CFT for multiple SLEs

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Based on joint work with
Tom Alberts (U. of Utah) and Nikolai Makarov (Caltech)

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Jeju Workshop

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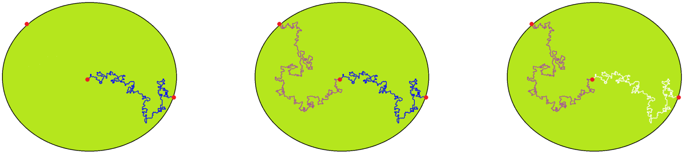
Jeju island has three UNESCO World Heritage sites. It is packed with museums and theme parks and has horses, mountains, lava tube caves, and waterfalls with a clear blue ocean lapping its beaches.



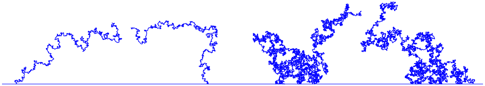

CFT and SLE

- (with N. Makarov) *Gaussian free field and conformal field theory*, Astérisque **353** (2013), viii+136 pages.



  - CFT is a provider for SLE MOs: Bauer-Bernard, Cardy, Kytölä, Rushkin-Bettelheim-Gruzberg-Wiegmann, etc.
  - (with N. Makarov) CFT on the Riemann sphere and its boundary version for SLE
  - (with S. Byun and H. Tak) CFT for annulus SLE: partition functions and MOs

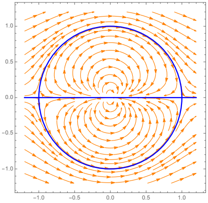
- CFT for multiple SLEs (with T. Alberts and N. Makarov)

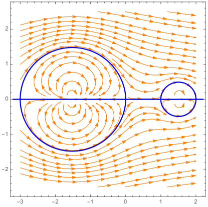


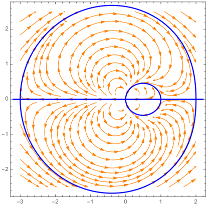
  - Physics literature: Bauer-Bernard-Kytölä, Cardy, Graham, etc.
  - Pole dynamics and an integral of motion for multiple SLE(0)

Outline

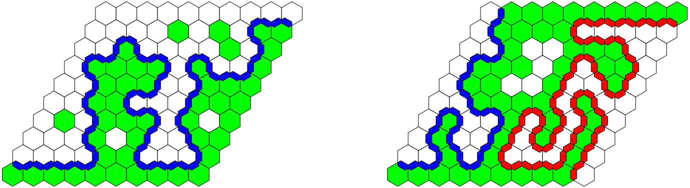
- to implement a version of CFT constructed from background charge modifications of Gaussian free field,
- to define N-leg operators $\phi_{1,2}(x_1)\phi_{2,1}(x_2)\dots$ producing multiple SLEs growing towards ∞ and insertion fields producing commuting multiple SLEs,
- to show that this version produces a collection of martingale-observables for commuting multiple SLEs,
- to explain how this theory is related to Tom Albert's talk on "Loewner Dynamics for Real Rational Functions and the Multiple SLE(0) Process."



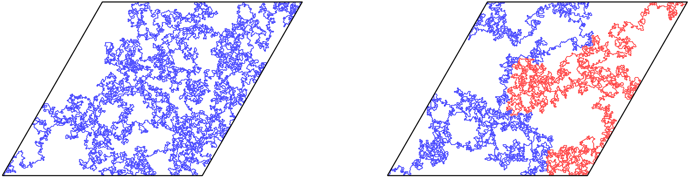




Single SLE v.s. Multiple SLEs



Single Interface v.s. Double Interfaces



Single SLE_6 v.s. Double SLE_6s

Multiple Schramm-Loewner Evolutions
with force points

Commuting multiple SLEs describe multiple interfaces:

$$\partial_t g_t(z) = \sum_{j=1}^{N} \frac{2\mu_j(t)}{g_t(z) - \xi_j(t)},$$

where the driving processes $\boldsymbol{\xi}(t)$ are given by

$$d\xi_j(t) = \sqrt{\kappa_j \mu_j(t)}\, dB_j(t) + 2b_j(\boldsymbol{\xi}(t), \boldsymbol{q}(t))\mu_j(t)\, dt + \sum_{k\neq j} \frac{2\mu_k(t)}{\xi_j(t) - \xi_k(t)}\, dt, \;\; b_j = \frac{\kappa_j}{2} \frac{\partial_{\xi_j} Z}{Z},$$

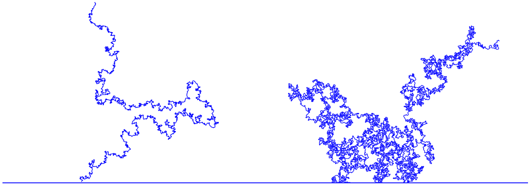
$\kappa_j = \kappa$, or $\tilde{\kappa} := 16/\kappa$ and a partition function Z satisfies the null vector equation:

$$\frac{\kappa_j}{4}\partial_{\xi_j}^2 Z = \sum_{k\neq j} \frac{\partial_{\xi_k} Z}{\xi_j - \xi_k} + \sum_{l} \frac{\partial_{q_l} Z}{\xi_j - q_l} + \sum_{k\neq j} \frac{6-\kappa_k}{2\kappa_k} \frac{1}{(\xi_j - \xi_k)^2} Z + \mathbf{E}\, T_{\boldsymbol{\beta}}(\xi_j) Z,$$

where $T_{\boldsymbol{\beta}}$ is the Virasoro field with background charge $\boldsymbol{\beta}$ placed at the force points $\boldsymbol{q}$:

$$\mathbf{E}\, T_{\boldsymbol{\beta}}(\xi) = \sum_{k} \frac{\lambda_{q_k}}{(\xi - q_k)^2} + \sum_{j<k} \frac{\beta_j \beta_k}{(\xi - q_j)(\xi - q_k)}, \qquad \lambda_{q_k} = \frac{1}{2}\beta_k^2 - b\beta_k.$$

Dubedát's Commutation Relations



Theorem (Dubedát) [Re-sampling symmetry] *Multiple SLE commutes if and only if*

$$[\mathcal{L}_j, \mathcal{L}_k] = \frac{2}{(\xi_j - \xi_k)^2}(\mathcal{L}_k - \mathcal{L}_j),$$

where the infinitesimal generators $\mathcal{L}_j$ are given by

$$\mathcal{L}_j := \frac{\kappa_j}{4}\partial^2_{\xi_j} + b_j(\boldsymbol{\xi}, \boldsymbol{q})\partial_{\xi_j} + \sum_{k\neq j}\frac{\partial_{\xi_k}}{\xi_k - \xi_j} + \sum_{l}\frac{\partial_{q_l}}{q_l - \xi_j}, \quad b_j = \frac{\kappa_j}{2}\frac{\partial_{\xi_j}Z}{Z}.$$

Remark. Null vector equation $\implies$ commutation relations.

Operator Product Expansion (OPE)

- BPZ equations hold for product of fields in the OPE family of Φ.
- We define the N-leg operators in terms of the OPE exponentials.

We write the OPE of two (*holomorphic*) fields $X(\zeta)$ and $Y(z)$ as

$$X(\zeta)Y(z) = \sum C_j(z)(\zeta - z)^j \quad (\zeta \to z, \zeta \neq z).$$

Write $X * Y$ for C_0. Meaning: $\mathbf{E}\,\mathrm{LHS}\mathcal{X} = \mathbf{E}\,\mathrm{RHS}\mathcal{X}$ for $\mathcal{X} = \Phi(z_1) \odot \cdots \odot \Phi(z_n)$.

Example for the current field $J = \partial\Phi$. In terms of a conformal map $w : D \to \mathbb{H}$,

$$J(\zeta)J(z) = \mathbf{E}[J(\zeta)J(z)] + J(\zeta) \odot J(z) = -\frac{w'(\zeta)w'(z)}{(w(\zeta) - w(z))^2} + J(\zeta) \odot J(z)$$

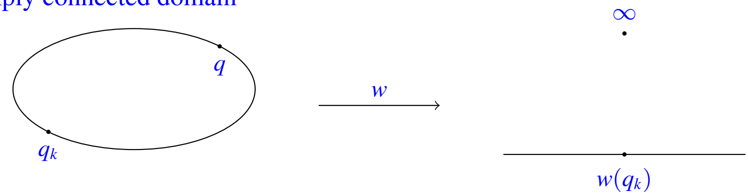
$$= -\frac{1}{(\zeta - z)^2} \underbrace{-\frac{1}{6}S_w(z) + J(z) \odot J(z)}_{J * J(z)} + \cdots \quad (\zeta \to z, \zeta \neq z).$$

Example.

$$\Phi^{*2} = \Phi^{\odot 2} + 2c, \quad c = \log C.$$

Background Charge Modifications of GFF

in a simply connected domain



Given a background charge $\boldsymbol{\beta} = \sum_k \beta_k \cdot q_k + \beta_\infty \cdot q$ placed at the marked boundary points q_k, q with the neutrality condition (NC_b: $\beta_\infty + \sum_k \beta_k = 2b$)

$$\Phi_{\boldsymbol{\beta}} := \Phi + \varphi_{\boldsymbol{\beta}}, \qquad \varphi_{\boldsymbol{\beta}} := -2b \arg w' + \sum_k 2\beta_k \arg(w - w(q_k)),$$

where w is a conformal map from (D, q) onto $(\mathbb{H}, \infty)$ and

$$b = a(\kappa/4 - 1), \quad a = \sqrt{2/\kappa}, \quad c = 1 - 12b^2.$$

Motivation. The OPE exponentials of $\Phi_{\boldsymbol{\beta}}$ are differentials.

Remark. The neutrality condition NC_b comes from the Gauss-Bonnet theorem.

Virasoro field

The OPE family $\mathcal{F}_{\boldsymbol{\beta}}$ of $\Phi_{\boldsymbol{\beta}}$ has the central charge $c = 1 - 12b^2$ and the Virasoro field

$$T_{\boldsymbol{\beta}} = -\frac{1}{2}J_{\boldsymbol{\beta}} * J_{\boldsymbol{\beta}} + ib\partial J_{\boldsymbol{\beta}}, \quad J_{\boldsymbol{\beta}} = J + j_{\boldsymbol{\beta}}, \quad J = \partial\Phi, \quad j_{\boldsymbol{\beta}} = \partial\varphi_{\boldsymbol{\beta}}.$$

It is well known in the algebraic literature that if the generators $\tilde{L}_n$ are constructed as

$$\tilde{L}_n = L_n - ib(n+1)J_n,$$

where J_n's and L_n's are the modes of the current field J and the Virasoro field $T = -\frac{1}{2}J * J$:

$$J_n(z) := \frac{1}{2\pi i}\oint_{(z)} (\zeta - z)^n J(\zeta)\,d\zeta, \qquad L_n(z) := \frac{1}{2\pi i}\oint_{(z)} (\zeta - z)^{n+1} T(\zeta)\,d\zeta.$$

then $\tilde{L}_n$ represent the Virasoro algebra with central charge $c = 1 - 12b^2$:

$$[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}.$$

One can identify $\tilde{L}_n$ with the modes of $T_{\boldsymbol{\beta}}$:

$$\tilde{L}_n(z) = \frac{1}{2\pi i}\oint_{(z)} (\zeta - z)^{n+1}\, T_{\boldsymbol{\beta}}(\zeta)\,d\zeta.$$

Chiral bosonic fields

Definition. $\Phi^+(\gamma) = \displaystyle\int_\gamma \partial\Phi(\zeta)\, d\zeta.$

Then $\Phi^+(z, z_0) := \{\Phi^+(\gamma) : z_0 \underset{\gamma}{\longrightarrow} z\}$ is a multivalued field.

Cf. Mikhail Sodin's talk on Random Weierstrass Zeta-Functions.

In $\mathbb{H}$,

$$\mathbf{E}[J(z)J(\zeta)] = -\frac{1}{(\zeta - z)^2}, \quad \mathbf{E}[\Phi^+(z, z_0)J(\zeta)] = -\frac{1}{\zeta - z} + \frac{1}{\zeta - z_0}$$

and

$$\mathbf{E}[\Phi^+(z, z_0)\Phi^+(\tilde{z}, \tilde{z}_0)] = \log\frac{(z - \tilde{z}_0)(z_0 - \tilde{z})}{(z - \tilde{z})(z_0 - \tilde{z}_0)}.$$

Chiral bosonic fields

For $\boldsymbol{\tau} = \sum \tau_j \cdot z_j$, $\boldsymbol{\tau}_* = \sum \tau_{*j} \cdot z_j$, we define formal fields

$$\Phi[\boldsymbol{\tau}, \boldsymbol{\tau}_*] = \sum \tau_j \Phi^+(z_j) - \tau_{*j} \Phi^-(z_j), \quad \Phi^+[\boldsymbol{\tau}] = \Phi[\boldsymbol{\tau}, \mathbf{0}],$$

where $\Phi^\pm$ have formal correlations:

$$\mathbf{E}[\Phi^+(\zeta)\Phi^+(z)] = \log \frac{1}{\zeta - z}, \quad \mathbf{E}[\Phi^+(\zeta)\Phi^-(z)] = \log(\zeta - \bar{z}) \quad \text{in } \mathbb{H}$$

and the relations $\Phi^- = \overline{\Phi^+}$, $\Phi = \Phi^+ + \Phi^-$.

Remark. $\Phi[\boldsymbol{\tau}, \boldsymbol{\tau}_*]$ is a well-defined Fock space field if and only if

$$\sum \tau_j + \tau_{*j} = 0 \, (\mathrm{NC}_0).$$

Examples. $\Phi^+(z) - \Phi^+(z_0) = \Phi^+(z, z_0)$, $\Phi^+(z) + \Phi^-(z) = \Phi(z)$.

OPE exponentials

Given a background charge $\boldsymbol{\beta} = \sum_k \beta_k \cdot q_k$ with the neutrality condition (NC_b) and a divisor $\boldsymbol{\tau} = \sum_j \tau_j \cdot z_j$ with the neutrality condition (NC_0), we define the OPE exponential $\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ of the bosonic field $i\Phi^+_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ by

$$\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}] := \frac{C_{(b)}[\boldsymbol{\tau} + \boldsymbol{\beta}]}{C_{(b)}[\boldsymbol{\beta}]} e^{\odot i\Phi^+[\boldsymbol{\tau}]},$$

where $\Phi^+[\boldsymbol{\tau}] := \sum \tau_j \, \Phi^+(z_j)$ and for $\boldsymbol{\sigma} = \sum \sigma_j \cdot z_j \in (\mathrm{NC}_b)$, $C_{(b)}[\boldsymbol{\sigma}]$ is a differential of conformal dimension $\frac{1}{2}\sigma_j^2 - b\sigma_j$ at z_j whose evaluation in the identity chart of $\mathbb{H}$ is given by

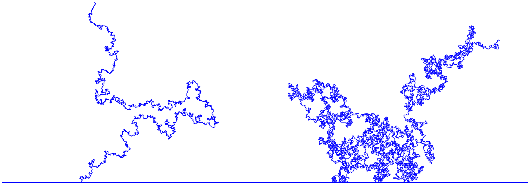
$$\prod_{j \neq k} (z_j - z_k)^{\sigma_j \sigma_k}.$$

If $\operatorname{supp} \boldsymbol{\tau} \cap \operatorname{supp} \boldsymbol{\beta} = \emptyset$, then

$$\mathbf{E}\, \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = \prod_{j \neq k} (z_j - z_k)^{\tau_j \tau_k} \prod_{j,k} (z_j - q_k)^{\tau_j \beta_k}$$

in the identity chart of $\mathbb{H}$. (No interactions between q_k's.)

N-leg operators



For the N-leg operators, we choose ($\boldsymbol{\beta} = 2b \cdot q$ and)

$$\boldsymbol{\tau} = \sum a_j \cdot \xi_j - \left(\sum a_j\right) \cdot q, \quad a_j = a := \sqrt{2/\kappa} \text{ or } \tilde{a} := -a - b = -\sqrt{\kappa/8} \equiv -\sqrt{2/\tilde{\kappa}},$$

where $b = a(\kappa/4 - 1), a = \sqrt{2/\kappa}$. With this choice, $\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ satisfies the level two degeneracy equations:

$$\left(\tilde{L}_{-2}(\xi_j) - \frac{\kappa_j}{4} \tilde{L}_{-1}(\xi_j)^2\right) \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = 0,$$

where $\kappa_j = \kappa$ if $a_j = a$ and $\kappa_j = \tilde{\kappa}$ if $a_j = \tilde{a}$.

Furthermore, the partition function $Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] := \mathbf{E}\, \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ satisfies the null vector equation.

Main Theorem

Let

$$\Psi(\boldsymbol{\xi},\boldsymbol{q}) := \frac{\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]}{\mathbf{E}\,\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]} = \frac{\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]}{Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}]}, \quad \boldsymbol{\tau} := \sum a_j \cdot \xi_j - (\sum a_j)\cdot\infty.$$

Theorem (Alberts-K-Makarov)

For any tensor product X of fields in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$ of $\Phi_{\boldsymbol{\beta}}$,

$$M_t = (\mathbf{E}\Psi(\boldsymbol{\xi},\boldsymbol{q})X \,\|\, g_t^{-1})\Big|_{\boldsymbol{\xi}=\boldsymbol{\xi}(t),\boldsymbol{q}=\boldsymbol{q}(t)}$$

is a local martingale.

The level two degeneracy equations + Ward's equations = BPZ-Cardy equations.

Theorem (Ward's equations)

For the tensor product X of fields in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$, in the identity chart of $\mathbb{H}$,

$$\mathbf{E}\,A_{\boldsymbol{\beta}}(\xi)X = \mathbf{E}\,\mathcal{L}_{v_\xi}X, \quad A_{\boldsymbol{\beta}} = T_{\boldsymbol{\beta}} - \mathbf{E}\,T_{\boldsymbol{\beta}}, \quad v_\xi(z) = \frac{1}{\xi - z}, \quad \xi\in\mathbb{R},$$

where the Virasoro field is given by $T_{\boldsymbol{\beta}} = -\frac{1}{2}J_{\boldsymbol{\beta}} * J_{\boldsymbol{\beta}} + ib\partial J_{\boldsymbol{\beta}}$.

Screening

It is well known that all solutions to the null vector equation can be obtained from the method of screening, see Dubedát, and Flores-Kleban. (Cf. Kytölä and Peltola used a quantum group technique to construct the pure partition functions of multiple SLEs.)

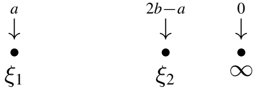
For example, in the identity chart of $\mathbb{H}$, up to a multiplicative constant,

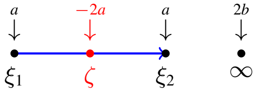
$$C_{(b)}[a\cdot\xi_1+(2b-a)\cdot\xi_2]=\int_{\xi_1}^{\xi_2} C_{(b)}[a\cdot\xi_1-2a\cdot\zeta+a\cdot\xi_2+2b\cdot\infty]\,d\zeta, \qquad (\kappa>4).$$

(Recall $C_{(b)}[\boldsymbol{\sigma}]$ is a differential of conformal dimension $\frac{1}{2}\sigma_j^2-b\sigma_j$ at z_j whose evaluation in the identity chart of $\mathbb{H}$ is given by

$$\prod_{j\neq k}(z_j-z_k)^{\sigma_j\sigma_k}.)$$

Charges:



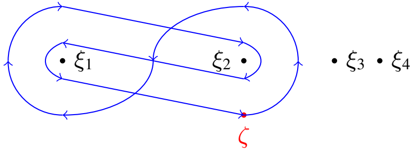


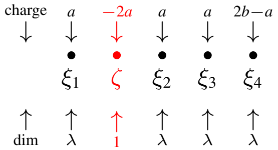
Dubedat's Screening

For example,

$$\int_{\mathcal{P}(\xi_1,\xi_2)} C_{(b)}[a\cdot\xi_1 - 2a\cdot\zeta + a\cdot\xi_2 + a\cdot\xi_3 + (2b-a)\cdot\xi_4]\, d\zeta$$

is a solution to the null vector equation and satisfies the conformal Ward identities, i.e. Moebius invariance. Here, Pochhammer contour $\mathcal{P}(\xi_1,\xi_2)$ and charges/dimensions of the integrand are given by





$$\lambda = \tfrac{1}{2}a^2 - ab = \tfrac{6-\kappa}{2\kappa}$$

Level Two Singular Vectors

For each b, there are exactly 4 exponents σ (except for the case $b = 0, \pm 1/2$ when some of σ's coincide) such that the vertex fields $\mathcal{O}^{(\sigma)}(z) = \mathcal{O}_{2b\cdot q}[\sigma \cdot z + \ldots]$ produce level two singular vectors:

$$\sigma = a, \quad \sigma = 2b - a,$$

and

$$\sigma = -a - b, \quad \sigma = 3b + a.$$

The vertex fields $\mathcal{O}^{(a)}(z)$ and $\mathcal{O}^{(-a-b)}$ are degenerate,

$$\Big(\tilde{L}_{-2}(\xi_j) - \frac{\kappa_j}{4}\tilde{L}_{-1}(\xi_j)^2\Big)\mathcal{O}^{(a_j)} = 0,$$

where $\kappa_j = \kappa$ if $a_j = a$ and $\kappa_j = \tilde{\kappa}$ if $a_j = \tilde{a}$, but $\mathcal{O}^{(2b-a)}$ and $\mathcal{O}^{(a+3b)}$ are not (unless $b = -1/2, 0, 1/2; \kappa = 2, 4, 8$).

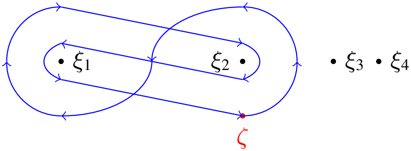
E.g., see Proposition 13.2 in *Gaussian free field and conformal field theory*, Astérisque **353** (2013).

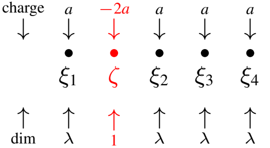
Screening

For example,

$$\int_{\mathcal{P}(\xi_1,\xi_2)} C_{(b)}[a \cdot \xi_1 - 2a \cdot \zeta + a \cdot \xi_2 + a \cdot \xi_3 + a \cdot \xi_4 + (2b - 2a) \cdot \infty] \, d\zeta$$

is a solution to the null vector equation. Here, Pochhammer contour $\mathcal{P}(\xi_1, \xi_2)$ and charges/dimensions of the integrand are given by





$$\lambda = \frac{1}{2}a^2 - ab = \frac{6-\kappa}{2\kappa}$$

Theorem (Alberts-K-Makarov)

CFTs associated with these solutions give rise to martingale-observables.

Screening

For example,

$$\int_{\mathcal{P}(\xi_1,\xi_2)} C_{(b)}[a\cdot\xi_1 - 2a\cdot\zeta + a\cdot\xi_2 + a\cdot\xi_3 + a\cdot\xi_4 + (2b-2a)\cdot\infty]\, d\zeta$$

is a solution to the null vector equation.

The integrand Z^{pre} satisfies the following equation

$$\begin{aligned}\frac{\kappa}{4}\partial_{\xi_j}^2 Z^{pre} &= \sum_{k\neq j}\frac{\partial_{\xi_k} Z^{pre}}{\xi_j-\xi_k} + \sum_{k\neq j}\frac{6-\kappa}{2\kappa}\frac{Z^{pre}}{(\xi_j-\xi_k)^2} + \frac{\partial_\zeta Z^{pre}}{\xi_j-\zeta} + \frac{Z^{pre}}{(\xi_j-\zeta)^2} \\ &= \sum_{k\neq j}\frac{\partial_{\xi_k} Z^{pre}}{\xi_j-\xi_k} + \sum_{k\neq j}\frac{6-\kappa}{2\kappa}\frac{Z^{pre}}{(\xi_j-\xi_k)^2} + \partial_\zeta \frac{Z^{pre}}{\xi_j-\zeta}.\end{aligned}$$

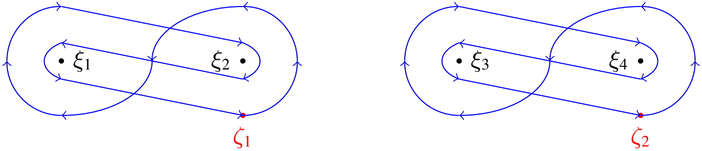
The red term becomes trivial after integrating along a Pochhammer contour.

Symmetric Screening
Catalan case

For $\alpha = \{\{1,2\},\{3,4\}\}$, $Z_\alpha(\boldsymbol{\xi}) =$

$$\int_{\mathcal{P}(\xi_1,\xi_2)} \int_{\mathcal{P}(\xi_3,\xi_4)} C_{(b)}[a\cdot\xi_1 - 2a\cdot\zeta_1 + a\cdot\xi_2 + a\cdot\xi_3 - 2a\cdot\zeta_2 + a\cdot\xi_4 + 2b\cdot\infty]\, d\zeta_1\, d\zeta_2$$

is a solution to the null vector equation and satisfies the conformal Ward identities, i.e. Möbius invariance.



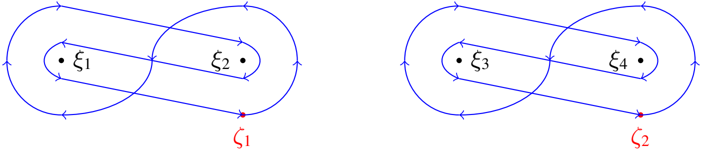
However, the Möbius invariance is not trivial.

Insertion Fields
Catalan case

Under the insertion of

$$\frac{\int_{\mathcal{P}(\xi_1,\xi_2)}\int_{\mathcal{P}(\xi_3,\xi_4)} \mathcal{O}_{\boldsymbol{\beta}}[a\cdot\xi_1 - 2a\cdot\zeta_1 + a\cdot\xi_2 + a\cdot\xi_3 - 2a\cdot\zeta_2 + a\cdot\xi_4 + 2b\cdot\infty]\, d\zeta_1\, d\zeta_2}{\text{its correlation function}},$$

the correlations of tensor products of fields in $\mathcal{F}_{\boldsymbol{\beta}}$ are martingale-observables for SLE associated to Z_α, $\alpha = \{\{1,2\},\{3,4\}\}$.



Cf. Eveliina Peltola's talk on log-CFT for UST and SLE(8):

$$Z_\alpha^{\text{pure}} = \sum_{\alpha'} \mathcal{M}^{-1}_{\alpha,\alpha'} Z_{\alpha'},$$

where $\mathcal{M}$ is the meander matrix.

Conformal Ward Identities

Catalan case

The partition function

$$\int_{\mathcal{P}(\xi_1,\xi_2)} \int_{\mathcal{P}(\xi_3,\xi_4)} C_{(b)}[a\cdot\xi_1 - 2a\cdot\zeta_1 + a\cdot\xi_2 + a\cdot\xi_3 - 2a\cdot\zeta_2 + a\cdot\xi_4 + 2b\cdot\infty]\, d\zeta_1\, d\zeta_2$$

is annihilated by the three differential operators

$$\sum_{k=1}^{2N} \partial_{\xi_k}, \quad \sum_{k=1}^{2N} \left(\xi_k \partial_{\xi_k} + \frac{6-\kappa}{2\kappa}\right), \quad \sum_{k=1}^{2N} \left(\xi_k^2 \partial_{\xi_k} + \frac{6-\kappa}{\kappa}\xi_k\right),$$

or

$$\mathcal{L}_{v_j} = \sum_k v_j(\xi_k)\partial_{\xi_k} + h v_j'(\xi_k), \quad h = \frac{6-\kappa}{2\kappa}, \quad v_j(z) = z^j, \quad j = 0, 1, 2.$$

The (local) flow $\psi_t^{(j)}$ of v_j ($\dot{\psi}_t^{(j)} = v_j \circ \psi_t^{(j)}$) is given by

$$\psi_t^{(0)}(z) = z + t, \quad \psi_t^{(1)}(z) = e^t z, \quad \psi_t^{(2)}(z) = \frac{z}{1 - tz}.$$

Conformal Ward Identities

Catalan case

Lemma

Let $Z^{pre} := C_{(b)}[\sum_{j=1}^{2N} a \cdot \xi_j + \sum_{k=1}^{N} (-2a) \cdot \zeta_k + 2b \cdot \infty]$. *Then we have*

$$\mathcal{L}_{v_2} Z^{pre} = \sum_k \frac{\partial}{\partial \zeta_k} (f_k Z^{pre}),$$

where

$$f_k = \frac{\prod_j^{2N} (\zeta_k - \xi_j)}{\prod_{l \neq k} (\zeta_k - \zeta_l)^2}.$$

Remark. We choose

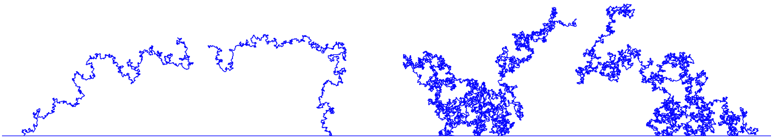
$$f_k = \frac{C_{(b)}[\sum a \cdot \xi_j + 2b \cdot \zeta_k + \sum_{l \neq k} (-2a) \cdot \zeta_l + (-2a) \cdot \infty]}{C_{(b)}[\sum a \cdot \xi_j + \sum (-2a) \cdot \zeta_k + 2b \cdot \infty]}.$$

Key identity for the proof of Lemma:

$$\sum_k \Big(\sum_j \frac{1}{\xi_j - \zeta_k} + \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l} \Big) f_k = \sum_j \xi_j - 2 \sum_k \zeta_k.$$

Cf. The null vector equations hold for Dubedat's screening with $\sigma = 2b - a$.

SLE and its Dual with Screening



Let $\tilde{a} = -(a+b)$ and $\tilde{\kappa} = 16/\kappa$.

For example,

$$\int_{\mathcal{P}(\xi_1,\xi_2)} \int_{\mathcal{P}(\xi_3,\xi_4)} C_{(b)}[a\cdot\xi_1 - 2a\cdot\zeta_1 + a\cdot\xi_2 + \tilde{a}\cdot\xi_3 - 2\tilde{a}\cdot\zeta_2 + \tilde{a}\cdot\xi_4 + 2b\cdot\infty]\, d\zeta_1\, d\zeta_2$$

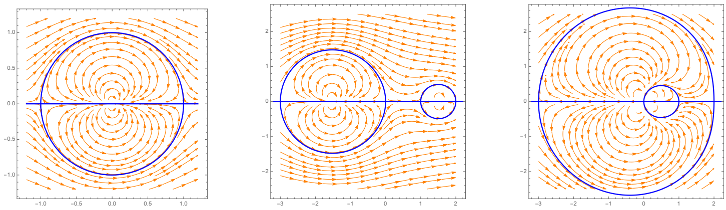
is a solution to the null vector equation

$$\frac{\kappa_j}{4}\partial_{\xi_j}^2 Z = \sum_{k\neq j} \frac{\partial_{\xi_k} Z}{\xi_j - \xi_k} + \sum_{k\neq j} \frac{6-\kappa_k}{2\kappa_k} \frac{1}{(\xi_j-\xi_k)^2} Z,$$

where $\kappa_1 = \kappa_2 = \kappa$, and $\kappa_3 = \kappa_4 = \tilde{\kappa}$.

Outline

- to implement a version of CFT constructed from background charge modifications of Gaussian free field,
- to define N-leg operators $\phi_{1,2}(x_1)\phi_{2,1}(x_2)\ldots$ producing multiple SLEs growing towards ∞ and insertion fields producing commuting multiple SLEs,
- to show that this version produces a collection of martingale-observables for commuting multiple SLEs,
- to explain how this theory is related to Tom Albert's talk on "Loewner Dynamics for Real Rational Functions and the Multiple SLE(0) Process."



Imaginary Geometry
Miller-Sheffield-…

We now consider non-chiral vertex fields

$$\mathcal{V}_{\boldsymbol{\beta}}^{i\sigma}(z) = e^{*i\sigma\Phi_{\boldsymbol{\beta}}(z)} = \mathcal{O}_{\boldsymbol{\beta}}[\sigma \cdot z - \sigma \cdot z^*]$$

with $\sigma \in \mathbb{R}$ and therefore with real conformal dimensions

$$\lambda = \frac{\sigma^2}{2} - \sigma b, \quad \lambda_* = \frac{\sigma^2}{2} + \sigma b.$$

The difference $\lambda - \lambda_* = -2\sigma b$ is called the conformal *spin* of the vertex field. If the spin is -1, then the direction of the field (in correlations with real Fock space fields) transforms as the direction of a vector field, and so the orbits of the ordinary differential equation

$$\dot{z} = \mathcal{V}_{\boldsymbol{\beta}}^{i\sigma}(z)$$

(if this can be defined appropriately) are natural conformally invariant objects.

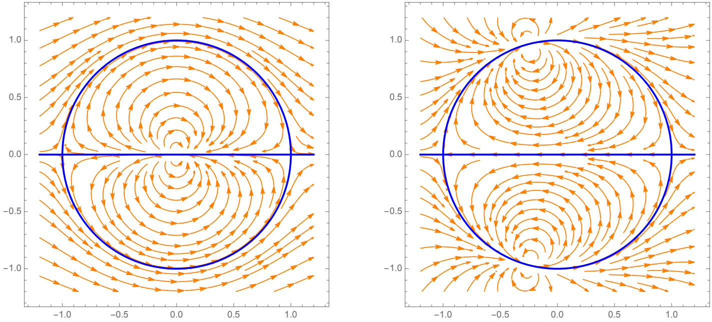
Classical Limits of Imaginary Geometry

Alberts-Byun-K-Makarov

We describe the Multiple SLE$(0; \boldsymbol{x}; \alpha)$ as the trajectories of the ordinary differential equation

$$\dot{z} = \frac{1}{R'_\alpha(z)},$$

where R_α is a real rational function with critical points $\boldsymbol{x}$ and link pattern α.
The trajectories are geodesics of $|R'_\alpha(z)|^2|dz|^2$.



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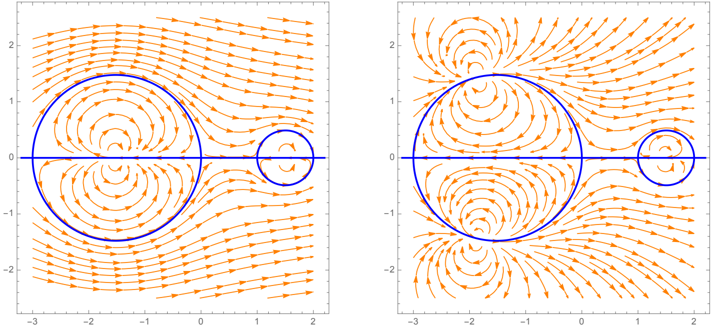

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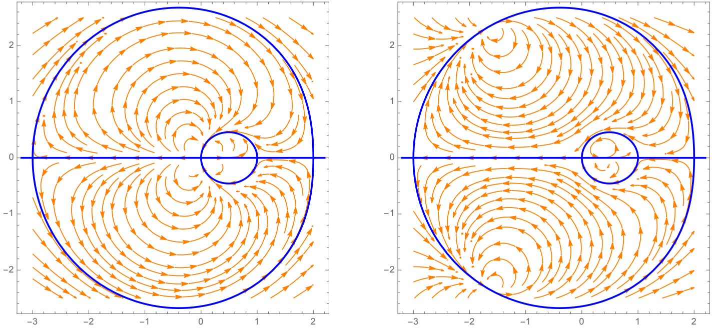

Classical Limits of Multiple Schramm-Loewner Evolutions

We now consider the multiple SLE(0) map described by

$$\dot{g}_t(z) = \sum_{j=1}^{2n} \frac{2\mu_j(t)}{g_t(z) - x_j(t)}, \quad \dot{x}_j(t) = U_j(\boldsymbol{x}(t))\mu_j(t) + \sum_{k \neq j} \frac{2\mu_k(t)}{x_j(t) - x_k(t)}, \quad (x_j(t) \in \mathbb{R}),$$

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where U_j satisfy the null vector equations

$$\frac{1}{2} U_j^2 + 2 \sum_{k \neq j} \frac{1}{x_k - x_j} U_k - 6 \sum_{k \neq j} \frac{1}{(x_k - x_j)^2} = 0, \quad j = 1, \ldots, 2n$$

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and the conformal Ward identities

$$\sum_{j=1}^{2n} U_j = 0, \quad \sum_{j=1}^{2n} x_j U_j = -6n, \quad \sum_{j=1}^{2n} x_j^2 U_j = -6\sum_{j=1}^{2n} x_j.$$

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We have found the Catalan number of solutions U_j and described the multiple SLE(0) curves as the real locus of real rational functions.

Classical Limits of Multiple Schramm-Loewner Evolutions

The commuting multiple SLEs (with $\kappa_j = \kappa$) describe multiple interfaces:

$$\partial_t g_t(z) = \sum_{j=1}^{N} \frac{2\mu_j(t)}{g_t(z) - x_j(t)},$$

where the driving processes $\boldsymbol{x}(t)$ are given by

$$dx_j(t) = \sqrt{\kappa \mu_j(t)}\, dB_j(t) + b_j(\boldsymbol{x}(t))\mu_j(t)\, dt + \sum_{k \neq j} \frac{2\mu_j(t)}{x_j(t) - x_k(t)}\, dt, \;\; b_j = \kappa \frac{\partial_{x_j} Z}{Z},$$

where Z satisfies the null vector equations:

$$\frac{\kappa}{4} \partial_{\xi_j}^2 Z = \sum_{k \neq j} \frac{\partial_{\xi_k} Z}{\xi_j - \xi_k} + \sum_{k \neq j} \frac{6-\kappa}{2\kappa} \frac{1}{(\xi_j - \xi_k)^2} Z.$$

The multiple SLE(0) map g_t is defined by

$$\dot{g}_t(z) = \sum_{j=1}^{N} \frac{2\mu_j(t)}{g_t(z) - x_j(t)}, \quad \dot{x}_j(t) = U_j(\boldsymbol{x}(t))\mu_j(t) + \sum_{k \neq j} \frac{2\mu_k(t)}{x_j(t) - x_k(t)}, \quad (x_j(t) \in \mathbb{R}),$$

where U_j satisfy the null vector equations

$$\frac{1}{2} U_j^2 + 2 \sum_{k \neq j} \frac{1}{x_k - x_j} U_k - 6 \sum_{k \neq j} \frac{1}{(x_k - x_j)^2} = 0, \quad j = 1, \ldots, N.$$

Conformal Ward Identities for Multiple SLE(0)

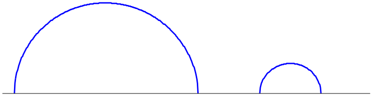
As $\kappa \to 0$, the conformal Ward identities

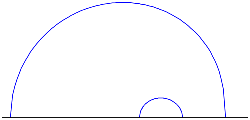
$$\sum_{j=1}^{2n} \partial_{x_j} Z = 0, \quad \sum_{j=1}^{2n} \left(x_j \partial_{x_j} + \frac{6-\kappa}{2\kappa} \right) Z = 0, \quad \sum_{j=1}^{2n} \left(x_j^2 \partial_{x_j} + \frac{6-\kappa}{\kappa} x_j \right) Z = 0$$

become

$$\sum_{j=1}^{2n} U_j = 0, \quad \sum_{j=1}^{2n} x_j U_j = -6n, \quad \sum_{j=1}^{2n} x_j^2 U_j = -6 \sum_{j=1}^{2n} x_j.$$

Peltola and Wang's characterization

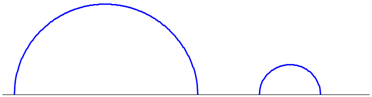


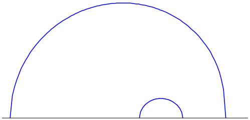


Peltola and Wang characterized the deterministic limit $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)$ of multiple $\mathrm{SLE}(\kappa; \boldsymbol{x}; \alpha)$ as $\kappa \to 0$,

- as a geodesic multichord, meaning that each η_j is the hyperbolic geodesic in the (unique) connected component of $\mathbb{H} \setminus \bigcup_{k \neq j} \eta_k$ that contains the endpoints of η_j,
- as curves generated by a single Loewner type evolution with only one non-trivial μ_j for each η_j, where the driving terms for the η_j are determined by special α-dependent solutions to the classical limit of the null vector equations as $\kappa \to 0$, and finally,
- as the real locus of a rational function of degree $n + 1$, with real coefficients and critical points at $x_1, \ldots, x_{2n}$.

Eremenko and Gabrielov's Theorem





For $\boldsymbol{x} = \{x_1, \ldots, x_{2n}\}$ with x_j distinct, real, and finite, let $\mathrm{CRR}_{n+1}(\boldsymbol{x})$ be the set of real rational functions $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $n+1$ whose critical points are $\boldsymbol{x}$.

An element $R = P/Q \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ is in canonical form[1] if $\deg Q = n$ and its derivative factors as

$$R'(z) = \frac{\prod_{j=1}^{2n}(z - x_j)}{\prod_{k=1}^{n}(z - \zeta_k)^2}.$$

Goldberg showed that the number of equivalence classes is at most C_n, and Eremenko and Gabrielov proved that it is exactly C_n (the n-th Catalan number):

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

[1] Since post-composition by elements of $\mathrm{PSL}(2, \mathbb{R})$ preserves the critical points it induces a natural equivalence relation on $\mathrm{CRR}_{n+1}(\boldsymbol{x})$.

Main Result: Stationary Relation

We say a rational function $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ with pole set $\boldsymbol{\zeta}$ is generic if $\boldsymbol{x} \cap \boldsymbol{\zeta} = \emptyset$.

Theorem

For distinct real points $\boldsymbol{x}$ let $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ be canonical and generic, and let $\boldsymbol{\zeta} = \boldsymbol{\zeta}(R) = \{\zeta_1, \ldots, \zeta_n\}$ be its set of finite poles. Then ζ_k satisfy the stationary relation

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \qquad k = 1, \ldots, n.$$

Theorem

For distinct real points $\boldsymbol{x}$ and a link pattern α connecting them define

$$U_{\alpha,j}(\boldsymbol{x}) := \sum_{k \neq j} \frac{2}{x_j - x_k} - \sum_{k=1}^{n} \frac{4}{x_j - \zeta_{\alpha,k}(\boldsymbol{x})}, \quad j = 1, \ldots, 2n.$$

Then $U_{\alpha,j}$ satisfy the system of null vector equations and conformal Ward identities.

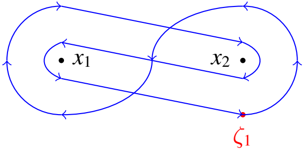
Main Result: Partition Function

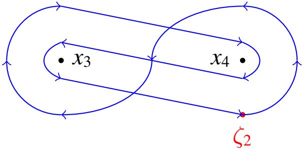
Theorem

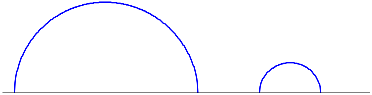
For distinct boundary points $\boldsymbol{x}$ *and a link pattern* α *connecting them define*

$$Z_\alpha(\boldsymbol{x}) := \prod_{1\le j<k\le 2n}(x_j - x_k)^2 \prod_{1\le l<m\le n}(\zeta_{\alpha,l}(\boldsymbol{x}) - \zeta_{\alpha,m}(\boldsymbol{x}))^8 \prod_{k=1}^{2n}\prod_{l=1}^{n}(x_k - \zeta_{\alpha,l}(\boldsymbol{x}))^{-4}.$$

Then Z_α *is strictly positive and* $U_{\alpha,j} = \partial_{x_j}\log Z_\alpha$ *for* $j = 1,\dots,2n$*.*







Main Result: an Integral of Motion

Let $\boldsymbol{x}$ be distinct real points and α be a link pattern connecting them. We consider

$$\dot{g}_t(z) = \sum_{j=1}^{2n} \frac{2\mu_j(t)}{g_t(z) - x_j(t)}, \quad \dot{x}_j(t) = U_{\alpha,j}(\boldsymbol{x}(t))\mu_j(t) + \sum_{k \neq j} \frac{2\mu_k(t)}{x_j(t) - x_k(t)}, \quad (x_j(t) \in \mathbb{R}),$$

where $U_{\alpha,j}$ is given by

$$U_{\alpha,j}(\boldsymbol{x}) := \sum_{k \neq j} \frac{2}{x_j - x_k} - \sum_{k=1}^{n} \frac{4}{x_j - \zeta_{\alpha,k}(\boldsymbol{x})}, \quad j = 1, \ldots, 2n.$$

Theorem

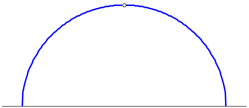
Under the Loewner system associated to $(\boldsymbol{x}; \alpha)$ *the quantity*

$$N_t(z) := \log\left(g_t'(z) \frac{\prod_{j=1}^{2n} (g_t(z) - x_j(t))}{\prod_{k=1}^{n} (g_t(z) - g_t(\zeta_{\alpha,k}(\boldsymbol{x})))^2} \right)$$

satisfies $\dot{N}_t(z) = 0$.

Main Result: an Integral of Motion

For example,



$$x_t = -\sqrt{1-t}, \qquad y_t = \sqrt{1-t}, \qquad \zeta_t = 0, \qquad (0 \le t \le 1).$$

and

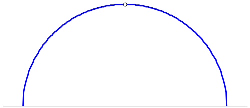
$$g_t(z) - \frac{t-1}{g_t(z)} = z + \frac{1}{z} = R(z), \qquad R_t(z) := R \circ g_t^{-1} = z - \frac{t-1}{z}.$$

Theorem

Let $\boldsymbol{x}$ be distinct boundary points, α be a link pattern connecting them, and $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x};\alpha)$ be canonical. Then under the Loewner system associated to $(\boldsymbol{x};\alpha)$ the map $R \circ g_t^{-1}$ is a canonical element in $\mathrm{CRR}_{n+1}(\boldsymbol{x}(t);\alpha)$ with poles $\zeta_{\alpha,k}(\boldsymbol{x}(t)) = g_t(\zeta_{\alpha,k}(\boldsymbol{x}))$.

Main Result: Real Locus

For example,



$$x_t = -\sqrt{1-t}, \qquad y_t = \sqrt{1-t}, \qquad \zeta_t = 0, \qquad (0 \le t \le 1).$$

and

$$g_t(z) - \frac{t-1}{g_t(z)} = z + \frac{1}{z} = R(z), \quad \gamma_1(t) = -\sqrt{1-t} + i\sqrt{t}, \quad \gamma_2(t) = \sqrt{1-t} + i\sqrt{t}.$$

Corollary

Let $\boldsymbol{x}$ be distinct boundary points and $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ be generic. Let $\gamma[0,t]$ be the trace of the curves generated by the Loewner flow corresponding to R. Then $\gamma[0,t]$ is a subset of the real locus $\Gamma(R)$.