Plumbing Liouville theory

Antti Kupiainen

joint work with C. Guillarmou, R. Rhodes, V. Vargas

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Complex analysis in the quantum domain

“We developed a general approach to CFTs, something like complex analysis in the quantum domain. It worked very well in the various problems of statistical mechanics but the Liouville theory remained unsolved."

Alexander Polyakov, From Strings to Quarks (2008)

This talk:

- What is "complex analysis in the quantum domain "?
- How to "solve" the Liouville theory

(Euclidean) Quantum Field Theory

- Random fields $\Psi(x)$, $x \in M$, M manifold, e.g. $\mathbb{R}^d$
- Expectation $\langle \cdot \rangle$
- **Correlation functions** $\langle \prod_{i=1}^{N} \Psi(x_i) \rangle$
- Axiomatizations describing **regularity** in x_i and behaviour under **symmetries** of M

Conformal Field Theory (CFT)

Euclidean QFT models **statistical physics**

At **critical temperature** such systems have **conformal symmetry** and the QFT is **conformal field theory**

This extra symmetry is believed to give rise to strong constraints on correlation functions expressed via **conformal bootstrap**

In 2 dimensions bootstrap was used by Belavin, Polyakov and Zamoldchicov (1984) to classify CFT's and find explicit predictions for the correlation functions in several cases

In more than 2 dimensions bootstrap has led to spectacular numerical predictions (e.g. 3d Ising model) by Rychkov and others.

Conformal invariance

Scaling fields $V_\Delta(x)$, $x \in \mathbb{R}^d$, $\Delta \in \mathbb{R}$

Correlation functions invariant under rotations and translations of $\mathbb{R}^d$ and under scaling

$$\langle \prod_i V_{\Delta_i}(\lambda x_i) \rangle = \prod_i \lambda^{-2\Delta_i} \langle \prod_i V_{\Delta_i}(x_i) \rangle \quad (*)$$

Δ_i scaling dimension or **conformal weight**.

Conformal invariance: $(*)$ extends to conformal maps $x \to \Lambda(x)$,

E.g. in $d = 2$: $\mathbb{R}^2 \simeq \mathbb{C}$

$$\Lambda(z) = \frac{az+b}{cz+c} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

and $\lambda^{-2\Delta_i} \to |\Lambda'(z)|^{-2\Delta_i}$.

Natural setup is the **Riemann sphere**: $z \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Structure Constants

Use conformal map to fix three points to $\{0, 1, \infty\}$.

3-point functions are determined up to constants

$$\langle \prod_{k=1}^{3} V_{\Delta_k}(z_k) \rangle = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} C(\Delta_1, \Delta_2, \Delta_3)$$

with $\Delta_{12} = \Delta_3 - \Delta_1 - \Delta_2$ etc.

$$C(\Delta_1, \Delta_2, \Delta_3) = \langle V_{\Delta_1}(0) V_{\Delta_2}(1) V_{\Delta_3}(\infty) \rangle$$

are called the **structure constants** of the CFT.

Bootstrap hypothesis

Operator Product Expansion Axiom:

$$V_{\Delta_1}(x_1)V_{\Delta_2}(x_2) = \sum_{\Delta \in \mathcal{S}} C^{\Delta}_{\Delta_1\Delta_2}(x_1, x_2, \partial_{x_2})V_{\Delta}(x_2)$$

assumed to hold when inserted to expectation:

$$\langle V_{\Delta_1}(x_1)V_{\Delta_2}(x_2)V_{\Delta_3}(x_3)\dots\rangle = \sum_{\Delta \in \mathcal{S}} C^{\Delta}_{\Delta_1\Delta_2}(x_1, x_2, \partial_{x_2})\langle V_{\Delta}(x_2)V_{\Delta_3}(x_3)\dots\rangle$$

- $C^{\Delta}_{\Delta_1\Delta_2}$ are **determined** by and **linear** in the structure constants
- $\mathcal{S}$ is called the **spectrum** of the CFT

Iterating OPE:

- All correlations are determined by $C(\Delta_1, \Delta_2, \Delta_3)$

Upshot: to “solve a CFT“ need to find its spectrum and structure constants.

Axioms for Weyl and Diff(Σ)

CFT extends naturally to surfaces Σ with **Riemannian metric g**

Diffeomorphism covariance axiom: For $\psi \in Diff(\Sigma)$

$$\langle \prod_i V_{\Delta_i}(x_i) \rangle_{\Sigma,g} = \langle \prod_i V_{\Delta_i}(\psi(x_i)) \rangle_{\Sigma,\psi^* g}$$

Weyl covariance axiom: For $\sigma \in C^\infty(\Sigma)$

$$\langle \prod_i V_{\Delta_i}(x_i) \rangle_{\Sigma,e^\sigma g} = e^{\frac{c}{96\pi} \int_\Sigma (|d\sigma|^2 + 2R_g \sigma) dv_g} \prod_i e^{-\Delta_i \sigma(x_i)} \langle \prod_i V_{\Delta_i}(x_i) \rangle_{\Sigma,g}$$

c **central charge** of the CFT, R_g scalar curvature, v_g volume

Hence correlations defined on **moduli space** of Riemann surfaces

$$g \sim e^\sigma \psi^* g \quad \psi \in Diff(\Sigma), \;\; \sigma \in C^\infty(\Sigma)$$

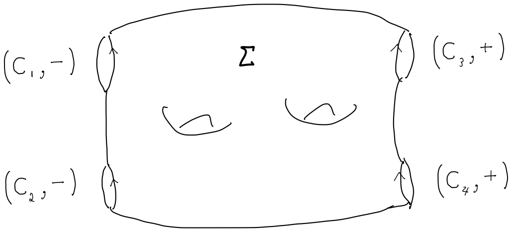
G. Segal's formulation of bootstrap

- Σ closed oriented Riemann surface with $n \geq 0$ marked points $z_1, \ldots, z_n$ and boundary

$$\partial\Sigma = \cup_i C_i$$

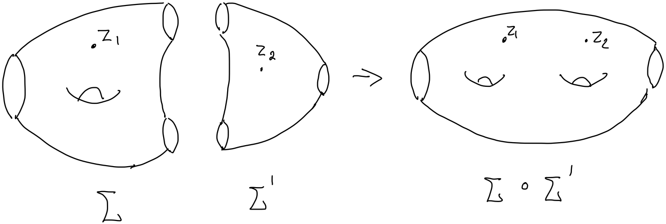
together with analytic parametrisations $\zeta_i : \mathbb{T} \to C_i$.

- Set $\sigma_i = \pm 1$ depending on whether orientation of $\zeta_i(\mathbb{T})$ agrees with that of Σ or not. Call them "in" and "out" boundaries.



Gluing surfaces

Glue "out" circles to "in" circles $(\Sigma, \Sigma') \to \Sigma \circ \Sigma'$



Segal's CFT functor

CFT consists of a **Hilbert space** $\mathcal{H}$ and an assignement

$$\Sigma \to \mathcal{A}_\Sigma$$

where

- $\mathcal{A}_\Sigma : \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes n}$ is a **Hilbert-Schmidt operator**
- Σ has *m* in-circles and *n* out-circles

Gluing Axiom

$$\mathcal{A}_{\Sigma \circ \Sigma'} = \mathcal{A}_\Sigma \mathcal{A}_{\Sigma'}$$

Semigroup of annuli

Let $v(z)\partial_z$ be an analytic vector field in a neighborhood of $\mathbb{D}$

$$v(z) = \sum_{n=0}^{\infty} v_n z^{n+1}$$

with $\mathrm{Re}\, v_0 < 0$ small enough. Its flow

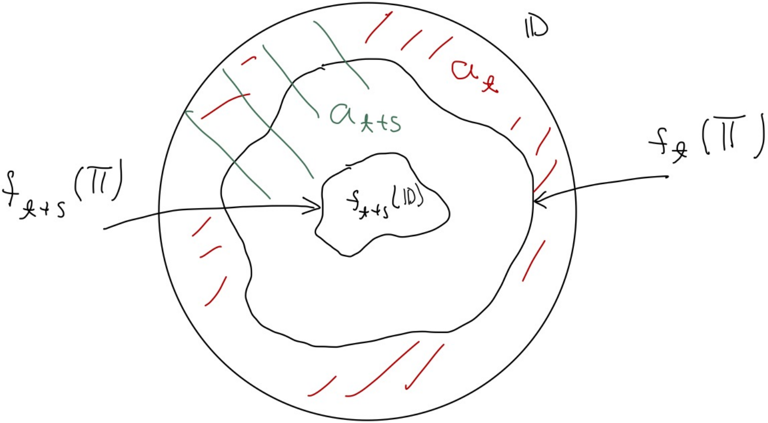
$$\frac{d}{dt} f_t(z) = v(f_t(z))$$

is univalent $f_t : \mathbb{D} \to f_t(\mathbb{D}) \subset \mathbb{D}$ and $a_t = \mathbb{D} \setminus f_t(\mathbb{D})$ are annuli with parametrised boundaries

$$e^{i\theta} \in \mathbb{T} \to \begin{cases} f_t(e^{i\theta}) & \text{on} \quad \mathcal{C}_1 \\ e^{i\theta} & \text{on} \quad \mathcal{C}_2 \end{cases}$$

and satisfying

$$a_t \circ a_s = a_{t+s}$$



$\mathbb{D}$
a_t
a_{t+s}
$f_t(\mathbb{T})$
$f_{t+s}(\mathbb{T})$
$f_{t+s}(\mathbb{D})$

Virasoro algebra

$\mathcal{A}_t := \mathcal{A}_{a_t}$ is a contraction semigroup on $\mathcal{H}$:

$$\mathcal{A}_t \mathcal{A}_s = \mathcal{A}_{t+s}$$

The generator $\mathcal{H}_v$ of $\mathcal{A}_t = e^{-t\mathcal{H}_v}$ is given by

$$\mathcal{H}_v = \sum_{n=0}^{\infty} (v_n L_n + \bar{v}_n \tilde{L}_n)$$

with $L_n, \tilde{L}_n$ densely defined operators in $\mathcal{H}$.

Setting $L_{-n} = L_n^*$, the adjoint in $\mathcal{H}$, one postulates for $n, n \in \mathbb{Z}$:

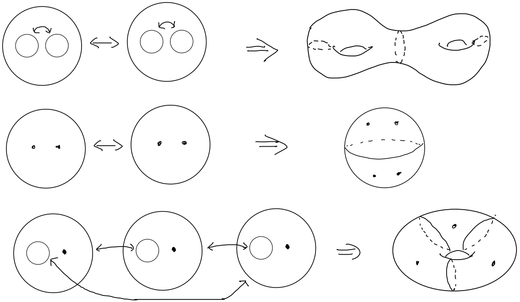
$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$
$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$
$$[L_n, \tilde{L}_m] = 0$$

where c is the central charge.

Building blocks

Build Σ by gluing simple topological building blocks $\mathcal{B}$:

- Pairs of pants $\mathcal{P} \sim \hat{\mathbb{C}} \setminus 3$ disks
- Annuli with one marked point $\hat{\mathbb{C}} \setminus \{2 \text{ disks}, \ 1 \text{ point}\}$
- Disks with two marked points $\hat{\mathbb{C}} \setminus \{1 \text{ disk}, \ 2 \text{ points}\}$



Plumbing

The moduli space $\mathcal{M}_{g,m}$ of Riemann surfaces of genus g and m marked points is a complex orbifold of dimension $3g - 3 + m$.

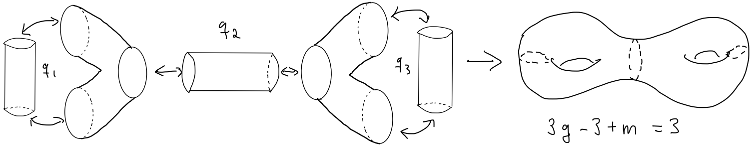
$\mathcal{M}_{g,m}$ can be parametrised by (Hinich-Vaintrob 2011)

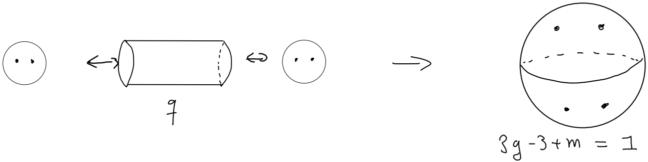
- Finite set of building blocks $\mathcal{B}_i$, $i = 1, \ldots N(g,m)$ where each $\mathcal{B}_i$ is a sphere with k punctures and $3 - k$ boundary circles, $k = 0, 1, 2$ equipped with a fixed conformal structure.
- **Plumbing parameters** $\mathbf{q} \in \mathbb{D}^{3g-3+m}$.
- Standard annulus a_q of modulus $|q|$, $q \in \mathbb{D}$:

$$e^{i\theta} \in \mathbb{T} \to \begin{cases} qe^{i\theta} & \text{on} \quad \mathcal{C}_1 \\ e^{i\theta} & \text{on} \quad \mathcal{C}_2 \end{cases}$$

  i.e. for $q = e^{-t+i\theta}$ cylinder of length t.
- Glue building blocks $\{\mathcal{B}_i\}$ together with annuli a_{q_j}

Plumbing





Bootstrap

Upshot:

Correlation function on Σ is given by compositions of the operators $\mathcal{A}_{\mathcal{B}_i}$ and $\mathcal{A}_{a_{q_j}}$

- $\mathcal{A}_{\mathcal{B}_i}$ determined by structure constants
- $\mathcal{A}_{a_q} : \mathcal{H} \to \mathcal{H}$ is the semigroup: $A_{a_q} = q^{L_0} \bar{q}^{\tilde{L}_0}$
- Composition of $\mathcal{A}_{\mathcal{B}_i}$ using **eigenfunctions** of $\mathcal{A}_q$
- Eigenfunctions of $\mathcal{A}_q$ determined by representation theory

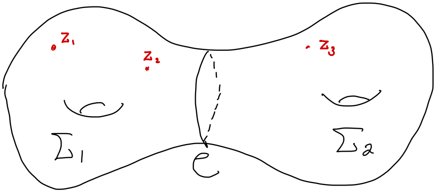
Path integrals

Motivation for axioms: let the QFT be given formally as a path integral, e.g. for a scalar field ϕ

$$\langle \prod_{i=1}^{n} V_{\Delta_i}(z_i) \rangle = \int_{\phi:\Sigma\to\mathbb{R}} \prod_{i=1}^{n} V_{\Delta_i}(\phi(z_i)) e^{-S_\Sigma(\phi)} D\phi$$

with local action functional $S_\Sigma(\phi)$

Let $\Sigma = \Sigma_1 \circ \Sigma_2$, $\partial\Sigma_i = \mathcal{C}$ so that $S_\Sigma = S_{\Sigma_1} + S_{\Sigma_2}$.



Path integrals

Let for $\varphi : \mathcal{C} \to \mathbb{R}$

$$\mathcal{A}_{\Sigma_j}(\varphi) = \int_{\phi|_{\Sigma_j = \varphi}} \prod_{i: z_i \in \Sigma_j} V_{\Delta_i}(z_i) e^{-S_{\Sigma_j}(\phi)} D\phi \quad j = 1, 2$$

Then formally get

$$\langle \prod_{i=1}^{n} V_{\Delta_i}(z_i) \rangle = \int_{\varphi:\mathcal{C}\to\mathbb{R}} \mathcal{A}_{\Sigma_1}(\varphi)\mathcal{A}_{\Sigma_2}(\varphi) D\varphi$$

This talk:

- Probabilistic construction of $\mathcal{A}_\Sigma$ for **Liouville CFT**
- Prove gluing $\mathcal{A}_{\Sigma \circ \Sigma'} = \mathcal{A}_\Sigma \mathcal{A}_{\Sigma'}$
- Use this to prove bootstrap and compute correlations.

Liouville Theory

Action functional

$$S_\Sigma(\phi) = \int_\Sigma (|d\phi|^2 + QR_g\phi + \mu e^{\gamma\phi})dv_g$$

- $\gamma \in (0, 2]$
- $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, $\mu > 0$.
- R_g Ricci curvature of the Riemannian metric g

Occurs among other places in

- Noncritical string theory (Polyakov 1981)
- 2d gravity Knizhnik, Polyakov, Zamolodchikov (1988)
- 4d SuSy Yang-Mills (Alday, Gaiotto, Tachikawa 2010)

Probablistic Liouville Theory

We define

$$\langle F\rangle_{\Sigma,g} := Z_g \int_{\mathbb{R}} \mathbb{E}\big(F(\phi_g)e^{-\int_\Sigma QR_g\phi_g dv_g + \mu M_\gamma(\Sigma)}\big)dc$$

- $\phi_g = c + X_g$
- X_g is **Gaussian free field**: $\mathbb{E}X_g(x)X_g(y) = -\Delta_g^{-1}(x,y)$
- **Gaussian multiplicative chaos** measure

$$M_\gamma = \lim_{\epsilon\to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma\phi_{g,\epsilon}} dv_g$$

- $Z_g = (\det'(\Delta_g)/\mathrm{v}_g(\Sigma))^{-1/2}$ with zeta function regularisation.

Primary fields are **vertex operators**

$$V_\alpha(z) = e^{\alpha\phi_g(z)}$$

defined through limits of regularised objects.

We want to compute their correlators

$$\langle\prod_i V_\alpha(z_i)\rangle_{\Sigma,g}$$

Existence and Structure constants

Theorem (David, K, Rhodes, Vargas, CMP 2016) *The correlation functions exist and are nontrivial if and only if the* ***Seiberg bounds*** *hold:*

$$(1)\quad \alpha_i < Q \ \ \forall i, \quad \text{and} \quad (2)\ \sum_{i=1}^{n} \alpha_i + \chi(\Sigma) Q > 0$$

V_α *are primary fields with scaling dimension* $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$

For the structure constants we take $\Sigma = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then

Theorem (K, Rhodes, Vargas, Annals of Mathematics **191**, 81) *Let* α_i *satisfy the Seiberg bounds. Then*

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\hat{\mathbb{C}}} = C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$$

where $C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$ *is an explicit formula conjectured by Dorn, Otto, Zamolodchicov, Zamolodchicov in 1995.*

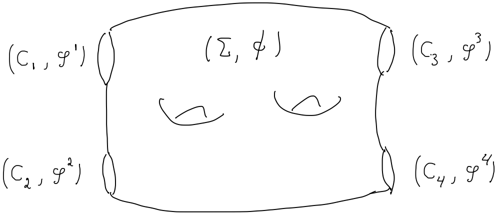
Amplitudes

Let $\partial\Sigma = \cup_{i=1}^{n} \mathcal{C}_i$. For $\phi : \Sigma \to \mathbb{R}$ set

$$\phi|_{\mathcal{C}_i} = \varphi_i, \quad \boldsymbol{\varphi} := (\varphi^1, \ldots, \varphi^n)$$

How to make sense of

$$\mathcal{A}_\Sigma(\boldsymbol{\varphi}) = \int_{\phi|_{\partial\Sigma} = \boldsymbol{\varphi}} \prod_i V_{\alpha_i}(z_i) e^{-S_\Sigma(\phi)} D\phi \ \ ?$$



Free Field Amplitudes

Free field action

$$S^0(\phi) := \int_\Sigma |d\phi|^2 dv_g, \quad \phi|_{\partial\Sigma} = \boldsymbol{\varphi}$$

Let ϕ_0 be the minimiser

$$\Delta_g \phi_0 = 0. \quad \phi_0|_{\partial\Sigma} = \boldsymbol{\varphi}$$

By Green formula

$$S^0(\phi) = S^0(\phi_0) + S^0(Z), \quad Z = \phi - \phi_0$$

and $S^0(\phi_0)$ reduces to a boundary term

$$S^0(\phi_0) = \int_{\partial\Sigma} \phi_0 \partial^\perp \phi_0 = (\boldsymbol{\varphi}, D_\Sigma \boldsymbol{\varphi})$$

where D_Σ is the **Dirichlet-Neumann** operator acting on the boundary fields

$$\phi|_{\partial\Sigma} = \boldsymbol{\varphi} = (\varphi^1, \varphi^2, \dots \varphi^n)$$

Free Field amplitudes

Let $\varphi^j(\theta) = \sum_{k\in\mathbb{Z}} \varphi^j_k e^{ik\theta}$. Then

$$S^0(\phi_0) = \frac{1}{4}\sum_{j=1}^{n}\sum_{k\in\mathbb{Z}} |k||\hat{\varphi}^j_k|^2 + (\varphi, \tilde{D}_\Sigma \varphi)$$

- $\tilde{D}_\Sigma$ is **smoothing**: $(\varphi, \tilde{D}_\Sigma \varphi)$ defined on $\varphi^i \in H^{-s}(\mathbb{T})\ \forall s > 0$.
- Z is the Dirichlet GFF on Σ

Definition. The free field amplitude is defined by

$$\mathcal{A}^0_\Sigma(\varphi) = \det(-\Delta_g^{dir})^{-\frac{1}{2}} e^{-(\varphi, \tilde{D}_\Sigma \varphi)}$$

where the determinant is zeta function regularised.

Liouville Amplitudes

Definition. The Liouville amplitude with vertex operators at z_i

$$\mathcal{A}_\Sigma(\varphi) = \mathcal{A}^0_\Sigma(\varphi)\mathbb{E}\big(\prod V_{\alpha_i}(z_i)e^{-\int_\Sigma QR_g\phi dv_g - \mu M_\gamma(\Sigma)}\big)$$

where $\phi = \phi_0 + Z$, and $\mathbb{E}$ is over the Dirichlet GFF Z.

Let μ be the measure on $\varphi = \sum_{k\in\mathbb{Z}} \varphi_k e^{ik\theta} \in H^s(\mathbb{T})$, $s < 0$

$$d\mu(\varphi) = d\varphi_0 \prod_{k>0} \tfrac{1}{\pi|k|} e^{-|k|\,|\hat{\varphi}_k|^2} d^2\varphi_k$$

View $\mathcal{A}_\Sigma(\varphi)$ as an integral kernel and take as **Liouville Hilbert space**

$$\mathcal{H} = L^2(H^s(\mathbb{T}), d\mu).$$

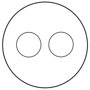
Then

Proposition (GKRV'21). $\mathcal{A}_\Sigma$ are Hilbert-Schmidt operators and

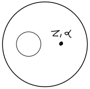
$$\mathcal{A}_{\Sigma\circ\Sigma'} = \mathcal{A}_\Sigma \mathcal{A}_{\Sigma'}$$

Examples

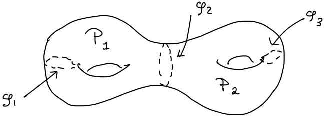
Building blocks: $\mathcal{P}$



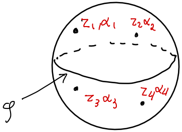
$(\mathbb{C}, z, \alpha)$



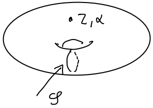
$(\mathbb{D}, \underset{\sim}{z}, \underset{\sim}{\alpha})$



$$\langle 1 \rangle_{\Sigma} = \int A_{\mathcal{P}_1}(\varphi_1, \varphi_1, \varphi_2) A_{\mathcal{P}_2}(\varphi_2, \varphi_3, \varphi_3) \prod_{i=1}^{3} d\mu(\varphi_i)$$



$$\left\langle \prod_{i=1}^{4} V_{\alpha_i}(z_i) \right\rangle_{\hat{\mathbb{C}}} = \int A_{\mathbb{D}, z_1 z_2 \alpha_1 \alpha_2}(\varphi) A_{\mathbb{D}, z_3 z_4 \alpha_3 \alpha_4}(\varphi) \, d\mu(\varphi)$$



$$\langle V_{\alpha}(z) \rangle_{\mathbb{T}^2} = \int A_{\mathbb{C}, z, \alpha}(\varphi, \varphi) \, d\mu(\varphi)$$

Spectrum of Liouville theory

Theorem (GKRV 2020) The semigroup of annuli (cylinders) $\{\mathcal{A}_q\}_{q\in\mathbb{D}}$ has a continuous spectrum and a **complete set of generalised eigenfunctions** $\Psi_{P,\nu,\tilde{\nu}}$:

$$\mathcal{A}_q\Psi_{P,\nu,\tilde{\nu}} = q^{\Delta_{Q+iP}+|\nu|}\bar{q}^{\Delta_{Q+iP}+|\tilde{\nu}|}\Psi_{P,\nu,\tilde{\nu}}$$

$$\Psi_{P,\nu,\tilde{\nu}} = L_{-\nu_1}\dots L_{-\nu_k}\tilde{L}_{-\tilde{\nu}_1}\dots\tilde{L}_{-\tilde{\nu}_k}\Psi_{P,0,0}$$

- $P\in\mathbb{R}$ and ν, $\tilde{\nu}$ are Young diagrams, $|\nu| := \sum\nu_i$.
- $\Psi_{P,0,0}$ is a **highest weight state** of weight Δ_{Q+iP}:

  $$L_0\Psi_{P,0,0} = \Delta_{Q+iP}\Psi_{P,0,0} = \tilde{L}_0\Psi_{P,0,0},\;\; L_n\Psi_{P,0,0} = 0 = \tilde{L}_n\Psi_{P,0,0},\;\; n>0$$

- $\Psi_{P,0,0}$ is amplitude of the disk $\mathbb{D}$ with $V_{Q+iP}(0)$ insertion
- **CFT spectrum** of LCFT is $\{\Delta_{Q+iP}\}_{P\in\mathbb{R}}$
- $\{L_n\}, \{\tilde{L}_n\}$ can be constructed by using the general annuli semigroup (Baverez, GKRV 2022)

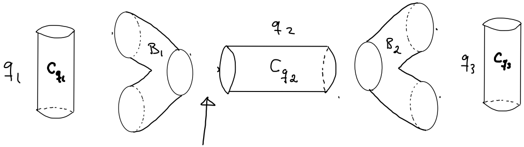
Plancharel

Completeness

$$\langle \Psi_{P,\nu|\tilde\nu} | \Psi_{P',\nu',\tilde\nu'} \rangle = \delta(P-P') F_{\nu,\nu'}(P) F_{\tilde\nu,\tilde\nu'}(P)$$

Apply to amplitude compositions:

$$\int A(\varphi)A'(\varphi,)d\mu(\varphi) = \sum_{\nu,\tilde\nu} \int_{\mathbb{R}} F^{-1}_{\nu,\nu'}(P) F^{-1}_{\tilde\nu,\tilde\nu'}(P) \langle A|\Psi_{P,\nu|\tilde\nu}\rangle\langle\Psi_{P',\nu',\tilde\nu'}|A'\rangle dP$$



$$\sum_{\nu,\tilde\nu} \int dp\, F^{-1}_{\nu\nu'}(p)\, F^{-1}_{\tilde\nu,\tilde\nu'}(p) |\psi_{p,\nu,\tilde\nu}\rangle\langle\psi_{p,\nu',\tilde\nu'}|$$

Holomorphic factorisation

Need to evaluate amplitudes of building blocks at eigenstates:

Proposition. Let $\mathcal{B}$ be a pair of pants. Then

$$A_{\mathcal{B}}(\otimes_{j=1}^{3}\Psi_{Q+iP_j,\nu_i,\tilde{\nu}_i}) = D_{\boldsymbol{\nu}}(\boldsymbol{Q}+\boldsymbol{iP})D_{\tilde{\boldsymbol{\nu}}}(\boldsymbol{Q}+\boldsymbol{iP})C_{DOZZ}(\boldsymbol{Q}+\boldsymbol{iP})$$

where $\boldsymbol{Q}+\boldsymbol{iP} = (Q+iP_1, Q+iP_2, Q+iP_3)$. Similar factorisation for other building blocks.

Proof is based on **probabilistic Ward identities**.

Integrability of Liouville theory

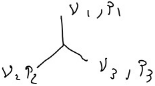
GKRV (2021). Let Σ have genus g. Then

$$\langle \prod_{i=1}^{m} V_{\alpha_i}(z_i) \rangle_\Sigma = \int_{\mathbb{R}_+^{3g+m-3}} |\mathcal{F}(\mathbf{q}, \mathbf{P})|^2 \rho(\mathbf{P}) d\mathbf{P}$$

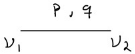
where

- $\mathbf{q}$ are plumbing parameters
- Conformal block $\mathcal{F}(\mathbf{q}, \mathbf{P})$ is purely representation theoretic and **holomorphic in the moduli** $\mathbf{q}$
- $\rho(\mathbf{P})$ is a product of structure constants $C(\alpha, \alpha', \alpha'')$ with $\alpha, \alpha', \alpha'' \in \{\alpha_i, Q \pm iP_j\}$

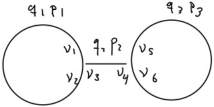
Conformal block Feynman rules



$= D_{\tilde\nu}(\tilde p, \Sigma)$

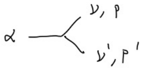


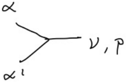
$= q^{|\nu|} F^{-1}_{\nu_1 \nu_2}(p)$

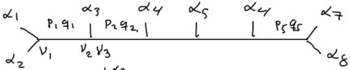


$$= \sum_{\nu_1, \dots, \nu_6} q_1^{|\nu_1|} q_2^{|\nu_3|} q_3^{|\nu_5|} F^{-1}_{\nu_1 \nu_2}(p_1) F^{-1}_{\nu_3 \nu_4}(p_2) F^{-1}_{\nu_5 \nu_6}(p_3) D_{\nu_1 \nu_2 \nu_3}(p_1, p_1, p_2) D_{\nu_4 \nu_5 \nu_6}(p_2, p_3, p_3)$$

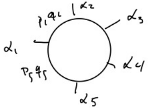
Insertions:







Sphere 8 point



Torus 5 point

Open questions

Correlation functions are modular invariant i.e. the same no matter how we cut the surface.

How about the conformal blocks? Suppose Σ is parametrised by $(\{\mathcal{B}_i\}, \mathbf{q})$ and $(\{\mathcal{B}'_i\}, \mathbf{q}')$. Are the blocks linearly related? True for $(g, n) = (0, 4)$ and $(g, n) = (1, 1)$.

Connection to quantisation of Teichmuller space?

Thank you!