A tale of two polytopes: the bipermutahedron and harmonic polytope

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Acknowledgment

I have benefitted enormously from my times at MSRI.

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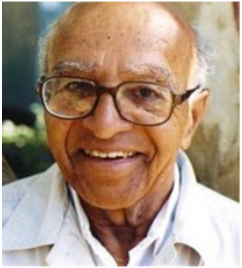
MSRI sits on the territory of xučyun (Huichin), the original landscape of the Chochenyo Ohlone people. Every member of the MSRI community benefits from the use and occupation of this land.

I also live and work in Ohlone land.

Sogorea Te' Land Trust: an urban Indigenous women-led land trust facilitating the return of Indigenous land to Indigenous people.

Acknowledgment

I have learned enormously from the mathematics and the leadership of David Blackwell, Richard Tapia, Tatiana Toro.



Thank you! Gracias! It’s an honor to celebrate you.

The plan

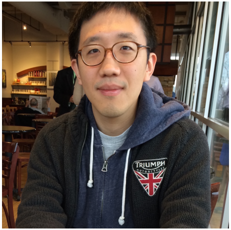
1. What is the permutahedron?
   Face enumeration
   Volume

2. What is the bipermutahedron?
   Face enumeration
   Volume?

3. What is the harmonic polytope?
   Face enumeration
   Volume

4. Where do they come from? A short origin story.

Bipermutahedron: with Graham Denham + June Huh (15-20).
Harmonic polytope: with Laura Escobar (20).

Lagrangian geometry of matroids. [ADH20]
`https://arxiv.org/abs/2004.13116`

The harmonic polytope. [AE20]
`https://arxiv.org/abs/2006.03078`

The bipermutahedron. [A20]
`https://arxiv.org/abs/2008.02295`

Permutations

A **permutation** of $[n] = \{1, \ldots, n\}$ is a reordering of $12\ldots n$.

$n = 3$: 123 132 213 231 312 321.

The set $[n]$ has $n!$ permutations. What structure do they have?

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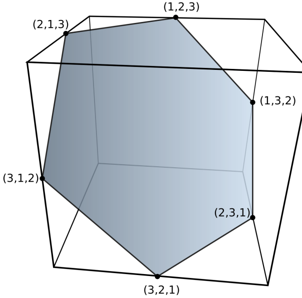
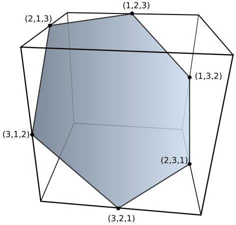
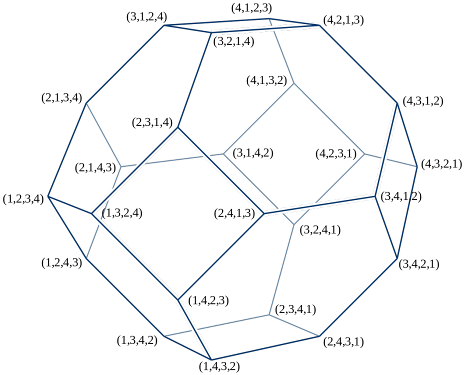
Algebra: the symmetric group! Geometry: the permutahedron!



The permutahedron

$$\Pi_n = \text{conv}\{(\sigma(1), \ldots, \sigma(n)) : \sigma \text{ permutation of } [n]\}$$





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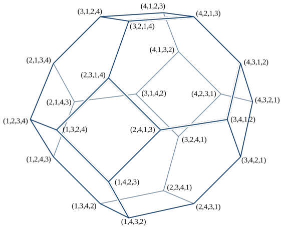

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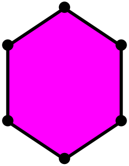
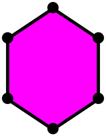


Prop. The inequalities defining the permutahedron are

$$\sum_{e \in [n]} x_e = n(n+1)/2,$$
$$\sum_{s \in S} x_s \geq |S|(|S|+1)/2 \qquad \emptyset \subsetneq S \subsetneq [n].$$

The f-vector of the permutahedron

- faces: **ordered set partitions** 12|47|368|5
- vertices: **permutations** 1|5|4|3|8|2|7|6 $n!$
- facets: **proper subsets** 12458 $2^n - 2$



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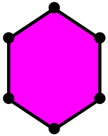

Theorem If $f_d(\Sigma_n) = \#$ of d-dimensional faces of Σ_n,

$$\sum_{d,n} f_d(\Sigma_n) \frac{x^d}{d!} \frac{y^n}{n!} = \frac{e^y - 1}{1 + xe^y - x}$$

The *h*-vector of the permutahedron

Encode the *f*-polynomial (counting faces) in the ***h*-polynomial**:

$$\begin{aligned} h_n(x) &= h_0 + h_1 x + \cdots + h_{n-1} x^{n-1} \\ &= f_0 + f_1(x-1) + \cdots + f_{n-1}(x-1)^{n-1} \end{aligned}$$

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Chavez-Yamzon 2017: The Dehn-Somerville matroid

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Prop. The h-vector of the permutahedron $\Pi_{n,n}$ is

$h_i(\Pi_{n,n}) =$ # of permutations of $[n]$ with i **descents.**

$h_3(x) = 1 + 4x + x^2$: 123 13.2 2.13 23.1 3.12 3.2.1

$h_n(x)$ is the n-th **Eulerian polynomial.**

The h-vector of the permutahedron

Prop.
The Eulerian polynomial is

$$\frac{x\,h_n(x)}{(1-x)^{n+1}} = \sum_{k\geq 0} k^n x^k$$

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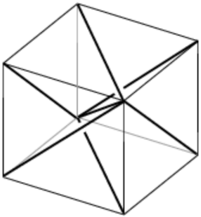
Prop.
- All roots of $h_n(x)$ are real and negative. (Frobenius)
- h-vector of permutahedron is log-concave: $h_i^2 \geq h_{i-1} h_{i+1}$

Why do combinatorialists care?
Combinatorics is full of log-concave sequences. The proof often requires a connection to a different area of mathematics.

The *h*-vector of the permutahedron via Ehrhart theory

Prop. There's a unimodular triangulation of the cube $\square_n$ that is combinatorially isomorphic to the cone over) Σ_n.

(Every simplex has volume $1/n!$)



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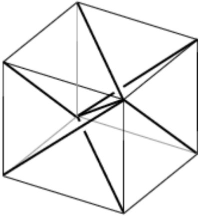

Then Ehrhart theory gives

$$\frac{x\,h_n(x)}{(1-x)^{n+1}} = \sum_{k\geq 0} k^n x^k$$

(LHS: faces of triangulation. RHS: lattice points of $k\square_n$)

Volume

Computing volumes is very hard! Can we do it for Π_n?

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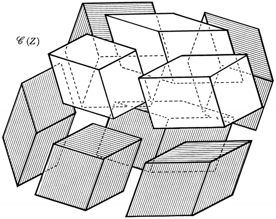
Good news: The permutahedron is a **zonotope**:

$$\Pi_n = \sum_{i<j} [e_i, e_j]$$

where $P + Q = \{p + q : p \in P, q \in Q\}$ is *Minkowski sum*.

This gives a tiling of Π_n into parallelepipeds!



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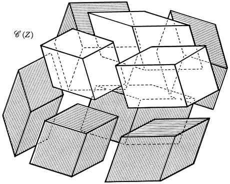
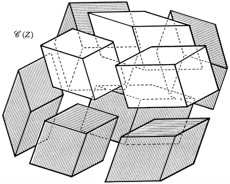


In the tiling of Π_n into parallelepipeds:

- each parallelepiped has volume 1, and
- # of parallelepipeds = # of trees on $[n]$ = $(n+1)^{n-1}$

Theorem. $Vol(\Pi_n) = (n+1)^{n-1}$

Volume



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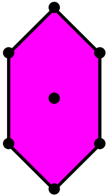
Equivariant setting:
FA-(Schindler/Supina)– Vindas-Meléndez 2020

The bipermutahedron



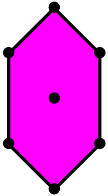
Prop. [FA-Denham-Huh 20, FA 20] The bipermutahedron is

$$\sum_{e\in[n]} x_e = \sum_{e\in[n]} y_e = 0,$$

$$\sum_{s\in S} x_s + \sum_{t\in T} y_t \geq -(|S| + |S-T|)(|T| + |T-S|) \qquad \text{for } S|T \sqsubseteq [n].$$

$S|T \sqsubseteq [n]$: subsets $S, T \neq \emptyset$, not both $[n]$, with $S \sqcup T = [n]$

Combinatorial structure of the bipermutahedron



- faces: **bisequences** 12|45|4|235
  (one number appears once, others once or twice)
- vertices: **bipermutations** 1|5|4|1|3|4|2|5|3. $(2n)!/2^n$
  (one number appears once, others twice)
- facets: **bisubsets** 1245|235 $3^n - 3$
  ($S, T \neq \emptyset$, not both $[n]$, with $S \cup T = [n]$)

The f-vector of the bipermutahedron

Prop. [FA 20] If $f_d(\Sigma_{n,n}) = \#$ of d-dim. faces of $\Sigma_{n,n}$,

$$\sum_{d,n} f_{d-2}(\Sigma_{n,n}) \frac{x^d}{d!} \frac{y^n}{n!} = \frac{F(x, e^y)}{e^x}$$

where

$$F(\alpha, \beta) = \sum_{n \geq 0} \frac{\alpha^n \beta^{\binom{n}{2}}}{n!}$$

is the two variable Rogers-Ramanujan function.

($F(\alpha, \beta)$ also arises in the generating functions for the (arithmetic) Tutte polynomials of root systems! (FA 02, De Concini-Procesi 08, FA-Castillo-Henley 15) Related: (Mphako, 2002). Connection?)

The h-vector of the bipermutahedron

Encode the f-polynomial (counting faces) in the h-**polynomial**:

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Prop. [FA 20] The h-vector of the bipermutahedron $\Pi_{n,n}$ is

$$h_i(\Pi_{n,n}) = \text{ \# of bipermutations of } [n] \text{ with } i \textbf{ descents.}$$

We call this the **biEulerian polynomial**.

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Observation: this is log-concave: $h_i^2 \geq h_{i-1}h_{i+1}$. How to prove it?

The h-vector of the bipermutahedron

Prop. [FA 20] The biEulerian polynomial is

$$\frac{h_n(x)}{(1-x)^{2n+1}} = \sum_{k \geq 0} \binom{k+2}{2}^n x^k$$

(LHS: faces of triangulation. RHS: lattice points of polytope???)

Let Δ = standard triangle in $\mathbb{R}^3$.

Prop. [FA 20] There's a unimodular triangulation of $\Delta \times \cdots \times \Delta$ that is combinatorially isomorphic to (the triple cone over) $\Sigma_{n,n}$.

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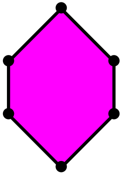
Prop. [FA 20] (thanks to Katharina Jochemko!)

- All roots of the biEulerian polynomial are real and negative.
- The h-vector of the bipermutahedron is log-concave.

The harmonic polytope

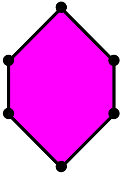


Def./Prop. [FA - Escobar 20] The harmonic polytope is

$$\sum_{e\in[n]} x_e = \sum_{e\in[n]} y_e = \frac{n(n+1)}{2} + 1,$$

$$\sum_{s\in S} x_s + \sum_{t\in T} y_t \geq \frac{|S|(|S|+1) + |T|(|T|+1)}{2} + 1 \qquad \text{for } S|T \sqsubseteq [n].$$

Combinatorial structure of the harmonic polytope



Prop. [FA-Escobar 20] Faces of polytope $\longleftrightarrow$ harmonic triples

- f-vector: we have a formula
- # of facets $= 3^n - 3$
- # of vertices $= (n!)^2 \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$!

Volume

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Good news: The harmonic polytope is a Minkowski sum!

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In dimension 2:

$Vol(P+Q) = Vol(P) + Vol(Q) + 2MVol(P,Q)$
$MVol(P,Q) =$ # of sols to 2×2 system of polynomial equations with support P and Q

$\text{conv}\{(0,0),(0,1),(1,0)\} \longrightarrow ax^0y^0 + bx^1y^0 + cx^0y^1 = 0$
$\text{conv}\{(1,0),(0,2),(0,3)\} \longrightarrow dx^1y^0 + ex^0y^2 + fx^0y^3 = 0$

Volume

Let $\mathbb{R}^n \times \mathbb{R}^n$ have basis $e_1, \ldots, e_n, f_1, \ldots, f_n$.

$$H_{n,n} = \sum_{i<j} [\mathrm{e}_i, \mathrm{e}_j] + \sum_{i<j} [\mathrm{f}_i, \mathrm{f}_j] + conv(\mathrm{e}_i + \mathrm{f}_i \, : \, 1 \leq i \leq n)$$

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Theorem. (FA - Escobar 20)

$$Vol(H_{n,n}) = \sum_{\Gamma} \frac{\deg(X_\Gamma)}{(v(\Gamma)-2)!} \prod_{v \in V(\Gamma)} \deg(v)^{\deg(v)-2}$$

Γ = connected bipartite multigraphs on edges $[n]$
$X_\Gamma =$ (embedded) toric variety given by toric ideal of Γ

Volume

Theorem. [AE 20] Summing over conn. bip. graphs on edges $[n]$

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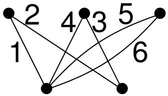
$\Gamma =$ 

Toric ideal $\langle z_1 z_3 - z_2 z_4, z_5 - z_6 \rangle$ has degree 2.
Polytope $P_\Gamma^- = (\Delta_{abc} + \Delta_{ab}) - \Delta_{abc} = \Delta_{ab}$ has 2 lattice points

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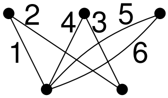
$\Gamma =$ 

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Polytope $P_\Gamma^- = (\Delta_{abc} + \Delta_{ab}) - \Delta_{abc} = \Delta_{ab}$ has 2 lattice points

(This is $\mathrm{MVol}(e_{12}, e_{34}, e_{56}, f_{14}, f_{23}, f_{45}, f_{56}, D_{123456}, D_{123456}, D_{123456}) = 2$.)

Minkowski quotients

Relationship: $\lambda H_{n,n}$ is a summand of $\Pi_{n,n}$.

Minkowski quotient $P/Q := max\{\lambda \ : \ P = \lambda Q + R \text{ for some } R\}$

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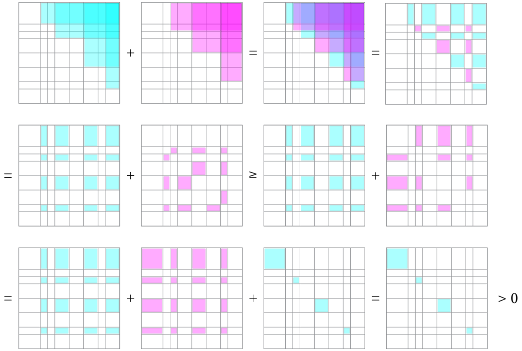
Proof:



A very brief origin story

[A21] The geometry of geometries: matroid theory, old and new. *Proceedings, International Congress of Mathematicians 2022*.

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Given a matroid M of rank r,

f**-vector** = |coeffs| of $\chi_M(q)$ h**-vector** = |coeffs| of $\chi_M(q+1)$

Theorem.

1. [Adiprasito-Huh-Katz '15] $f_0, f_1, \ldots, f_r$ **is log-concave.**
Conjectured by Rota 71, Welsh 71, 76, Heron 72, Mason 72.

2. [Ardila-Denham-Huh '20] $h_0, h_1, \ldots, h_r$ **is log-concave.**
Conjectured by Brylawski 82, Dawson 83, Hibi 89.

A very brief origin story

[A21] The geometry of geometries: matroid theory, old and new. *Proceedings, International Congress of Mathematicians 2022*.

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Theorem.

1. [Adiprasito-Huh-Katz '15] $f_0, f_1, \ldots, f_r$ **is log-concave.**
Conjectured by Rota 71, Welsh 71, 76, Heron 72, Mason 72.

2. [Ardila-Denham-Huh '20] $h_0, h_1, \ldots, h_r$ **is log-concave.**
Conjectured by Brylawski 82, Dawson 83, Hibi 89.

Tropical geometry, combinatorics, Hodge theory in Chow ring of

[AHK 15]: Bergman fan Σ_M in permutahedral fan Σ_n [AK 06]

[ADH 20]: conormal fan $\Sigma_{M,M^\perp}$ in **bi**permutahedral fan $\Sigma_{n,n}$

muchas gracias

[ADH20]: https://arxiv.org/abs/2004.13116

[AE20]: https://arxiv.org/abs/2006.03078

[A20]: https://arxiv.org/abs/2008.02295