FROBENIUS ALGEBRAS
galore

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CONGRATS PROF. TORO!

STORY TIME

VARIOUS CHARACTERIZATIONS OF "FROBENIUS ALGEBRAS" OVER THE LAST 100+ YEARS

LINEAR-ALGEBRAIC $\rightsquigarrow$ REPRESENTATION-THEORETIC $\rightsquigarrow$ CATEGORICAL

... 10+ DEFINITIONS THAT DON'T ALIKE
... ALL EQUIVALENT!

Setting the scene: Algebraic structures...

Abelian group
Set, +, 0, −
commutative

Ring
Set, +, 0, −, ×, 1
com
multiplication
(associative)
(could be non com.)

Field
Set, +, 0, −, ×, 1, ÷
com
com
multip. inverses

Module over a ring R
Set, +, 0, −, *
com
scalar (R-)multip.

Vector space over a field $\Bbbk$
Set, +, 0, −, *
com
scalar ($\Bbbk$-)multip.

Algebra over a field $\Bbbk$
Set, +, 0, −, ×, 1, *
com
assoc.
assoc
could be non com

A unital ring R that is given the structure of an $\Bbbk$-VS (see Dummit-Foote...)

$\Longleftrightarrow$

A $\Bbbk$-vector space V that is given the structure of a unital ring (categorically, this is nicer...)

We'll eventually get here

Linear-algebraic $\rightsquigarrow$ representation-theoretic $\rightsquigarrow$ categorical

But let's start here

Frobenius algebras via linear algebra

A 1903-STYLE GAME —

DUE TO F.G. FROBENIUS (1849–1917)

- START WITH A FINITE-DIM'L $\Bbbk$-ALGEBRA A
- PICK BASIS $\{v_1, \dots, v_n\}$ OF A
- GET SCALARS $\beta_{ij}^{(a)}, \gamma_{ij}^{(a)} \in \Bbbk$ SUCH THAT

$$v_i\, a = \sum_{j=1}^{n} \beta_{ij}^{(a)} v_j \quad \& \quad a v_i = \sum_{j=1}^{n} \gamma_{ji}^{(a)} v_j \qquad \forall a \in A$$

- GET $\Bbbk$-LINEAR MAPS

$$\beta : A \longrightarrow \mathrm{Mat}_n(\Bbbk) \quad \& \quad \gamma : A \longrightarrow \mathrm{Mat}_n(\Bbbk)$$
$$a \longmapsto (\beta_{ij}^{(a)}) \qquad\qquad a \longmapsto (\gamma_{ij}^{(a)})$$

Q: WHEN DOES $\exists P \in GL_n(\Bbbk)$ s.t. $P\,\beta(a) = \gamma(a)\,P \quad \forall a \in A$?

IN THE MODERN LANGUAGE OF REP. THEORY:

Q $\Leftrightarrow$ WHEN ARE THE REG. REPS β, γ OF A EQUIVALENT?

BACK IN 1903:

NO REPS, NO MODULES,

& "ALGEBRAS" = "HYPERCOMPLEX SYSTEMS"

JUST HAD LINEAR ALGEBRA

CAN ANSWER QUESTION VIA MATRICES...

FINITE-DIM'L $\Bbbk$-ALGEBRA A

BASIS $\{v_1, \dots, v_n\}$ OF A

$\beta_{ij}^{(a)}, \gamma_{ij}^{(a)} \in \Bbbk$ SUCH THAT

$$v_i\, a = \sum_{j=1}^{n} \beta_{ij}^{(a)} v_j$$

$$\& \quad a\, v_j = \sum_{i=1}^{n} v_i\, \gamma_{ij}^{(a)} \quad \forall a \in A$$

GET $\Bbbk$-LINEAR MAPS

$$\beta : A \longrightarrow \mathrm{Mat}_n(\Bbbk), \quad a \longmapsto (\beta_{ij}^{(a)})$$

$$\gamma : A \longrightarrow \mathrm{Mat}_n(\Bbbk), \quad a \longmapsto (\gamma_{ij}^{(a)})$$

Q: WHEN DOES $\exists\, P \in GL_n(\Bbbk)$ $\ni$. $P\,\beta(a) = \gamma(a)\, P \quad \forall a \in A$?

GET SCALARS $p_{ij}^{k} \in \Bbbk$ SUCH THAT

$$v_i v_j = \sum_{k=1}^{n} p_{ij}^{k}\, v_k$$

FOR $\underline{c} = (c_1, \dots, c_n) \in \Bbbk^n$, DEFINE PARATROPHIC MATRIX OF A AT $\underline{c}$

$$P_{\underline{c}} = \left(\textstyle\sum_{k=1}^{n} c_k\, p_{ij}^{k} \right)_{i,j} \in \mathrm{Mat}_n(\Bbbk)$$

A: [FROBENIUS, 1903]:

WHEN $\exists\, \underline{c} \in \Bbbk^n$ $\ni$. $P_{\underline{c}}$ IS INVERTIBLE

(IN THIS CASE, $P_{\underline{c}} = P$ ABOVE)

DEFINITION 1

A FINITE DIM'L $\Bbbk$-ALG. A IS **FROBENIUS***

$$\Updownarrow$$

$$\exists\, \underline{c} \in \Bbbk^n \ \ni. \ \det(P_{\underline{c}}) \neq 0$$

* NOT SELF NAMED

(NON)EXAMPLES

TRY!

- $\Bbbk[x,y]/(x^2, y^2)$ IS FROB.
- $\Bbbk[x,y]/(x^2, xy, y^2)$ IS <u>NOT</u> FROB

FINITE-DIM'L $\Bbbk$-ALGEBRA A

BASIS $\{v_1, \dots, v_n\}$ OF A

Q: WHEN DOES $\exists\, P \in GL_n(\Bbbk)$ $\ni$. $P\,\beta(a) = \gamma(a)\, P \quad \forall a \in A$?

GET SCALARS $p_{ij}^{k} \in \Bbbk$ SUCH THAT

$$v_i v_j = \sum_{k=1}^{n} p_{ij}^{k}\, v_k$$

FOR $\underline{c} = (c_1, \dots, c_n) \in \Bbbk^n$, DEFINE PARATROPHIC MATRIX OF A AT $\underline{c}$

$$P_{\underline{c}} = \left(\textstyle\sum_{k=1}^{n} c_k\, p_{ij}^{k} \right)_{i,j} \in \mathrm{Mat}_n(\Bbbk)$$

A: [FROBENIUS, 1903]:

WHEN $\exists\, \underline{c} \in \Bbbk^n$ $\ni$. $P_{\underline{c}}$ IS INVERTIBLE

(IN THIS CASE, $P_{\underline{c}} = P$ ABOVE)

≡ AND THEN THE GAME WAS OVER FOR A WHILE ≡

≡ UNTIL ~35 YEARS LATER ≡

FROBENIUS ALGEBRAS VIA REPRESENTATION THEORY

/// MODULE THEORY

DEFINITION 1

A FINITE DIM'L $\Bbbk$-ALG. A

IS FROBENIUS *

$\Updownarrow$

$\exists\, \underline{c} \in \Bbbk^n$ s.t. $\det(P_{\underline{c}}) \neq 0$

PARATROPHIC MATRIX

* NOT SELF NAMED

MORE TOOLS OF ABSTRACT ALG. NOW

RINGS, ALGEBRAS, MODULES ...

≡ ABLE TO WORK BASIS-FREE ≡

(THANKS, EMMY NOETHER!)

BRAUER, NAKAYAMA, & NESBITT REVIVED FROBENIUS ALGEBRAS

FROM T.Y. LAM'S TEXT "LECTURES..."

C. Nesbitt wrote: "The writer, in collaborating with T. Nakayama, adopted the term Frobeniusean algebra, but now, quailing before our critics, we return to simply Frobenius algebra." So apparently, people were not attracted by a six-syllable word.

DEFINITION 1

FINITE DIM. $\Bbbk$-ALG. A IS FROBENIUS

$\Updownarrow$

$\exists\, \underline{c} \in \Bbbk^n \ni.\ \det(P_{\underline{c}}) \neq 0$

PARATROPHIC MATRIX

EXAMPLES

- MATRIX ALGEBRAS $Mat_n(\Bbbk)$
- EXTERIOR ALGEBRAS $\Lambda(V)$
- GROUP ALGEBRAS $\Bbbk G$
- COHOMOLOGY ALGEBRAS OF ORIENTED MANIFOLDS $H^*(X)$

DEFINITIONS *galore*

THEOREM [BRAUER-NESBITT, NAKAYAMA, 1937-1942]

LET A BE A FINITE DIM'L $\Bbbk$-ALGEBRA

THEN THE FOLLOWING ARE EQUIVALENT:

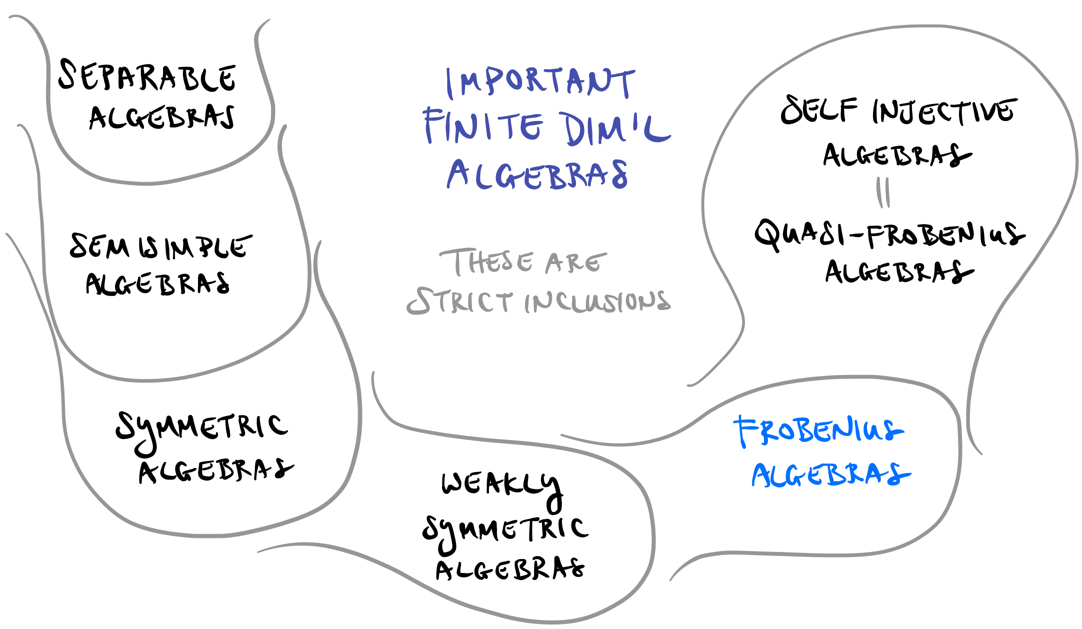
1. A IS FROBENIUS
2. $\exists$ NONDEG. ASSOC. $\Bbbk$-BILINEAR FORM $(-,-) : A \times A \longrightarrow \Bbbk$
3. $\exists$ ISOM. OF LEFT A-MODULES: $A \cong A^*$

3'. $\exists$ ISOM. OF RIGHT A-MODULES: $A \cong A^*$

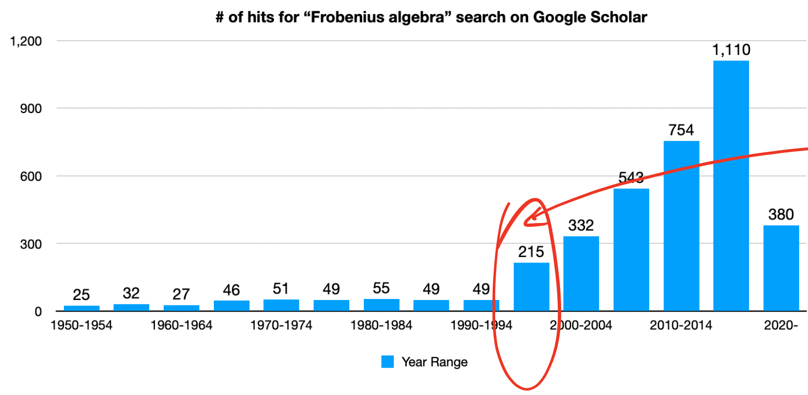
4. $\exists$ $\Bbbk$-LINEAR FORM $\nu : A \longrightarrow \Bbbk \ni.\ \ker(\nu)$ DOES NOT CONTAIN $\neq 0$ LEFT IDEAL OF A

4'. $\exists$ $\Bbbk$-LINEAR FORM $\nu : A \longrightarrow \Bbbk \ni.\ \ker(\nu)$ DOES NOT CONTAIN $\neq 0$ RIGHT IDEAL OF A

≡ RELATED GAMES OF THE 1940'S & BEYOND INVOLVE THE PLAYERS... ≡

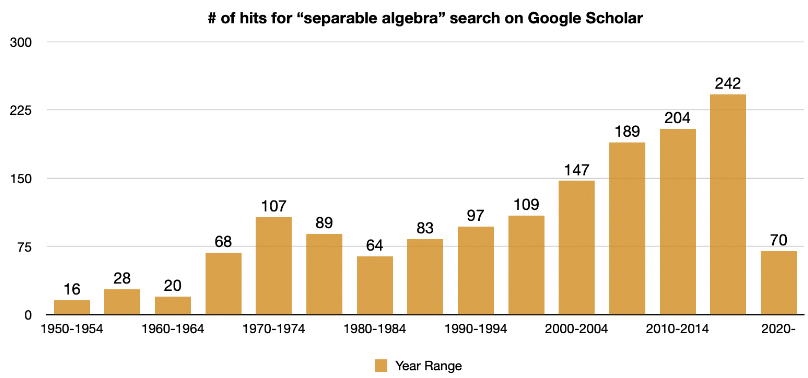


POPULARITY THROUGHOUT THE YEARS



SOMETHING HAPPENED HERE

(JUST FOR COMPARISION)



(EXAMPLES INCLUDE IMPORTANT FROB. ALGS LIKE $Mat_n(k)$, kG, $|G| \in k^{\times}$)

FROBENIUS ALGEBRAS VIA CATEGORY THEORY

A LATE 1980'S/1990'S STYLE GAME –

DUE TO G. SEGAL, E. WITTEN, M. ATIYAH ORIGINALLY...

BUILD A CATEGORICAL MACHINE

TO PRODUCE INVARIANTS OF LOW-DIM'L MANIFOLDS

CATEGORY $\mathscr{C}$ ⇢ MONOIDAL CATEGORY $(\mathscr{C}, \otimes, \mathbb{1})$

≡ COLLECTION OF OBJECTS
& MAPS BETWEEN THESE OBJECTS.
(SUBJECT TO ADD'L NICE CONDITIONS)

≡ CATEGORY $\mathscr{C}$ EQUIPPED WITH
BIFUNCTOR $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$
DISTINGUISHED OBJECT $\mathbb{1}$
THAT MIMIC THE STRUCTURE OF A MONOID
(SUBJECT TO COMPATIBILITY CONDITIONS)

SYMMETRIC MON'L CAT. $(\mathscr{C}, \otimes, \mathbb{1}, c)$
≡ MONOIDAL CATEGORY EQUIPPED WITH
NATURAL ISOMORPHISM : $\forall X, Y \in \mathscr{C}$
$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ ∋. $c^2 = \mathrm{id}$

EX. $(\mathrm{Vec}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk, c)$ ← FLIP MAP
IS A SYM. MON'L CATEGORY

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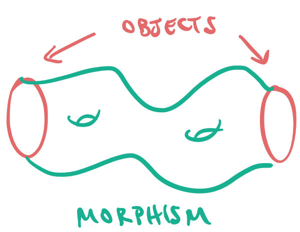
d-TQFT (TOP'L QUANTUM FIELD THY) ≡ A SYMMETRIC MONOIDAL FUNCTOR

$$Z : \text{d-Cobord} \longrightarrow \mathrm{Vec}_{\Bbbk}$$

OBJECTS: ORIENTED, CLOSED SMOOTH (d−1)-DIM. MFLDS | OBJECTS: $\Bbbk$-VECTOR SPACES

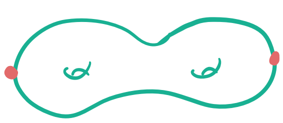
MORPHISMS: (BOUNDARY & ORIENTATION PRES. DIFFEOM. CLASSES OF) COMPACT d-DIM. MFLDS. | MORPHISMS: $\Bbbk$-LINEAR MAPS

d = 2



$\xrightarrow{Z}$ $V \xrightarrow{\Bbbk\text{-LIN. MAP}} V$ ($\Bbbk$-VS)

TOP'L INV. ≡ MATRIX



$\xrightarrow{Z}$ $\Bbbk \longrightarrow \Bbbk$

TOP'L INV. ≡ SCALAR

d-TQFTs PRODUCE $\Bbbk$-LINEAR ALGEBRAIC INVARIANTS OF d-MFLDS

THEOREM ["FOLKLORE": A. VORONOV (1994)
"FOLKLORE NO MORE!" : L. ABRAMS (1996)]

> THERE IS AN EQUIVALENCE OF MONOIDAL CATEGORIES
>
> MON. CATEGORY OF 2-TQFTs WITH VALUE IN $\mathrm{VEC}_{\Bbbk}$ & MON. CATEGORY OF COMMUTATIVE FROBENIUS ALGEBRAS / $\Bbbk$

DEFINITION 5 [ABRAMS, QUINN] FROBENIUS ALGEBRA/$\Bbbk$ IS A 5-TUPLE:

FORMS AN ALG/$\Bbbk$

M ASSOCIATIVE & UNITAL WRT u

- A, f.d. $\Bbbk$-VECTOR SPACE
- $m: A \otimes A \longrightarrow A$, $\Bbbk$-LINEAR MAP
- $u: \Bbbk \longrightarrow A$, $\Bbbk$-LINEAR MAP
- $\Delta: A \longrightarrow A \otimes A$, $\Bbbk$-LINEAR MAP
- $\varepsilon: A \longrightarrow \Bbbk$, $\Bbbk$-LINEAR MAP

FORMS A COALG/$\Bbbk$

Δ COASSOCIATIVE & COUNITAL WRT ε

SO THAT $(m \otimes id_A)(id_A \otimes \Delta) = \Delta m = (id_A \otimes m)(\Delta \otimes id_A)$

ABRAMS SHOWED EQUIV. TO DEF2 INVOLVING NONDEG PAIRING

CATEGORICAL UPGRADE REPLACE $\mathrm{Vec}_{\mathbb{k}}$ WITH ANY SYM MON'L CATEG...

FROBENIUS ALGEBRA IN $\mathcal{C}$ IS A 5-TUPLE:

$A \in \mathcal{C}$ OBJECT

$m: A \otimes A \to A \in \mathcal{C}$

$u: \mathbb{1} \to A \in \mathcal{C}$

$\Delta: A \to A \otimes A \in \mathcal{C}$

$\varepsilon: A \to \mathbb{1} \in \mathcal{C}$

WITH (A, m, u) IS AN ALGEBRA IN $\mathcal{C}$

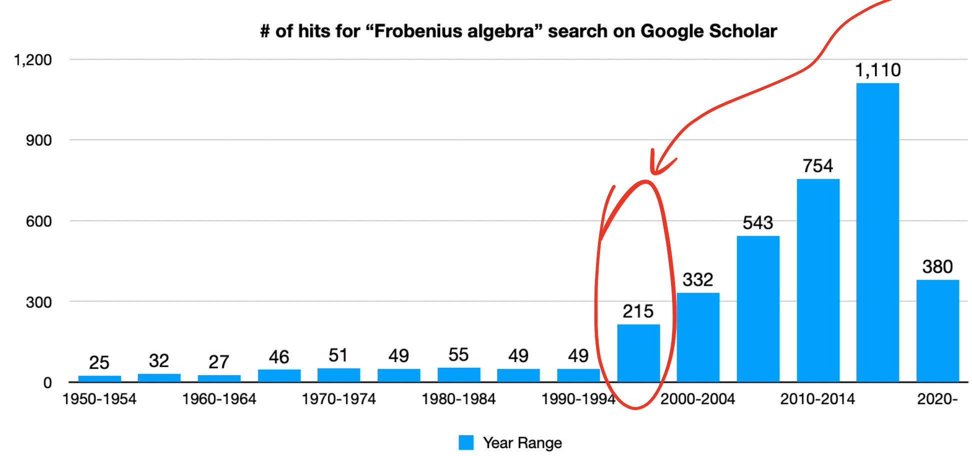
(A, Δ, ε) IS A COALGEBRA IN $\mathcal{C}$

SATISFYING SIMILAR DIAGRAMS AS FOR $\mathcal{C} = \mathrm{Vec}_{\mathbb{k}}$

THEOREM FOR A SYMMETRIC MONOIDAL CATEG. $(\mathcal{C}, \otimes, \mathbb{1}, c)$

> THERE IS AN EQUIVALENCE OF MONOIDAL CATEGORIES
>
> MON. CATEGORY OF 2-TQFTs WITH VALUE IN $\mathcal{C}$ & MON. CATEGORY OF COMMUTATIVE FROBENIUS ALGEBRAS IN $\mathcal{C}$

(WITH SAME PROOF AS FOR $\mathcal{C} = \mathrm{Vec}_{\mathbb{k}}$)



2D-TQFTs MOSTLY CAUSED THIS

DUBROVIN'S (1996) "FROBENIUS MANIFOLDS" WAS ALSO A CAUSE ... QUANTUM COHOMOLOGY... (SKIPPING HERE...)

... AND FROBENIUS ALGEBRAS IN MONOIDAL CATEGORIES HAVE BEEN OF GREAT INTEREST EVER SINCE...

INCLUDING APPLICATIONS IN
- MORITA THEORY
- QFTs & CFTs
- COMPUTER SCIENCE
- RECENTLY: 3D-TQFTs ARXIV: 2105.04613 [KMRS]

CATEGORICAL FROBENIUS ALGEBRAS *galore*

LET A BE A FINITE DIM'L $\Bbbk$-ALGEBRA
THEN THE FOLLOWING ARE EQUIVALENT:

1. $\exists$ INVERTIBLE PARATROPHIC MATRIX
2. $\exists$ NONDEG. ASSOC. $\Bbbk$-BILINEAR FORM $(-,-) : A \times A \longrightarrow \Bbbk$
3. $\exists$ ISOM. OF LEFT A-MODULES : $A \cong A^*$

3′. $\exists$ ISOM. OF RIGHT A-MODULES : $A \cong A^*$

4. $\exists$ $\Bbbk$-LINEAR FORM $\nu : A \longrightarrow \Bbbk$ $\ni$. $\ker(\nu)$ DOES NOT CONTAIN $\neq 0$ LEFT IDEAL OF A

4′. $\exists$ $\Bbbk$-LINEAR FORM $\nu : A \longrightarrow \Bbbk$ $\ni$. $\ker(\nu)$ DOES NOT CONTAIN $\neq 0$ RIGHT IDEAL OF A

5. $\exists$ $\Bbbk$-LINEAR MAPS $\Delta : A \to A \otimes_{\Bbbk} A$, $\varepsilon : A \to \Bbbk$ $\ni$. (A, Δ, ε) IS A $\Bbbk$-COALG. + COMPATIBILITY

LET $\mathcal{C}$ BE A RIGID MONOIDAL CATEG.
$\{\exists$ DUAL OBJECTS$\}$. TAKE AN ALG IN $\mathcal{C}$.

1. NO CATEGORICAL ANALOGUE TO DATE
2. ←
3. ←

3′. ←

4. $\exists$ CATEGORICAL ANALOGUE

4′. ...NOW! [W-YADAV] 2106.01999

5. ←

$\exists$ CATEGORICAL ANALOGUE.

[FUCHS-STIGNER] 2009

RECENT WORK INVOLVING FROBENIUS ALGEBRAS

GENERALIZES WORK OF BONGALE (1967) : $\mathcal{C} = \mathrm{Vec}_{\Bbbk}$

THEOREM [W-YADAV, 2106.01999]

TAKE $\mathcal{C}$ A MONOIDAL CATEG. + NICE CONDITIONS

IF A IS A "FILTERED" ALGEBRA IN $\mathcal{C}$

& ITS ASSOC. GRADED ALGEBRA $gr(A)$ IS FROBENIUS IN $\mathcal{C}$

THEN SO IS A.

* FURTHER DEVELOP FILTERED-GRADED TECHNIQUES
  $\rightsquigarrow$ MONOIDAL ASSOC. GRADED FUNCTOR

MAIN EXAMPLE $\quad$ A = "BRAIDED CLIFFORD ALGEBRA"
$\mathrm{gr}(A)$ = "BRAIDED EXTERIOR ALGEBRA"

OTHER RECENT WORKS:

[MMPRTW] (WINARTZ GROUP) (ARXIV: 2001.03837)
USES "SPECIAL" FROBENIUS ALGEBRAS TO
STUDY MORITA EQUIVALENCE OF ALGEBRAS
IN NICE MONOIDAL CATEGORIES

BOTH WORKS ARE WRITTEN FOR NEWCOMERS
PUBLISHED VERS. WILL BE SHORTER.

[W-WICKS-WON] (ARXIV: 1911.12847)
DEVELOPS / RECONCILES FORMULAIC VS CATEGORICAL
NOTIONS OF
- ALGEBRAS
- COALGEBRAS
- FROB. ALGS

WITH "WEAK HOPF ALG." SYMMETRY...

VARIOUS CHARACTERIZATIONS OF "FROBENIUS ALGEBRAS" OVER THE LAST 100+ YEARS

LINEAR-ALGEBRAIC $\rightsquigarrow$ REPRESENTATION-THEORETIC $\rightsquigarrow$ CATEGORICAL

$\underbrace{\text{LINEAR-ALGEBRAIC} \rightsquigarrow \text{REPRESENTATION-THEORETIC}}_{\text{ALGEBRAIC}}$ $\rightsquigarrow$ $\underbrace{\text{CATEGORICAL}}_{\text{APPLICATION TO TOPOLOGY}}$

... BUT WHAT ABOUT **ANALYSIS**?

FROBENIUS ALGEBRAS IN FUNCTIONAL ANALYSIS

LET A BE A FINITE DIM'L $\mathbb{R}$-ALGEBRA

THEN THE FOLLOWING ARE EQUIVALENT:

1. A IS FROBENIUS
2. $\exists$ NONDEG. ASSOC. $\mathbb{R}$-BILINEAR FORM $(-,-): A \times A \longrightarrow \mathbb{R}$

[ALESKER] (GAFA, 2004; ICM 2002)

V: $\mathbb{R}$-VECTOR SPACE OF DIM n

$\mathcal{K}(V)$: CLASS OF ALL CONVEX COMPACT SUBSETS OF V

A FUNCTION $\phi: \mathcal{K}(V) \longrightarrow \mathbb{C}$ IS A VALUATION IF

$\forall K_1, K_2 \in \mathcal{K}(V)$ s.t. $K_1 \cup K_2$ CONVEX, WE GET:

$$\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).$$

IT IS TRANSLATION-INVARIANT IF $\phi(K+x)=\phi(K)$ $\forall x\in V,\ K\in \mathcal{K}(V)$.

IT IS CONTINUOUS IF CONTINUOUS WRT HAUSDORFF METRIC ON $\mathcal{K}(V)$, AND FURTHER SMOOTH IF THE MAP $A \longmapsto A\phi$ IS ∞ DIFF'BLE (FOR $A\phi(K) := \phi(A^{-1}K)$ $\forall A\in GL(V),\ K\in\mathcal{K}(V)$)

LET A BE A FINITE DIM'L $\mathbb{R}$-ALGEBRA.
THEN A IS FROBENIUS $\Longleftrightarrow$
★ $\exists$ NONDEG. ASSOC. $\mathbb{R}$-BILINEAR FORM
$(-,-): A\times A \longrightarrow \mathbb{R}$

THE ∞-DIM'L SPACE OF SMOOTH TRANSLATION-INV. VALUATIONS FORMS AN ALGEBRA/$\mathbb{C}$: $\mathrm{Val}(V)^{sm}$ THAT SATISFIES A POINCARÉ DUALITY

IF $G \leq GL(V)$ IS A COMPACT SUBGROUP, THEN THE SUBSPACE OF G-INVARIANT VALUATIONS $\mathrm{Val}^{G}(V)^{sm}$ IS FROBENIUS!

Ex. $\mathrm{Val}^{O(n)}(V)^{sm} \cong \mathbb{C}[x]/(x^{n+1})$ (WELL-KNOWN FROBENIUS ALG.)

ANALYTIC DATA IS ENCODED IN THE FROBENIUS STRUCTURE (EG. VOLUMES...)

SEE ALESKER, BERNIG, SCHUSTER (GAFA, 2011) ON TIES TO HARMONIC ANALYSIS AS WELL

FROBENIUS ALGEBRAS

galore

CONGRATS AGAIN, PROF. TORO!

≡ THANKS FOR LISTENING ≡