Concordance and Instantons

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Plan

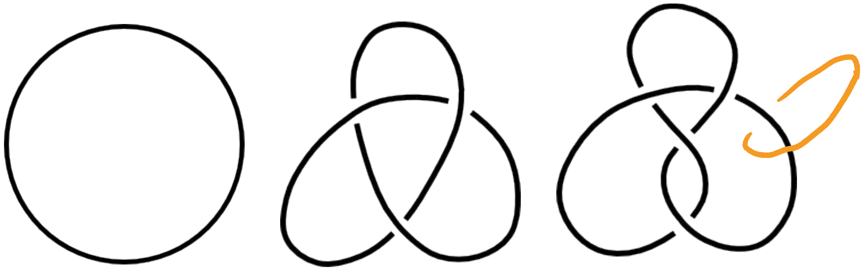
1. Concordance
2. Satellites
3. 3- and 4-manifolds
4. Instantons

Knots in 3D

Definition: A knot K is a smooth embedding

$$K: S^1 \to S^3$$

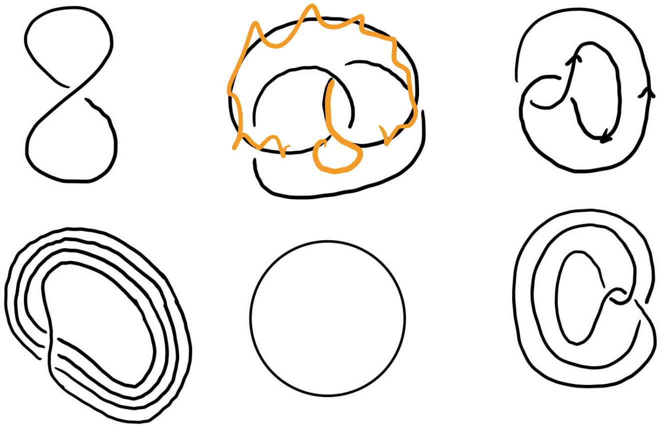
(start with a rope, tie it into a knot, fuse the ends together)



Unknot Trefoil Figure 8

Def: Two knots are isotopic if one can be deformed into the other via embeddings in S^3.

Different diagrams of the unknot:



Why knots?

Knots allow us to study arbitrary closed 3-manifolds

Let M be a closed, orientable, connected 3-manifold.

Theorem (Lickorish, Wallace) M can be described as Dehn surgery on a framed link.

Theorem (Alexander) M can be described as a cover of S^3 branched over a link.

$f : M \to S^3$

$f^{-1}(L) \cong L$

$M \setminus f^{-1}(L) \to S^3 \setminus L$

covering space

Knots in 4D

Two knots K and K’ are concordant if there exists an annulus A

- A: $S^1 \times [0,1] \longrightarrow S^3 \times [0,1]$
- with boundary $\partial A = K \sqcup -K'$.

A special class of knots:

equivalent

- K is concordant to the unknot
- K bounds a disk D in B^4
- K is a slice knot

$A: S^1 \times [0,1] \hookrightarrow S^3 \times [0,1]$

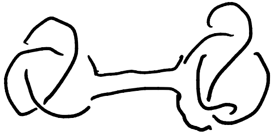
smoothly $\rightsquigarrow$ ℓsm

top. locally flat

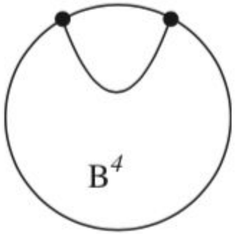
$\forall\ (z,t)\ \exists$ coordinates

$A: (x,y) \longrightarrow (x,y,0,0)$

$-K'$: changing every crossing



K # K'



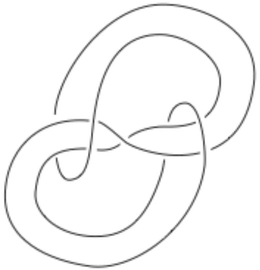
smooth slice

Slice knots

concordant

K not isotopic to U but K=U in $\mathscr{C}_{sm}$

Ribbon knots



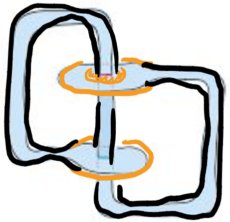
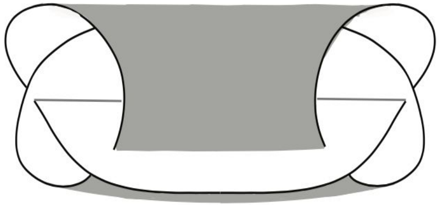


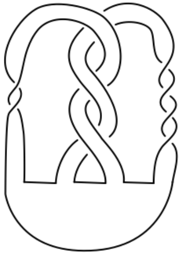
Image of Ben Ruppik

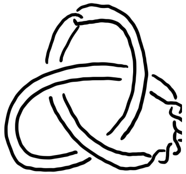


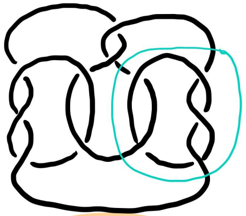
Picture taken from Livingston-Naik
“Introduction to Knot Concordance”

K not isotopic to U but K=U in $\mathscr{C}_{top}$

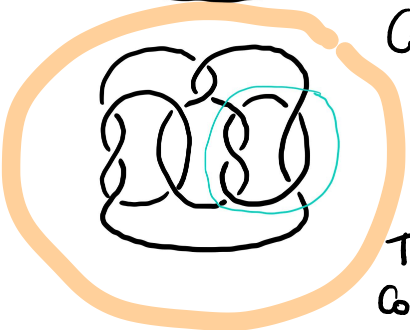
Theorem (Freedman) If $\underbrace{\Delta_K(t)=1}_{\text{Alexander polynomial}}$, then K is topologically slice.







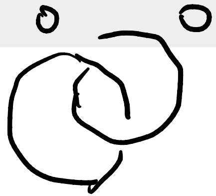
KT trivial in both $\mathscr{C}_{Top}$ & $\mathscr{C}_{sm}$



Conway knot trivial $\mathscr{C}_{TOP}$

THM (Piccirillo) Conway knot is non-trivial in $\mathscr{C}_{sm}$

Why concordance?



: $S^2 \times S^2$

Thm: Every 3-mfd bounds a 4-mfd

$\Omega^3_* = \{0\}$

Knot concordance allows us to study

1. Smooth 4-manifolds
2. Bordism of 3-manifolds

Let X be a smooth 4-manifold with boundary:

Theorem (Alexander) X can be described as a cover of B^4 branched over a surface F.

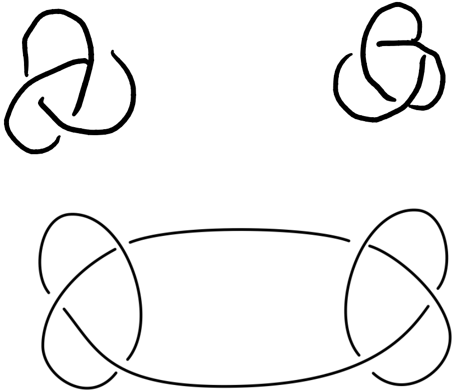
Theorem X can be described via a Kirby diagram.

Theorem Smooth structure of X is related to embedded surfaces $F \subset B^4$

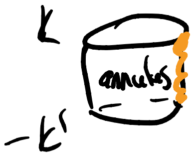
In dimension 4: smooth 4-mfds $\neq$ topological 4-mfds

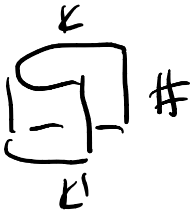
Group Structure on $\mathscr{C}$

Theorem (Fox-Milnor 1966) Knot concordance is an equivalence relation and the set $\mathscr{C}$ of concordance classes of knots is an abelian group with connected sum as the binary operation.

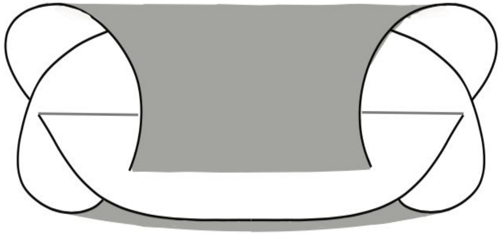


Two knots K, K′ ⊂ S^3 are smoothly concordant if K# − K′=U





For any knot K, K# − K is a slice knot



Picture taken from Livingston-Naik
“Introduction to Knot Concordance”

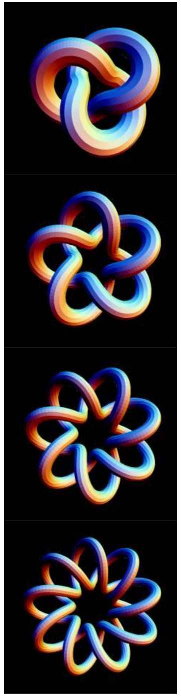
Remark: $\mathcal{C}_{sm} \longrightarrow \mathcal{C}_{Top}$

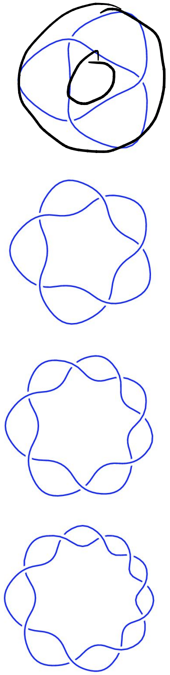
Concordance: 4-dimensional approach to knots

$\mathbb{Z}^{\infty} \subset \mathscr{C}$

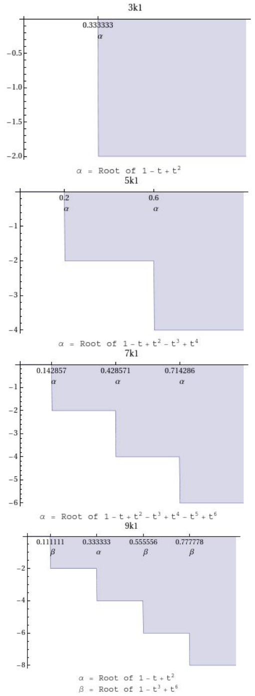
An infinite family of torus knots: $T_{2,2k+1}$

k	$T_{2,2k+1}$
1	2,3
2	2,5
3	2,7
4	2,9





Theorem (Litherland) The group generated by the family $\{T_{2,2k+1}\}_k$ is isomorphic to $\mathbb{Z}^{\infty}$



Knots ⇝ Function

$K \to \sigma_K : S^1 \to \mathbb{Z}$

σ defined in terms of 3D.

Thm: if σ_K is not the trivial function, then K is non-trivial in concordance

Smooth vs Topological

Theorem (Levine) The following maps are surjective homomorphisms

$$\mathscr{C}_{sm} \longrightarrow \mathscr{C}_{top} \longrightarrow \mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty$$

Q: How big is $\ker(\mathscr{C}_{sm} \longrightarrow \mathscr{C}_{top})$?

Theorem (Hedden-PC) $\mathbb{Z}^\infty \subset \ker(\mathscr{C}_{sm} \longrightarrow \mathscr{C}_{top})$

WARNING: We are not the first ones to prove this.

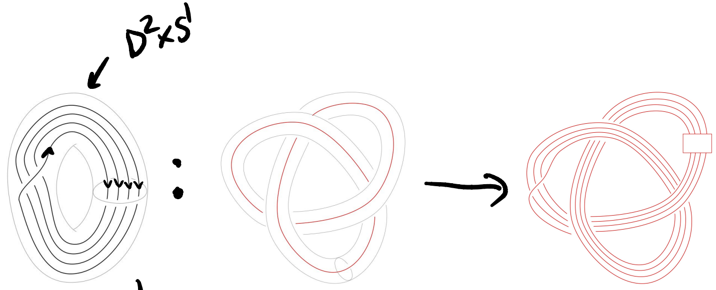
Innovation: satellite knots
Instantons

Strategy:

1. Give a recipe to produce knots $\{K_n\}_n$ trivial in $\mathscr{C}_{top}$
   spoiler: $\Delta_K(t) = 1$
2. Use an Instanton obstruction to show that no combination of the K_n's is trivial in $\mathscr{C}_{sm}$

Satellite Operations

Definition



$P \subseteq D^2 \times S^1$ Pattern

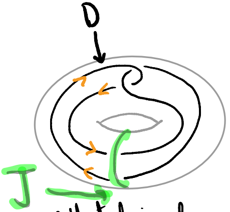
K Companion

$P(K)$ Satellite knot

$$P : \text{Knot}/\text{conc} \longrightarrow \text{Knot}/\text{conc}$$

Winding number $lk(P,J) = w$ in the example $w=4$

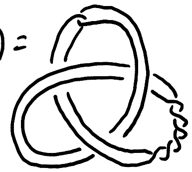
More examples of patterns:

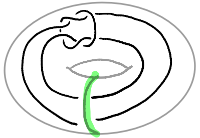


Whitehead

$w=0$

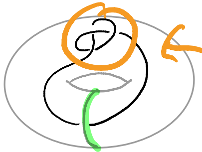
$D(T_{2,3}) =$





Square knot

$w=0$



"swallow-follow"

$w=1$

Gives connected sum with the trefoil

$\mathrm{Ker}(\mathscr{C}_{sm} \longrightarrow \mathscr{C}_{top})$

winding number

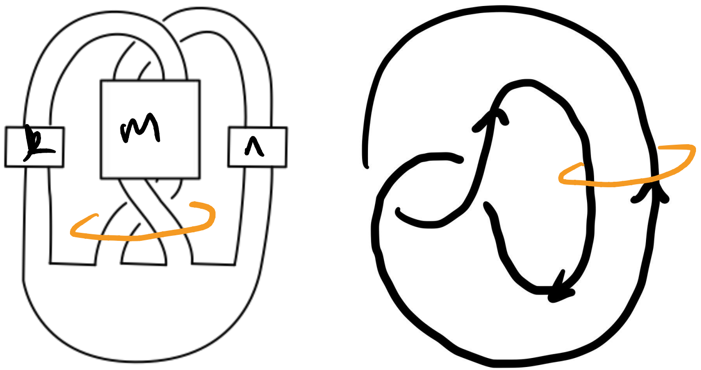
Theorem $\Delta_{P(K)}(t)=\Delta_{P(U)}(t)\Delta_K(t^w)$

Observation: $\Delta_K(1)=1$ for any knot K

As a consequence, if $\Delta_{P(U)}(t)=1$ and $w=0$, then $P(K)=0$ in $\mathscr{C}_{top}$ for any knot K!

Back to the strategy:
Show that no sum $\#c_iP(T_{2,2ki+1})$ is trivial in $\mathscr{C}_{sm}$

Examples of P(U) with $\Delta_{P(U)}(t)=1$ and w=0 for which sums of $P(T_{2,2ki+1})$ are non-trivial:



A condition on m, k, n gives Alex pol. 1.

Bordism of 3-manifolds

Theorem (Casson-Gordon) If K is slice, the 2-fold cover of S^3 branched over K bounds the $\mathbb{Z}/2$-homology ball formed as the 2-fold cover of B^4 branched over the slice disk.

This allows me to obstruct cobordism of 3-mfds instead of knots

Back to the strategy:

No sum $\#c_i\Sigma(P(T_{2,2ki+1})))$ bounds a $\mathbb{Z}/2HB^4$

In other words, obstruct the branched covers from bounding smooth 4-manifolds with the same $\mathbb{Z}/2$-homology as B^4

Our theorem

<u>Thm</u>: $P : \mathcal{C} \to \mathcal{C}$ $\quad w_P = 0 \quad$ $\mathrm{lk}_{\Sigma_2(P(U))}\left(\begin{matrix}\text{lifts} \\ \text{of } \partial D^2 \times \{1\}\end{matrix}\right) \neq 0$

Then $\exists$ $\{T_{2,2k_i+1}\}_{i \in \mathbb{N}}$ s.t

$\{P(T_{2,2k_i+1})\}_{i \in \mathbb{N}}$ generates a subgroup isomorphic to $\mathbb{Z}^\infty$

TQFT's 3–4 mfds

4-mfd with boundary $\longrightarrow$ 3-mfd

$X \longrightarrow \partial X = Y$

Anti self dual connections $\longleftrightarrow$ Flat connections

D_X ~~Donaldson polynomial~~ element $\qquad I_*(Y)$ vector space

$$D_X \in I_*(Y)$$

We can capture obstructions to bounding from $I_*(Y)$

Instanton Obstruction

Theorem (Furuta p = 1, Hedden–Kirk p > 1) Consider a family $\{\Sigma_i\}_{i=1}^N$ of oriented $\mathbb{Z}/2$-homology 3-spheres. Let (p,q) be a pair of relatively prime and positive integers. $\Sigma_N = S^3_{p/q}(T_{2,2k+1})$. If

$$\frac{p}{2(2k+1)(2(2k+1)p-q)} < \min\{\min(cs(\pm\Sigma_1)), \ldots, cs(\pm\Sigma_{N-1}))\}$$

$\Sigma_1 = \Sigma(T_{2,3})$

Then there does not exist a smooth 4-manifold X s.t.

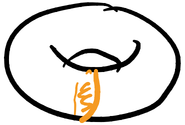
- $H^1(X;\mathbb{Z}/2)=0$
- X has negative definite intersection form
- $\partial X = \sqcup c_i\Sigma_i$ with c_i in $\mathbb{Z}$, $c_N>0$

choose $T_{2,2k_i+1}$ one by one

$S^3_{p/q}(T_{2,2k+1}) =$

$S^3 \setminus N(T_{2,2k+1}) \cup D^2 \times S^1$

(p,q)-curve in $\partial N(T_{2,2k+1})$



$\partial D^2 \times \{1\}$

Our Theorem

For example:

Theorem (Hedden-PC) Let $P \subset S^1 \times D^2$ be a pattern with winding number zero, and consider the branched double cover $\Sigma(P(U))$.

If ∂D^2, equipped with the Seifert framing from D^2, has framed lifts in $\Sigma(P(U))$ with non-zero rational linking number, then there exists an infinite family of knots $\{K_i\}_i$ for which $\{P(K_i)\}$ is a $\mathbb{Z}$-independent family in $\mathscr{C}_{sm}$

THANK YOU !!!