Woodson Research Center, Fondren Library, Rice University.

 

David Blackwell (left) and Richard Tapia were honored for their achievements, both mathematical and personal, at a conference held last spring at Cornell University. With the conference, Cornell inaugurated the David Blackwell and Richard Tapia Distinguished Lecture Series in the Mathematical and Statistical Sciences, to be given every other year, beginning in 2002, by a distinguished African American, Latino, or Native American mathematician.

The Remarkable Journey of The Isoperimetric Problem: A Completion of Euler's Approach

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"A Unified Approach to Infinite Dimensional Optimization Theory for Scientists and Engineers"

Outline

Part I	Preliminaries
Part II	The state of the art:
	What others have done to solve the isoperimetric problem
Part III	Our contribution:
	A completion of Euler approach to the isoperimetric problem

Part I Preliminaries

The Isoperimetric Problem

Determine, from all simple closed planar curves of the same perimeter, the one that encloses the greatest area.

Visibility of the Isoperimetric Problem Promoted by Queen Dido

- The Aeneid: written by Virgil in period 29-19 BC
- Dido – Life in danger flees her homeland with wealth and entourage
- Finds new land and bargains with local king for a piece of land that she can mark out with the hide of a bull
- The Dido trick: cut hide into as many thin strips as possible to form a long cord, using the seashore as one edge, lay out the cord in the form of a semicircle in order to maximize area.

Engraving in German Museum circa 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

The Dido Maximum Principle in Action?

Medieval map of Cologne

Again the Dido Maximum Principle in Action?

Medieval map of Paris

Claim

The isoperimetric problem has been the most impactful mathematical problem of all time.

- Euler constructed multiplier theory in an attempt to solve this problem.
- Weierstrass first introduced parametric representation of a curve to solve this problem.

Part II: The State of the Art: What Others Have Done

Solution Attempt Classifications

Our study has shown us that all isoperimetric problem solution attempts can be put into one of the following three categories:

(1) The Euler Approach (1744): Characterized by Cartesian coordinate functional representation $y = y(x)$

(2) The Steiner Approach (1838): Characterized by complete use of geometry

(3) The Weierstrass Approach (1879): Characterized by parametric function representation,

$$y = y(t), x = x(t)$$

The Completion Process

Necessity: If the problem has a solution then it is the circle.

Sufficiency: The circle solves the problem.

By **a completion** of each category we mean the activity of adding to the initial approach that gave necessity an appropriately short proof of sufficiency.

Completion of Weierstrass – Hurwitz (1902)

Hardy-Littlewood-Polya (1934)

Lax (1995)

Completion of Steiner – Lawlor (1999)

Siegel (2003)

Completion of Euler – Today's task

Concerning the Isoperimetric Problem
What Did Euler Do (Or Not Do)
In His Opus Magnum of 1744?

Euler Considered What We Will Call
The Incomplete Isoperimetric Problem

For a given arc length $\ell > 0$ and for a **fixed** $a > 0$ consider

$$\underset{y}{\text{maximize}} \int_{-a}^{a} y(x)dx \qquad \text{(Area)}$$

$$\text{subject to } \int_{-a}^{a} \sqrt{1 + y'(x)^2}\, dx = \ell, \qquad \text{(Arc Length)}$$

$$y(-a) = y(a) = 0. \qquad \text{(Boundary Conditions)}$$

Observation: We call this problem incomplete because both $y: [-a, a] \to R$ and the base parameter a should be variables.

Euler's Contribution to the Isoperimetric Problem

He constructed the **Euler Multiplier Rule** auxiliary problem

$$\underset{y}{\text{maximize}} \quad J(\mathrm{y}) = \int_{-a}^{a} \left(y(x) - \boldsymbol{\lambda}\sqrt{1 + y'(x)^2} \right)$$

$$\text{subject to} \qquad y(-a) = y(a) = 0.$$

He showed that a solution of the incomplete isoperimetric problem must satisfy the Euler (Lagrange) equation for the auxiliary problem and therefore must be arcs of the λ-parametric family of circles

$$y(x) = \sqrt{\lambda^2 - x^2} - \sqrt{\lambda^2 - a^2} \qquad -a \leq x \leq a.$$

The radius parameter $\lambda \geq a$ is determined from the arc length condition.

Corollary (Euler's Principle of Reciprocity)

He observed that the following two problems have the same solution:

- Determine from all simple closed planar curves of the same perimeter, the one that encloses the greatest area.
- Determine from all simple closed planar curves that enclose the same area, the one that has the smallest perimeter.

Observe that the semicircle

$$y(x) = \sqrt{a^2 - x^2} \qquad -a \leq x \leq a$$

satisfies the Euler's necessary conditions; it has the necessary form

$$y(x) = \sqrt{\lambda^2 - x^2} - \sqrt{\lambda^2 - a^2} \qquad \text{with} \quad \lambda = a.$$

Remark: Euler and many contemporary authors leave us to conclude that the semicircle satisfies the necessary conditions for the isoperimetric problem. But this is a giant mathematical sham.

An Obstacle in Euler's Road

While the semicircle does satisfy Euler's necessary conditions, it is not in the domain of definitions of the optimization problem. It does not satisfy the arc length constraint.

Claim: Consider the semicircle curve

$$y_c(x) = \sqrt{a^2 - x^2} \qquad -a \leq x \leq a.$$

For this semicircle curve y_c the arc length integral

$$\int_{-a}^{a} \sqrt{1 + y'(x)^2}\,dx$$

does not exist as a Riemann integral.

Proof. Notice

$$y_c' = \frac{-x}{\sqrt{a^2 - x^2}}$$

is not bounded, for $x \in [-a, a]$;

Hence the integrand is not bounded and therefore is not Riemann integrable.

Euler's 1744 Approach to the Isoperimetric Problem

- He showed that solutions to the isoperimetric problem in incomplete form are necessarily circular arcs.
- He said absolutely nothing about the semicircle.
- He said absolutely nothing about the complete isoperimetric problem, i.e. variable a, also gave no restrictions on a.
- Majority of texts on the calculus of variations follow Euler's presentation of the isoperimetric problem. Hence students cannot find a demonstration that the semicircle satisfies even necessary conditions for either the incomplete or the complete isoperimetric problem. It is interesting how textbooks subtly avoid this shortcoming.

Euler’s Naiveté Concerning the Euler-Lagrange Equation.

H.H. Goldstine states

“It is interesting that Euler did not completely understand the fact that satisfaction of the Euler (Lagrange) equation is a necessary condition but not a sufficient one.”

Euler’s Genius.

Carathéodory who edited Euler’s great works of 1744 said “Euler’s multiplier rule is an achievement of the first class and a major accomplishment that even an Euler did not achieve very often.”

Euler’s Generousity.

Euler was so impressed with 19 year old Lagrange‘s derivation of his multiplier rule using variations that he dropped his own methods, promoted those of Lagrange, named the subject the calculus of variations, and called multiplier theory Lagrange multiplier theory. Indeed today the Euler auxiliary functional given above is called the Lagrangian: how fair can that be?

We ask, does Euler have the right to give up credit for his original contributions simply because Lagrange came up with a cleaner way of deriving the theory?

Steiner (1838)

In 1838 Steiner, very aware of the short comings of the Euler approach, gave the first of his five equivalent proofs that the circle solved the isoperimetric problem. His proofs used synthetic geometry and were mathematically quite elegant.

However, he fell into the use of necessity as sufficiency trap and made the trap rather infamous. The analysts of the time, led by his colleague Peter Dirichlet, pointed out to Steiner that his proof is not valid unless he assumes that the isoperimetric problem has a solution, i.e. existence. Steiner did not accept the criticism. He said that it is obvious that the problem has a solution. He was very critical of analysis.

A Completion of Steiner

Lawlor (1999) and Siegel (2003)

Weierstrass (1879)

Weierstrass believed that the short comings of the Euler approach (primarily infinite slope) could be eliminated by turning to parametric function representation.

So he for the first time introduced such representation. He then constructed an elegant and sophisticated sufficiency theory for problems from the calculus of variations. His theory could be used to show that the circle solved the isoperimetric problem.

Weierstrass (1879)

So, some 135 years after Euler's approach we have the first sufficiency proof. While this notable work gave the world its first sufficiency proof for the isoperimetric problem, Weierstrass really used a sledge hammer to pound a nail. His sophisticated sufficiency theory is not needed to merely demonstrate that the circle solves the isoperimetric problem.

A Completion of Weierstrass

Hurwitz (student of Weierstrass)	1902	(Properly Rated)
Hardy-Littlewood-Polya	1934	(Over Rated)
Peter Lax	1995	(Under Rated)

A Qualification for Authors from Euler to the Present

Carathéodory (1957), editor of Euler's works, wrote:

"Weierstrass succeeded in removing many of the difficulties that were contained in the investigations of Euler, Lagrange, Legendre, and Jacobi by insisting that first the class of curves in which the minimizing curve is sought be **rigorously defined**. Euler's treatment of the isoperimetric problem is the prime example of the lack of rigor."

Part III: Our Contribution:
A Completion of Euler's Approach

Our Definition of the Proper Class of Functions

Definition. By $E(-a,a)$, the **Euler Class** of curves for the incomplete isoperimetric problem, we mean the collection $y\colon [-a,a] \to R$ satisfying the following conditions

A. $y(-a) = y(a) = 0$,

B. y is continuous on $[-a,a]$,

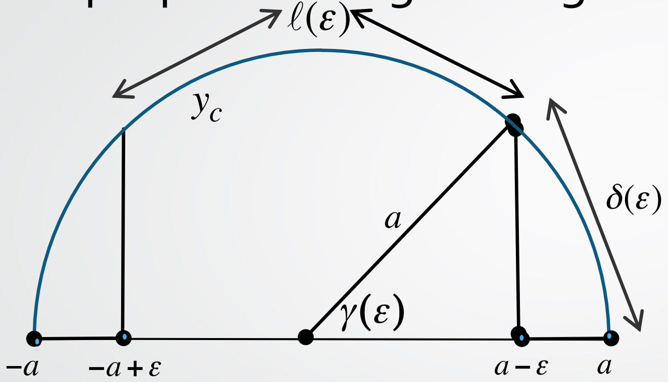
C. y is differentiable except possible on a countable subset of $[-a,a]$,

D. The curve y is rectifiable and the arc length can be obtained from the formula

$$\ell(y) = \lim_{\varepsilon \to 0} \int_{-a+\varepsilon}^{a-\varepsilon} \sqrt{1 + y'(x)^2}\, dx$$

Proposition: The Semicircle is in the Euler Class

A picture proof that the improper arc length integral exists and gives arc length



$$\hat{\ell}(\varepsilon) = \int_{-a+\varepsilon}^{a-\varepsilon} \sqrt{1 + y_c'(x)^2}\,dx = a\pi - 2\delta(\varepsilon).$$

with

$$\delta(\varepsilon) = \mathrm{a}\gamma(\varepsilon) = a\cos^{-1}\left(\frac{a-\varepsilon}{a}\right) \to a\cos^{-1}(1) = 0 \text{ as } \varepsilon \to 0.$$

So improper integral exists and gives arc length.

The Complete Isoperimetric Problem

For a given arc length $\ell > 0$

$$\underset{(a,y)}{\text{maximize}} \; J(a,y) = \int_{-a}^{a} y(x)dx$$

$$\text{subject to} \; \int_{-a}^{a} \sqrt{1 + y'(x)^2}\, dx = \ell,$$

$$y(-a) = y(a) = 0,$$
$$0 < a.$$

Observation: **Both** a and the function $y: [-a, a,] \to R$ are **variables.**

Bilevel Mathematical Model of The Complete Isoperimetric Problem

For a given arc length $\ell > 0$

$$\left\{ \begin{array}{c} \underset{a>0}{\text{maximize}} \int_{-a}^{a} y_a(x)dx \\ \text{subject to } y_a \text{ solves} \end{array} \right\}$$

Level 1 Problem: Complete Isoperimetric Problem

$$\left\{ \begin{array}{c} \underset{y}{\text{maximize}} \quad \int_{-a}^{a} y(x)\, dx \\ \text{subject to} \quad \int_{-a}^{a} \sqrt{1 + y'(x)^2} dx = \ell, \\ y(-a) = y(a) = 0, \\ \text{for a fixed } a \text{ satisfying } a > 0. \end{array} \right\}$$

Level 2 Problem: Incomplete Isoperimetric Problem

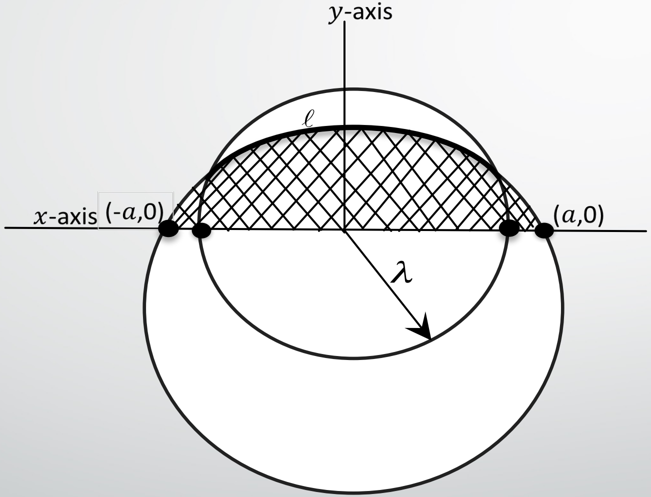
Note: We use the qualifier "extended" to mean that the arc length integral may be improper and the domain is the Euler class.

Solution of the Level 2 Problem: A Subtle but Critical Observation

A graphical interpretation of the Euler family of circular arcs

$$y(x) = \sqrt{\lambda^2 - x^2} - \sqrt{\lambda^2 - a^2}, \qquad -a \leq x \leq a,$$

with arc length ℓ, varying radius parameter λ, and varying base parameter a is shown below



For fixed arc length ℓ, the choice of λ determines the base parameter a. The choice of $\lambda = a$ leads to $a = \ell/\pi$, while the choice of $\lambda = \infty$ leads to $a = \ell/2$. So membership in the Euler family of circular arcs requires $\frac{\ell}{\pi} \leq a < \frac{\ell}{2}$.

Solution of the Level 2 Problem

Theorem. *Consider the extended incomplete isoperimetric problem (the level 2 problem) with arc length* ℓ *and base interval parameter* a *satisfying* $\ell/\pi \le a < \ell/2$*. Then there exists a radius parameter* $\lambda_* \ge a$ *so that the Euler family circular arc*

$$y(x) = \sqrt{\lambda_*^2 - x^2} - \sqrt{\lambda_*^2 - a^2}, \qquad -a \le x \le a \qquad (1)$$

has length ℓ *and is the unique solution of the extended incomplete isoperimetric problem for this choice of* a*.*

Proof. For each $a \in \left[\frac{\ell}{\pi}, \frac{\ell}{2}\right)$ there exists a corresponding $\lambda_* > 0$ leading to a circular arc of the form (1) with arc length ℓ.

We call this member of the family of circular arcs (1) y_*.

Consider Euler's ε −perturbed auxiliary function with choice of multiplier equal to $-\lambda_*$ where $\lambda_* > 0$ is the radius of the circle that determines the circular arc y_*

$$J_\varepsilon(y) = \int_{-a+\varepsilon}^{a-\varepsilon} \left(y(x) - \lambda_* \sqrt{1 + y'(x)^2}\right) dx. \qquad (2)$$

Our idea: Maximize the perturbed auxiliary function (2) (dare we say Lagrangian) and let $\varepsilon \to 0$. However, it is a little bit more tricky. Our main tool will be Taylor's theorem.

Remark. Recall from optimization theory, that which maximizes the Lagrangian solves the maximization constrained optimization problem. The so-called fundamental theorem of nonlinear programming.

So consider any y contained in the Euler class $E(-a, a)$ and let η denote $y - y_*$. Define

$$\phi_\varepsilon(t) = \int_{-a+\varepsilon}^{a-\varepsilon} \left[y_* + t\eta - \lambda_* \sqrt{1 + (y_*' + t\eta')^2}\, dx\right]$$

for $t \in [0,1]$ and $\varepsilon \in (0, a)$.

Notice that $\phi_0(0) = J_0(y_*)$ and $\phi_0(1) = J_0(y)$.

So we want to show

$$\phi_0(1) < \phi_0(0)$$

Using Taylor's theorem.

Straightforward differentiation gives

$$\phi_\varepsilon'(t) = \int_{-a+\varepsilon}^{a-\varepsilon} \left[\eta - \lambda_* \frac{(y_*' + t\eta')\eta'}{\sqrt{1 + (y_*' + t\eta')^2}} \right] dx, \quad (3) \qquad \phi_\epsilon''(t) = -\lambda_* \int_{-a+\varepsilon}^{a-\varepsilon} \left[\frac{(\eta')^2}{\left[1 + (y_*' + t\eta')^2\right]^{3/2}} \right] dx$$

Taylor's theorem tells us that

$$\phi_\varepsilon(1) = \phi_\varepsilon(0) + \phi_\varepsilon'(0) + \frac{1}{2}\phi_\varepsilon''(\theta_\varepsilon) \text{ for some } \theta_\varepsilon \in (0,1) \quad (4)$$

Step 1. We can show from (3) that $\boldsymbol{\phi_\varepsilon'(0) \to 0}$ **as** $\boldsymbol{\varepsilon \to 0}$.

Step 2. Since θ_ε is bounded, turning to a subsequence if necessary, we have that $\theta_\varepsilon \to \theta^* \in [0,1]$ therefore $\phi_\varepsilon''(\theta_\varepsilon) \to \phi_0''(\theta^*) < 0$ as $\varepsilon \to 0$.

Step 3. Letting $\epsilon \to 0$ in (4) and using our Euler class D condition gives

$$\text{Area}(y) - \lambda_* \text{ length}(y) < \text{Area}(y_*) - \lambda_* \text{length}(y_*).$$

Hence if length(y) = length(y_*), then Area(y) < Area (y_*).

Conclusion: y_* uniquely solves the extended incomplete isoperimetric problem for any $a \in [\ell/\pi, \ell/2)$.

Remark: The pieces fit together so remarkably well. We have turned Euler's necessity condition for the incomplete isoperimetric problem into a sufficiency condition.

Question:

Could Euler have made our observation at the time of his 1744 writing? The foundation of our observation is Taylor's theorem with remainder. The literature tells us that Taylor published his theorem in 1715. So Euler most likely was aware of Taylor's theorem in 1744. However, the rub is that Taylor's theorem with remainder was not known at that time. It is somewhat ironic that the form of the remainder that we used in our proof is credited to Lagrange in 1797, and is actually referred to today as the Lagrange form of the remainder. So Euler would not have been in good position to make our observation.

Remark

Lagrange could have made this proof because he was familiar with the form of Taylor's theorem that we used, indeed it is due to him. While this hypothesized proof would have been made 50 years after Euler, it would still have been some 80 years before Weierstrass. But Lagrange did not consider the isoperimetric problem.

We now move on to Level 1 of the bilevel problem and the solution of the isoperimetric problem.

Theorem. For a given arc length ℓ the semicircle

$$y(x) = \sqrt{\frac{\ell^2}{\pi} - x^2}, \qquad -\frac{\ell}{\pi} \leq x \leq \frac{\ell}{\pi},$$

uniquely solves the extended bilevel complete isoperimetric problem, for base parameter a contained in the interval $[{}^{\ell}/_{\pi}, {}^{\ell}/_{2})$.

Proof. A graphical interpretation of the circular arcs with arc length ℓ and varying radius parameter λ and varying base parameter a contained in the Euler family of circles in (1) is shown in Figure 1 below.

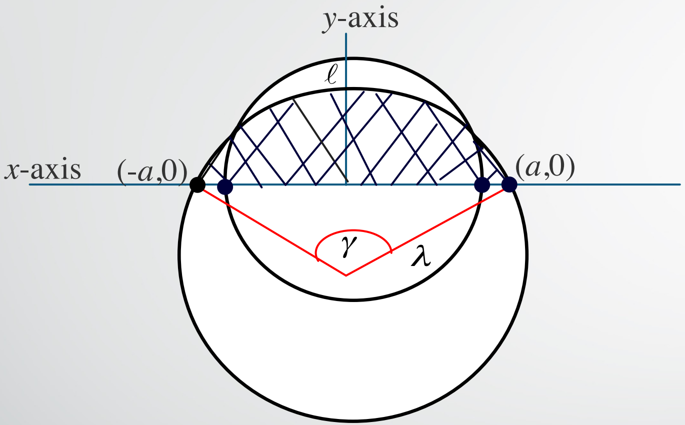


Figure 1. Euler Family of Circles with arc length ℓ above the x-axis.

Note: The dots depict the various values of $(-a,0)$ and $(a,0)$.

Note: Since the radius parameter satisfies $a \le \lambda < \infty$ in this class of circular arcs we must have $\frac{\ell}{\pi} \le a \le \frac{\ell}{2}$.

To show: From our Euler family of circular arcs with arc length ℓ the semicircle gives the largest cross-hatched area.

For fixed ℓ and a varying in the interval $\ell/_\pi \le a < \ell/_2$ the area of the cross-hatched secant-region in Figure 1 as a function of the radius parameter λ is

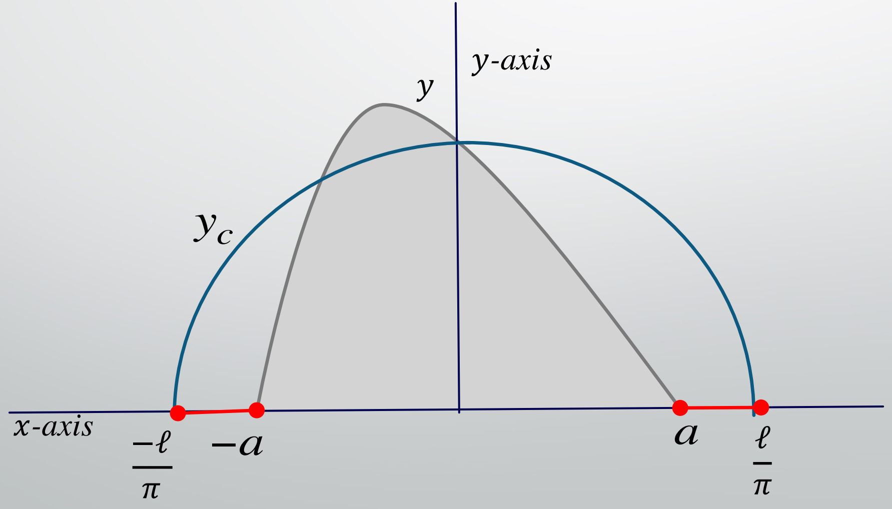
$$A(\lambda) = \frac{\ell\lambda}{2} - \frac{\lambda^2}{2}\sin\left(\frac{\ell}{\lambda}\right).$$

Direct differentiations show that for $\lambda^* = \ell/_\pi$ we have

$$A'(\lambda^*) = 0 \;\text{ and } A''(\lambda^*) < 0$$

so λ^* is a maximizer of the function $A(\lambda)$. Observe that $\lambda^* = \ell/_\pi$ can only happen for the semicircle. Hence the area of the cross-hatched secant-regions is uniquely maximized when its upper boundary is the semicircle that results from the choice $a = \ell/_\pi$. But this area is the objective function of the optimization problems under consideration.

But wait, we are not through yet. Our concern is $a > 0$. Now for $0 < a < {}^{\ell}/_{\pi}$ the level 2 problem cannot have a solution because the Euler necessary condition will not be satisfied, i.e., it will not be an arc from the Euler family of circular arcs. But we need to show that for for $0 < a < {}^{\ell}/_{\pi}$ there does not exist a curve with given arc length ℓ and area larger than that of the semicircle of radius ${}^{\ell}/_{\pi}$.



Theorem

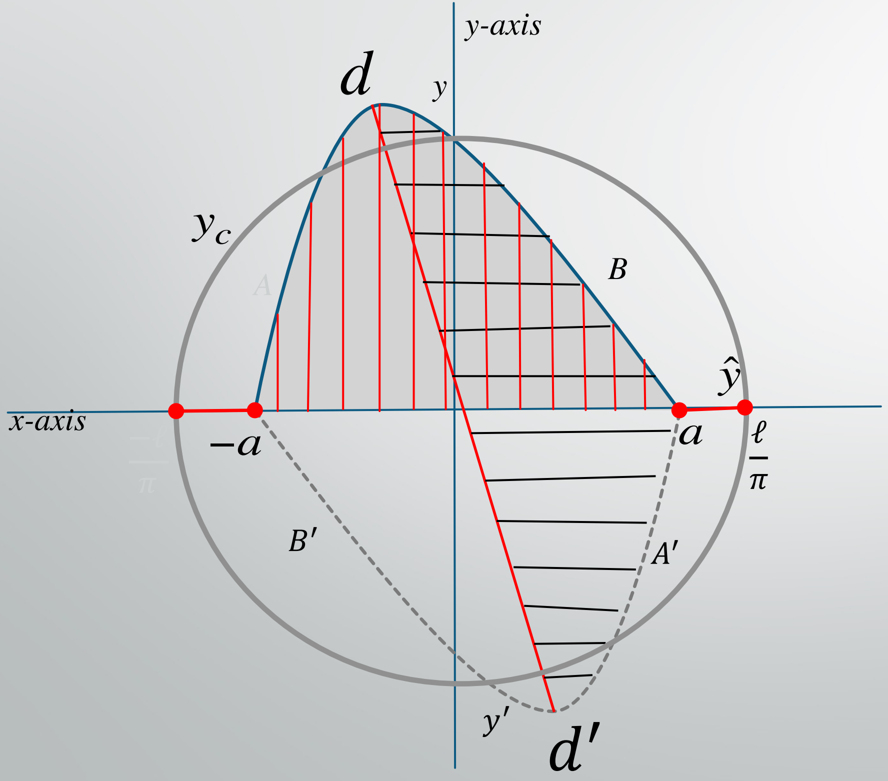
In the extended complete isoperimetric problem where we have $a > 0$ it is sufficient to restrict the variable a to the interval $\left[{}^{\ell}/_{\pi}, {}^{\ell}/_{2}\right)$. Hence for a given arc length ℓ the semicircle

$$y(x) = \sqrt{\left(\frac{\ell}{\pi}\right)^2 - x^2}, \qquad \frac{\ell}{\pi} \leq x \leq \frac{\ell}{2}$$

uniquely solves the extended bilevel complete isoperimetric problem with $a > 0$.

Proof. $a > {}^{\ell}/_{2}$ implies no feasible curves, while $a = {}^{\ell}/_{2}$ implies the only feasible curve is $y \equiv 0$. Now if $a < {}^{\ell}/_{\pi}$ then the following picture shows that any curve $y = [-a, a] \rightarrow R$ with arc length ℓ cannot be have greater area than the semicircle.

"The Jimmy Flip"
The Reflected Equal-Area Curve



- The proof of this theorem is due to Rice Undergraduate Zhe (Jimmy) Zhang and was produced while he was a student in my graduate optimization theory class some years back.
- Curve above vertical-hatched region and curve above horizontal-hatched region have same area and same arclength.

Observe:

If $\text{distance}(d, d') \leq \frac{\ell}{\pi}$ we are through

If $\text{distance}(d, d') > \frac{\ell}{\pi}$ previous theorem applies

Corollary

The circle solves the isoperimetric problem.

Thank You