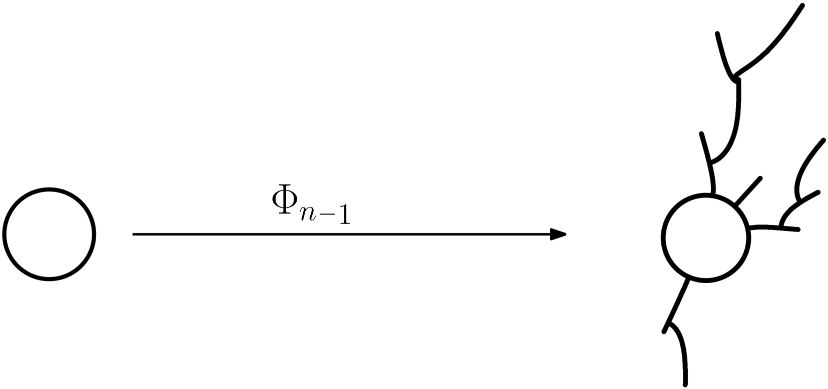
for some $c > 0$. We use F as a mathematical description of the particle. The (log of the) capacity, c, is a measure of the size of the particle.



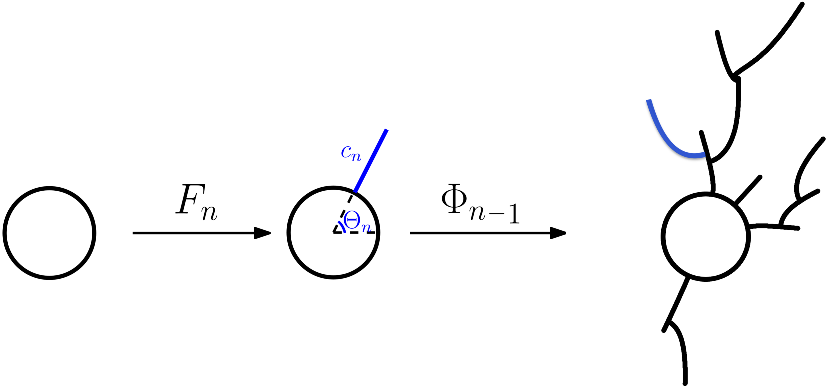
Conformal mapping representation of a cluster

- Suppose $P_1, P_2, \ldots$ is a sequence of particles, where P_n has capacity c_n and attachment angle Θ_n, $n = 1, 2, \ldots$. Let F_n be the particle map corresponding to P_n.
  - Set $\Phi_0(z) = z$.
  - Recursively define $\Phi_n(z) = \Phi_{n-1} \circ F_n(z)$, for $n = 1, 2, \ldots$.
- This generates a sequence of conformal maps $\Phi_n : D_0 \to K_n^c$, where $K_{n-1} \subset K_n$ are growing compact sets, which we call clusters.

Cluster formed by iteratively composing mappings



Cluster formed by iteratively composing mappings



$$\Phi_n = \Phi_{n-1} \circ F_n = F_1 \circ F_2 \circ \cdots \circ F_n$$

Parameter choices for physical models

- By varying the sequences $\{\Theta_n\}$ and $\{c_n\}$, it is possible to describe a wide class of growth models.
- For biological growth (Eden model)

$$\mathbb{P}(\Theta_n \in (a, b)) \propto \int_a^b |\Phi'_{n-1}(e^{i\theta})| d\theta$$

and

$$c_n \approx c|\Phi'_{n-1}(e^{i\Theta_n})|^{-2}$$

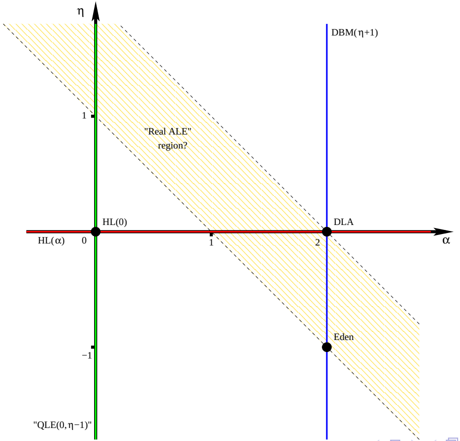
- For DLA, c_n is as above and

$$\mathbb{P}(\Theta_n \in (a, b)) = \mathbb{P}(\Phi_{n-1}^{-1}(B_\tau) \in (a, b)) \propto (b - a)$$

where B_t is Brownian motion started from ∞ and τ is the hitting time of the cluster K_{n-1}.

Aggregate Loewner Evolution, ALE(α, η, σ)

- Θ_n distributed $\propto |\Phi'_{n-1}(e^{\sigma+i\theta})|^{-\eta} d\theta$; $\quad c_n = c|\Phi'_{n-1}(e^{\sigma+i\Theta_n})|^{-\alpha}$.



Regularization for ALE

- Even after the arrival of a single slit particle, the map $\theta \mapsto |\Phi_n'(e^{i\theta})|$ is badly behaved and takes the values 0 and ∞.
- For some values of η,

$$\int_{-\pi}^{\pi} |\Phi_{n-1}'(e^{i\theta})|^{-\eta} d\theta = \infty,$$

  so regularization is necessary to even define the measure.
- A solution is to let Θ_n have distribution

$$\propto |\Phi_{n-1}'(e^{\sigma+i\theta})|^{-\eta} d\theta$$

  for $\sigma > 0$ and take the limit $\sigma \to 0$.
- Models are very sensitive to the rate at which $\sigma \to 0$. Can be argued that $\sigma \sim c^{1/2}$ is natural from a physical point of view.

Universality of particle shapes

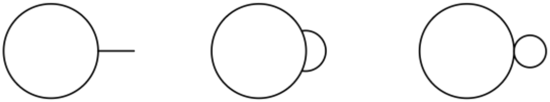
The model depends on the choice of a family of basic particles $(P^{(c)} : c \in (0,\infty))$ with $P^{(c)}$ of capacity c. We will require that

$$P^{(c_1)} \subset P^{(c_2)} \quad \text{for } c_1 < c_2$$

and, for some $\Lambda \in [1,\infty)$,

$$\sup\{|z-1| : z \in P^{(c)}\} \leq \Lambda \sup\{|z|-1 : z \in P^{(c)}\} \quad \text{for all } c.$$

For small c, the second condition forces the particles to concentrate near the point 1 while never becoming too flat against the unit circle.



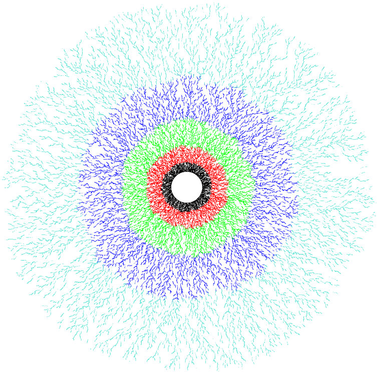
Phase transition

Open Problem:

Does ALE(α, η, σ) exhibit a phase transition from disks to non-disks along the line $\alpha + \eta = 1$ in the limit as $c \to 0$ (for 'broad' choices of the regularization parameter σ)?

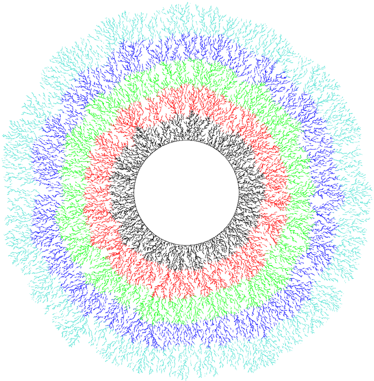
- Longstanding conjectures:
  - HL(α) has a phase transition at $\alpha = 1$.
  - DBM(η) has a phase transition at $\eta = 0$.

ALE(0,0,0) cluster with 25,000 particles for $c = 10^{-4}$



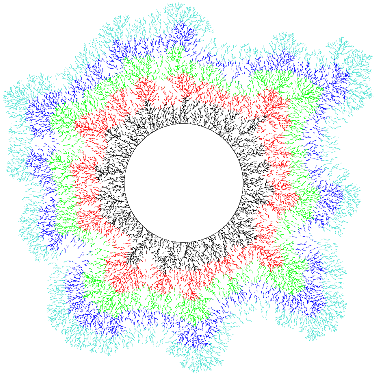
Simulation by Alan Sola

ALE(1,0,0.02) cluster with 25,000 particles for $c = 10^{-4}$



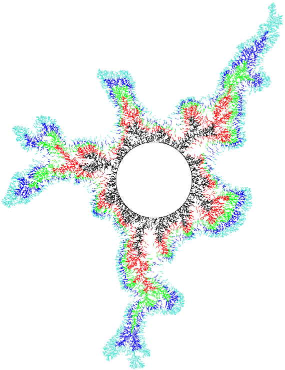
Simulation by Alan Sola

ALE(1.5,0,0.02) cluster with 25,000 particles for $c = 10^{-4}$



Simulation by Alan Sola

ALE(2,0,0.02) cluster with 25,000 particles for $c = 10^{-4}$



Simulation by Alan Sola

Loewner chain representation

Define the driving measure $\mu_t = \delta_{e^{i\xi_t}}$, where

$$\xi_t = \sum_{k=1}^{N} \Theta_k 1_{(C_{k-1}, C_k]}(t),$$

with $C_k = \sum_{j=1}^{k} c_k$, for angles $\{\Theta_k\}$ and capacities $\{c_k\}$ as above.

Consider the solution to the Loewner equation

$$\partial_t \Psi_t(z) = z\Psi_t'(z) \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} d\mu_t(e^{i\theta}),$$

with initial condition $\Psi_0(z) = z$.

Then (for slit particles, but general case similar)

$$\Phi_n = \Psi_{C_n}, \quad n = 0, 1, 2, \ldots .$$

Continuity properties of the Loewner equation

- Solutions to the Loewner equation are close if the driving measures are close in some suitable sense.
  - Suppose $\mu^n = \{\mu_t^n\}_{t\geq 0}$, $n = 1, 2, \ldots$, and $\mu = \{\mu_t\}_{t\geq 0}$ are families of measures on the unit circle $\mathbb{T}$.
  - Let Ψ_t^n be the solution to the Loewner equation corresponding to μ^n and Ψ_t be the solution corresponding to μ.
  - To show that $\Psi_t^n \to \Psi_t$ uniformly on compact subsets of D_0, it is enough to show that

$$\int_{\mathbb{T}\times[0,\infty)} f(e^{i\theta}, t) d\mu_t^n(e^{i\theta}) dt \to \int_{\mathbb{T}\times[0,\infty)} f(e^{i\theta}, t) d\mu_t(e^{i\theta}) dt$$

    for all continuous functions f in $\mathbb{T} \times [0, \infty)$ with compact support.

Example: Anisotropic Hastings-Levitov

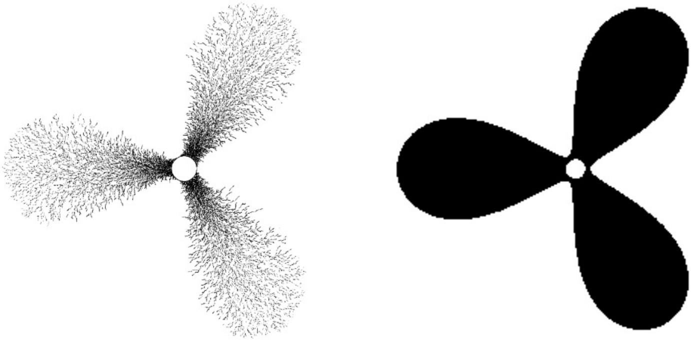
- Suppose Θ_n are i.i.d. with density $h(\theta)$ on $[0, 2\pi)$.
- Suppose $c_n = cg(\Theta_n)$, for some bounded continuous function g on $[0, 2\pi)$.
- Let Ψ_t solve

$$\partial_t \Psi_t(z) = z\Psi_t'(z) \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} g(\theta)h(\theta)d\theta,$$

with initial condition $\Psi_0(z) = z$.

Theorem (Viklund, Sola, T. '12): Fix $T > 0$. As $c \to 0$, $\Phi_{\lfloor T/c \rfloor} \to \Psi_T$ in probability.

Clusters with non-uniform attachment angles



Simulations by Alan Sola

Heuristic for ALE scaling limit

- ALE does not fit into the framework above as the attachment densities and capacities are random and depend on the cluster.
- Nevertheless, the same heuristic suggests that a candidate scaling limit for $\Phi_{\lfloor T/c \rfloor}$ is the solution Ψ_t to

$$\partial_t \Psi_t(z) = \frac{z\Psi_t'(z)}{Z_t} \int_0^{2\pi} \frac{z+e^{i\theta}}{z-e^{i\theta}} |\Psi_t'(e^{i\theta})|^{-(\alpha+\eta)} d\theta,$$

with initial condition $\Psi_0(z) = z$, where

$$Z_t = \int_0^{2\pi} |\Psi_t'(e^{i\theta})|^{-\eta} d\theta.$$

- It is straightforward to check that $\Psi_t(z) = (1+\alpha t)^{1/\alpha} z$.

Stability of the dynamics

- The randomness in ALE introduces perturbations around the disk solution.
- Depending on the stability of the Loewner equation, these perturbations can be suppressed or amplified by the PDE dynamics.
- The factor Z_t just induces a time-change, so does not affect the stability.
- Stability therefore depends only on $\alpha + \eta$.
- Simulations suggest a transition between stable and unstable dynamics at $\alpha + \eta = 1$.

Analysis of the stability

Set

$$a(\phi)(z) = \frac{z\phi'(z)}{2\pi} \int_0^{2\pi} \frac{z+e^{i\theta}}{z-e^{i\theta}} \left|\phi'(e^{i\theta})\right|^{-\zeta} d\theta.$$

Then

$$a(\phi+\varepsilon\psi)(z) = a(\phi) + \varepsilon\left(z\psi'(z)h(z) - \zeta z\phi'(z)g(z)\right) + o(\varepsilon)$$

where

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{z+e^{i\theta}}{z-e^{i\theta}} |\phi'(e^{i\theta})|^{-\zeta} d\theta$$

and, setting $\rho = \psi'/\phi'$,

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{z+e^{i\theta}}{z-e^{i\theta}} |\phi'(e^{i\theta})|^{-\zeta} \Re\rho(e^{i\theta}) d\theta.$$

Linear stability of disk solutions in the subcritical case

Suppose $\phi_t(z) = e^{\tau_t} z$ where $\tau_t = \zeta^{-1} \log(1 + \zeta t)$.

Then ϕ_t solves $\partial_t \phi_t = a(\phi_t)$ with initial condition $\phi_0(z) = z$.

The integrals h and g can be explicitly evaluated as

$$h(z) = e^{-\zeta \tau_t} \quad \text{and} \quad g(z) = e^{-(1+\zeta)\tau_t} \psi'(z)$$

so, by equating coefficients of ε in $\partial_t(\phi_t + \varepsilon \psi_t) = a(\phi_t + \varepsilon \psi_t)$, the first order variations ψ_t around the solution ϕ_t can be expected to satisfy the linearized equation

$$\partial_t \psi_t = (1 - \zeta) z \psi_t'(z) e^{-\zeta \tau_t} = (1 - \zeta) z \psi_t'(z) \dot{\tau}_t.$$

Linear stability of disk solutions in the subcritical case

- Formally, this has solution

$$\psi_t(z) = \psi_0\left(e^{(1-\zeta)\tau_t} z\right).$$

- In the case $\zeta > 1$, ψ_t can be holomorphic in $\{|z| > 1\}$ only if ψ_0 extends to a holomorphic function in the larger domain $\{|z| > e^{-(\zeta-1)\tau_t}\}$.
- In particular, if ψ_0 has singularities on the boundary $\{|z| = 1\}$, then the variation blows up immediately.
- When $\zeta \in [0, 1]$, the variation ψ_t is holomorphic in $\{|z| > 1\}$ for all t and, when $\zeta < 1$, gets more regular as t increases.

Disk theorem for ALE(α, η, σ) when $\alpha + \eta < 1$

Theorem (Norris, Silvestri, T.):

For all $T \in [0, \infty)$, $\epsilon \in (0, 1/2)$ and $e^\sigma \geq 1 + c^{1/2-\epsilon}$, there exists a constant C such that, with high probability, for all $n \leq T/c$ and $|z| \geq 1 + c^{1/2-\epsilon}$,

$$|\Phi_n(z) - (1 + \alpha cn)^{1/\alpha} z| \leq \frac{C}{|z|} \left(c^{1/2-\epsilon} + \frac{c^{1-\epsilon}}{(e^\sigma - 1)^2} \right).$$

Disk theorem for ALE(α, η, σ) when $\alpha + \eta = 1$

Theorem (Norris, Silvestri, T.):

For all $T \in [0, \infty)$, $\epsilon \in (0, 1/5)$ and $e^\sigma \geq 1 + c^{1/5-\epsilon}$, there exists a constant C such that, with high probability, for all $n \leq T/c$ and $|z| \geq 1 + c^{1/5-\epsilon}$,

$$|\Phi_n(z) - (1 + \alpha c n)^{1/\alpha} z| \leq \frac{C}{|z|} \left(c^{1/2-\epsilon} \left(\frac{|z|}{|z| - 1} \right)^{1/2} + \frac{c^{1-\epsilon}}{(e^\sigma - 1)^3} \right).$$

Fluctuations for ALE(α, η, σ) when $\alpha + \eta \leq 1$

Set

$$\mathcal{F}_n^{(c)}(z) = c^{-1/2}((1 + \alpha c n)^{-1/\alpha}\Phi_n(z) - z)$$

and let $n(t) = \lfloor t/c \rfloor$.

Under the assumptions above (but with slightly stronger restrictions on σ), $\mathcal{F}_{n(t)}^{(c)}(z) \to \mathcal{F}_t(z)$ where

$$\dot{\mathcal{F}}_t(z) = \frac{1}{1 + \alpha t}\left((1 - \alpha - \eta)z\mathcal{F}_t'(z) - \mathcal{F}_t(z) + \sqrt{2}\xi_t(z)\right).$$

Here $\xi_t(z)$ is complex space-time white noise on the circle, analytically continued to the exterior unit disk.

Fluctuations for ALE(α, η, σ) when $\alpha + \eta \leq 1$

Specifically

$$\mathcal{F}_t(z) = \sum_{m=0}^{\infty} (A_t^m + iB_t^m) z^{-m}$$

where

$$dA_t^m = -\frac{(m(1-\alpha-\eta)+1)\, A_t^m}{1+\alpha t} dt + \frac{\sqrt{2}}{1+\alpha t} d\beta_t^m$$

$$dB_t^m = -\frac{(m(1-\alpha-\eta)+1)\, B_t^m}{1+\alpha t} dt + \frac{\sqrt{2}}{1+\alpha t} d\beta'^m_t.$$

Here $\beta_t^m, \beta_t'^m$ are i.i.d. Brownian motions for $m = 0, 1, \ldots,$ so

$$A_t^m, B_t^m \sim \mathcal{N}\left(0, \frac{1 - e^{-2(m(1-\alpha-\eta)+1)\tau_t}}{m(1-\alpha-\eta)+1}\right).$$

Remarks

- The map $z \mapsto \mathcal{F}_t(z)$ is determined (by analytic extension) by the boundary process $\theta \mapsto \mathcal{F}_t(e^{i\theta})$.
- When $\alpha = \eta = 0$, these boundary fluctuations are the same as for internal diffusion limited aggregation (IDLA).
- As $t \to \infty$, $\mathcal{F}_t(e^{i\theta})$ converges to a Gaussian field.
  - When $\alpha + \eta = 0$, $\mathcal{F}_\infty(e^{i\theta})$ is known as the augmented Gaussian Free Field.
  - When $\alpha + \eta < 1$, $\mathrm{Cov}\left(\mathcal{F}_\infty(e^{ix}), \mathcal{F}_\infty(e^{iy})\right) \asymp \log|x-y|$.
  - When $\alpha + \eta = 1$, $\mathcal{F}_\infty(e^{i\theta})$ is complex white noise.

Idea behind the proof

Using the particle assumptions one can show that

$$\log \frac{F_n(z)}{z} = c_n \frac{ze^{-i\Theta_n} + 1}{ze^{-i\Theta_n} - 1} + O(c_n^{3/2}).$$

Therefore

$$\begin{aligned}\Phi_n(z) &= \Phi_{n-1}(F_n(z)) \\ &= \Phi_{n-1}(z) + z\Phi'_{n-1}(z)(\log F_n(z) - \log z) + \cdots \\ &= \Phi_{n-1}(z) + cz\Phi'_{n-1}(z)|\Phi'_{n-1}(e^{\sigma+i\Theta_n})|^{-\alpha} \frac{ze^{-i\Theta_n} + 1}{ze^{-i\Theta_n} - 1} + O(c^{3/2}).\end{aligned}$$

Idea behind the proof

Now

$$\mathbb{E}\left(z\Phi'_{n-1}(z)|\Phi'_{n-1}(e^{\sigma+i\Theta_n})|^{-\alpha}\frac{ze^{-i\Theta_n}+1}{ze^{-i\Theta_n}-1}\middle| \mathcal{F}_{n-1}\right)$$
$$= \frac{z\Phi'_{n-1}(z)}{Z_n}\int_0^{2\pi} |\Phi'_{n-1}(e^{\sigma+i\theta})|^{-(\alpha+\eta)}\frac{ze^{-i\theta}+1}{ze^{-i\theta}-1}d\theta$$
$$:= a_\sigma(\Phi_{n-1})(z)$$

so

$$\frac{\Phi_n(z)-\Phi_{n-1}(z)}{c} = a_\sigma(\Phi_{n-1})(z) + M_n(z) + O(c^{1/2})$$

where $M_n(z)$ is a martingale difference term.

Idea behind the proof

For $\phi_t(z) = (1+\alpha t)^{1/\alpha} z = e^{\tau_t} z$, write

$$\Phi_n(z) = \phi_{nc}(z) + \mathcal{M}_n(z).$$

Using the variational solution from earlier with $\zeta = \alpha + \eta$ gives

$$\begin{aligned} \mathcal{M}_n(z) &\approx \mathcal{M}_{n-1}\left(e^{(1-\zeta)(\tau_{nc}-\tau_{(n-1)c})}\right) + M_n(z) \\ &= \sum_{k=1}^{n} M_k\left(e^{(1-\zeta)(\tau_{nc}-\tau_{kc})} z\right). \end{aligned}$$

The scaling limit and fluctuation results follow from an analysis of the martingale and estimates on the errors.

References

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