Modular forms on exceptional groups

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Modular forms

Let

$$\mathrm{SL}_2(\mathbf{R}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbf{R}) : ad - bc = 1 \right\}$$

and

$$\mathrm{SL}_2(\mathbf{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbf{Z}) : ad - bc = 1 \right\}.$$

Recall:

$\mathrm{SL}_2(\mathbf{R})$ acts on $\mathfrak{h} = \{z \in \mathbf{C} : Im(z) > 0\}$ as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$

Suppose $\ell > 0$ is a positive integer.

Modular forms

A modular form of weight ℓ for $\mathrm{SL}_2(\mathbf{Z})$ is a holomorphic function $f : \mathfrak{h} \to \mathbf{C}$ satisfying

1. Invariance: $f(\gamma z) = (cz+d)^\ell f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$;
2. Moderate growth: The $\mathrm{SL}_2(\mathbf{Z})$-invariant function $|y^{\ell/2} f(z)|$ on $\mathfrak{h}$ is bounded by $C(y + y^{-1})^N$ for some $C, N > 0$.

Fourier expansion

Note that

$$\mathrm{SL}_2(\mathbf{Z}) \supseteq \Gamma_\infty = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbf{Z}\}.$$

As

$$\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \cdot z = \frac{1 \cdot z + n}{0 \cdot z + 1} = z + n,$$

$f(z+n) = f(z)$ for all $n \in \mathbf{Z}$.

Fourier expansion

The holomorphy and the invariance imply

$$f(z) = \sum_{n \in \mathbf{Z}} a_f(n) e^{2\pi i n z}$$

for some $a_f(n) \in \mathbf{C}$. The moderate growth implies $a_f(n) = 0$ if $n < 0$, i.e.,

$$f(z) = \sum_{n \geq 0} a_f(n) e^{2\pi i n z} = \sum_{n \geq 0} a_f(n) q^n$$

Examples

1. (Eisenstein series) For an even integer $\ell \geq 4$, define
$$E_\ell(z) = \sum_{m,n \in \mathbf{Z}^2 \setminus \{(0,0)\}} \frac{1}{(mz+n)^\ell}.$$
2. (Poincare series) Fix $D > 0$, and an integer $k \geq 2$. Define
$$f_{D,k}(z) = \sum_{(a,b,c) \in \mathbf{Z}^3 : b^2 - 4ac = D} \frac{1}{(az^2+bz+c)^k}.$$
3. (Ramanujan Δ function) Define
$$\Delta(q) = q \prod_{n \geq 1} (1-q^n)^{24}.$$

Eisenstein series

- The condition $\ell \geq 4$ implies the sum converges absolutely to a holomorphic function
- For the invariance condition:

$$
\begin{aligned}
E_\ell\left(\frac{az+b}{cz+d}\right) &= \sum_{m,n} \frac{1}{\left(m\left(\frac{az+b}{cz+d}\right)+n\right)^\ell} \\
&= (cz+d)^\ell \sum_{m,n} \frac{1}{(m(az+b)+n(cz+d))^\ell} \\
&= (cz+d)^\ell \sum_{m,n} \frac{1}{((ma+nc)z+(mb+nd))^\ell} \\
&= (cz+d)^\ell E_\ell(z)
\end{aligned}
$$

by rearranging the sum.

Broadening the definition

If $N \geq 1$ is a positive integer, set

$$\Gamma(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(Z) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \right\}.$$

Congruence subgroup

A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ is called a **congruence subgroup** if $\Gamma \supseteq \Gamma(N)$ for some N.

Modular forms for congruence subgroups

A modular form of weight ℓ for Γ is a holomorphic function $f : \mathfrak{h} \to \mathbf{C}$ satisfying

1. Invariance: $f(\gamma z) = (cz+d)^{\ell} f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$;
2. Moderate growth: The Γ-invariant function $|y^{\ell/2} f(z)|$ on $\mathfrak{h}$ is bounded by $C(y+y^{-1})^N$ for some $C, N > 0$.

Generating functions

Sums of squares

For an integer $k \geq 0$ set

$$r_k(n) = \#\{(x_1, x_2, \ldots, x_k) \in \mathbf{Z}^k : x_1^2 + \cdots + x_k^2 = n\}$$

the number of ways of expressing n as a sum of k squares. Then if $k > 0$ is even, $\theta_k(z) = \sum_{n \geq 0} r_k(n) q^n$ is a modular form of weight $k/2$.

More generally, modular forms are known to be generating functions of interesting arithmetic quantities: E.g.,

1. class numbers of imaginary quadratic fields (Cohen, Zagier)
2. intersection numbers of curves on certain surfaces (Hirzebruch-Zagier, Kudla-Millson)

Some standard facts

Let $M_\ell(\Gamma; \mathbf{C})$ be the $\mathbf{C}$-vector space of modular forms of weight ℓ for Γ.

Finite dimensionality

The $\mathbf{C}$-vector space $M_k(\Gamma; \mathbf{C})$ is **finite dimensional**.

Denote by $M_\ell(\mathrm{SL}_2(\mathbf{Z}); \mathbf{Q})$ the $\mathbf{Q}$-vector space of modular forms with rational Fourier coefficients, i.e.,

$$M_\ell(\mathrm{SL}_2(\mathbf{Z}); \mathbf{Q}) = \{f \in M_\ell(\mathrm{SL}_2(\mathbf{Z}); \mathbf{C}) : f = \sum_{n \geq 0} a_f(n) q^n \text{ with all } a_f(n) \in \mathbf{Q}\}.$$

Rational structure

One has $M_\ell(\mathrm{SL}_2(\mathbf{Z}); \mathbf{Q}) \otimes \mathbf{C} = M_\ell(\mathrm{SL}_2(\mathbf{Z}); \mathbf{C})$, i.e., $M_\ell(\mathrm{SL}_2(\mathbf{Z}); \mathbf{C})$ has a basis consisting of modular forms with rational Fourier coefficients.

Why modular forms

Modular forms, their generalizations, and their degenerate versions:

1. **Generating functions**: are generating functions of interesting arithmetic quantities
2. **Iwasawa theory**: Can be used to understand the class groups of cyclotomic fields (Ribet, Mazur-Wiles)
3. **Elliptic curves I**: Parametrize elliptic curves over $\mathbf{Q}$ (Shimura, Eichler-Shimura, Wiles, Taylor-Wiles)
4. **Elliptic curves II**: Can be used to understand the rank of an elliptic curve over $\mathbf{Q}$ (Gross-Zagier)
5. **The Tate conjecture**: conjecturally predict the existence of nontrivial algebraic cycles on algebraic varieties
6. **The Langlands program**: conjecturally control the category of motives

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The factor of automorhpy

Define $j : \mathrm{SL}_2(\mathbf{R}) \times \mathfrak{h} \to \mathbf{C}$ as $j(\gamma, z) = cz + d$ if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

The invariance condition can be rewritten as $f(\gamma z) = j(\gamma, z)^{\ell} f(z)$ for all $\gamma \in \Gamma$.

Transition to the group

If $f \in M_{\ell}(\Gamma; \mathbf{C})$ a modular form for Γ of weight ℓ, define $\varphi_f : \mathrm{SL}_2(\mathbf{R}) \to \mathbf{C}$ as $\varphi_f(g) = j(g, i)^{-\ell} f(g \cdot i)$.

The function φ_f is left-Γ-invariant:

$$\varphi_f(\gamma g) = \varphi_f(g) \text{ for all } \gamma \in \Gamma.$$

If $k_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, then because $k_\theta \cdot i = i$,

$$\varphi_f(g k_\theta) = e^{-\ell i \theta} \varphi_f(g).$$

Proof of invariance

- $j(\gamma, z)$ satisfies $j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z)j(\gamma_2, z)$ for all $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbf{R})$ and $z \in \mathfrak{h}$.

Thus

Γ-Invariance

$$\begin{aligned}\varphi_f(\gamma g) &= j(\gamma g, i)^{-\ell} f(\gamma g \cdot i)\\ &= (j(\gamma, g\cdot i)j(g,i))^{-\ell} j(\gamma, g\cdot i)^{\ell} f(g\cdot i)\\ &= j(g,i)^{-\ell} f(g\cdot i)\\ &= \varphi_f(g).\end{aligned}$$

Also:

$$\begin{aligned}\varphi_f(gk_\theta) &= j(gk_\theta, i)^{-\ell} f(gk_\theta \cdot i)\\ &= (j(g, k_\theta \cdot i)j(k_\theta, i))^{-\ell} f(g\cdot i)\\ &= e^{-i\ell\theta}\varphi_f(g).\end{aligned}$$

Modular form: alternate definition

Suppose $\ell > 0$ is a positive integer and $\Gamma \subseteq SL_2(\mathbf{Z})$ is a congruence subgroup. Then a function $\varphi : \Gamma \backslash SL_2(\mathbf{R}) \to \mathbf{C}$ is a modular form of weight ℓ if

- $\varphi(gk_\theta) = e^{-i\ell\theta}\varphi(g)$ for all $k_\theta \in SO(2)$
- $D_{\ell,CR}\varphi \equiv 0$, for a certain linear differential operator $D_{\ell,CR}$ that can be defined entirely from the pair $SO(2) \subseteq SL_2(\mathbf{R})$
- $|\varphi(g)| \leq C||g||^N$ for some $C, N > 0$, where $||g||^2 = tr(gg^t)$.

Here $D_{\ell,CR}\varphi = 0$ is a condition equivalent to $f_\varphi(g \cdot i) = j(g,i)^\ell\varphi(g)$ satisfies the Cauchy-Riemann equations.

- **Upshot**: One can make a definition of modular forms entirely from the group theory of the pair $SO(2) \subseteq SL_2(\mathbf{R})$!

Let

- $\mathfrak{g} = Lie(\mathrm{SL}_2(\mathbf{R})) \otimes \mathbf{C}$ and
- $\mathfrak{k} = Lie(\mathrm{SO}(2)) \otimes \mathbf{C}$

Lie algebra decomposition

Then as a representation of SO(2),

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_- \oplus \mathfrak{p}_+$$

where $\mathfrak{p}_\pm$ are one-dimensional.

- Let X_- be a basis of $\mathfrak{p}_-$

Then $D_{CR,\ell}\varphi(g) = X_-\varphi(g)$.

The $G \supseteq K$ setup

The relationship of SO(2) to $SL_2(\mathbf{R})$ is that SO(2) is a **maximal compact subgroup** of $SL_2(\mathbf{R})$

Maximal compact subgroup

- If G is a simple Lie group (i.e., its Lie algebra has no nontrivial ideals)
- Then G has a (unique conjugacy class of) maximal compact subgroups K:
  1. K is compact
  2. maximal with respect to inclusion among compact subgroups
  3. any other such subgroup L is conjugate to K in G

Automorphic forms

- Suppose G is a simple Lie group, and
- $\Gamma \subseteq G$ is an appropriate discrete subgroup.
- Let $K \subseteq G$ be a fixed maximal compact subgroup, corresponding to the Cartan involution θ.

Automorphic forms

A smooth function $\varphi : \Gamma \backslash G \to \mathbf{C}$ is an **automorphic form** if

- φ is K-finite, i.e., the right K-translates of φ generate a finite-dimensional subspace of $C^\infty(\Gamma \backslash G; \mathbf{C})$
- φ is $Z(\mathfrak{g})$-finite, i.e., φ is annihilated by a finite codimension ideal J of the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.
- φ is of moderate growth, i.e., $|\varphi(g)| \leq C||g||^N$ for some $C, N > 0$, where $||g||^2 = \mathrm{tr}(Ad(g)Ad(\theta(g)^{-1}))$.

Holomorphic modular forms

Suppose G is a simple Lie group and $K \subseteq G$ is a maximal compact subgroup. **Sometimes**, G/K has complex structure.

Some examples

1. $\mathrm{Sp}_{2n}(\mathbf{R}) = \{g \in \mathrm{GL}_{2n}(\mathbf{R}) : gJ_ng^t = J_n\}$, $J_n = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$.
2. $\mathrm{SO}(2,n) = \{g \in \mathrm{SL}_{2+n}(\mathbf{R}) : g \begin{pmatrix} 1_2 & \\ & -1_n \end{pmatrix} g^t = \begin{pmatrix} 1_2 & \\ & -1_n \end{pmatrix}\}$
3. $U(p,q) = \{g \in \mathrm{GL}_{p+q}(\mathbf{C}) : g \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix} g^* = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}\}$

Let $\Gamma \subseteq G$ be an appropriate discrete subgroup of G.

Holomorhpic modular forms

If G has G/K with $\mathbf{C}$-structure, can consider

- those holomorphic functions $f : G/K \to \mathbf{C}$ that satisfy $f(\gamma z) = j(\gamma, z)^\ell f(z)$ for all $\gamma \in \Gamma$ and $z \in G/K$.
- Equivalently, smooth, moderate growth functions $\varphi : \Gamma \backslash G \to \mathbf{C}$ that satisfy a K-equivariance condition and a certain very special differential equation.

Holomorphic modular forms

- These special functions are called holomorphic modular forms
- They again have[1] a semiclassical Fourier expansion
- Are connected with algebraic geometry, and have special rationality properties

Motivating question

If G is a simple Lie group with G/K not Hermitian, is there a **special class** of automorphic forms for G which deserve to be called "modular forms"?

[1] When G/K is a tube domain

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Lie algebras and Lie groups

Suppose L is a real Lie algebra.

Lie algebra

A real Lie algebra L is a finite-dimensional $\mathbf{R}$ vector space, that comes equipped with a Lie bracket $[\, ,\,] : L \times L \to L$ satisfying

1. $[\, ,\,]$ is bilinear, i.e., $[x+y, z] = [x,z] + [y,z]$
2. $[\, ,\,]$ is antisymmetric, i.e., $[y,x] = -[x,y]$
3. $[\, ,\,]$ satisfies the Jacobi identity, i.e., $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$.

Associated to L one can define a group $G(L)$ as

$$G(L) = \{g \in Aut_{\mathbf{R}}(L) : g[x,y] = [gx, gy] \; \forall x, y \in L\}.$$

Then $G(L)$ is a real Lie group with Lie algebra L.

The Lie algebra $\mathfrak{sl}_n$

- Let $V_n = \mathbf{R}^n$ and $V_n^\vee$ the dual vector space
- The Lie algebra of $\mathfrak{gl}_n$:

$$\mathfrak{gl}_n = M_n(\mathbf{R}) = End(V_n) = V_n \otimes V_n^\vee$$

  The Lie bracket is $[X, Y] = XY - YX$.
- The Lie algebra $\mathfrak{sl}_n$ is the trace 0 matrices: $\mathfrak{sl}_n = M_n(\mathbf{R})^{\mathrm{tr}=0}$.
- There is a projection $V_n \otimes V_n^\vee \to \mathfrak{sl}_n$ given as $v \otimes \phi \mapsto v \otimes \phi - \frac{\phi(v)}{n} 1_n$.
- If $\delta \in \mathfrak{sl}_n$, $v \in V_n$ and $\phi \in V_n^\vee$, then $\delta(v) \in V_n$ and $\delta(\phi) \in V_n^\vee$ are defined
- There is an identification $\wedge^{n-1} V_n \simeq V_n^\vee$ and $\wedge^{n-1} V_n^\vee \simeq V_n$

Notation as above. Set

$$\mathfrak{g}_2 = \mathfrak{sl}_3 \oplus V_3 \oplus V_3^\vee.$$

This is a 14-dimensional real vector space. One defines $[\,,\,]$ on $\mathfrak{g}_2$ as follows:

1. If $\delta_1, \delta_2 \in \mathfrak{sl}_3$, then $[\delta_1, \delta_2]$ is the usual commutator in $\mathfrak{sl}_3$
2. If $\delta \in \mathfrak{sl}_3$, $v \in V_3$, $\phi \in V_3^\vee$, then $[\delta, v] = \delta(v) \in V_3$, $[\delta, \phi] = \delta(\phi) \in V_3^\vee$
3. If $v_1, v_2 \in V_3$, then $[v_1, v_2] = 2v_1 \wedge v_2 \in \wedge^2 V_3 \simeq V_3^\vee$ and similarly if $\phi_1, \phi_2 \in V_3^\vee$ then $[\phi_1, \phi_2] = 2\phi_1 \wedge \phi_2$ considered in $\wedge^2 V_3^\vee \simeq V_3$
4. If $v \in V_3$ and $\phi \in V_3^\vee$ then $[\phi, v] = 3v \otimes \phi - \phi(v) 1_3 \in \mathfrak{sl}_3$.

The Lie group G_2 and other exceptional Lie groups

G_2

The group $G_2 = G(\mathfrak{g}_2)$ is the group associated to the Lie algebra $\mathfrak{g}_2$. It is a noncompact Lie group of dimension 14.

One can define a discrete subgroup $\Gamma = G_2(\mathbf{Z}) \subseteq G_2$ as follows:

1. Let $\mathfrak{g}_2(\mathbf{Z}) = M_3(\mathbf{Z})^{\mathrm{tr}=0} \oplus \mathbf{Z}^3 \oplus (\mathbf{Z}^3)^\vee$.
2. Then $\mathfrak{g}_2(\mathbf{Z}) \subseteq \mathfrak{g}_2$ is a lattice, closed under the Lie bracket.
3. Let $G_2(\mathbf{Z}) = \{g \in G_2 : g(\mathfrak{g}_2(\mathbf{Z})) \subseteq \mathfrak{g}_2(\mathbf{Z})\}$.

Other Lie groups

There are Lie groups, which can be defined similarly, with the names F_4, E_6, E_7, E_8. They have dimensions

- $\dim F_4 = 52$
- $\dim E_6 = 78$
- $\dim E_7 = 133$
- $\dim E_8 = 248$.

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Motivating question

If G is a simple Lie group with G/K not Hermitian, is there a **special class** of automorphic forms for G which deserve to be called "modular forms"?

If G is such that G/K is Hermitian, then K admits a surjection to $U(1)$, and conversely.

- For example, if $G = \mathrm{Sp}_{2n}(\mathbf{R})$, then $K \simeq U(n)$

Gross-Wallach

Consider G which have K that admits a surjection to $\mathrm{SU}(2)/\mu_2 = \mathrm{SO}(3)$

Examples:

- G_2, $K = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2$
- F_4, $K = (\mathrm{SU}(2) \times \mathrm{Sp}_6)/\mu_2$
- $E_{8,4}$, $K = (\mathrm{SU}(2) \times E_7)/\mu_2$.

Exceptional groups have 'modular forms'

The groups

$G: G_2 \subseteq D_4 \subseteq F_4 \subseteq E_{6,4} \subseteq E_{7,4} \subseteq E_{8,4}$

- $K \subseteq G$ the maximal compact. $K \twoheadrightarrow \mathrm{SU}(2)/\mu_2$.
- G/K: no Hermitian structure

Definition of modular forms on G

Let $\ell \geq 1$ be an integer. A modular form on G of weight ℓ is

- an automorphic form $\varphi : \Gamma\backslash G \to Sym^{2\ell}(\mathbf{C}^2)$
- satisfying $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $g \in G$, $k \in K$
- and $\mathcal{D}_\ell \varphi = 0$ for a certain special linear differential operator $\mathcal{D}_\ell$

- Definition is a paraphrase of one due to Gross-Wallach, Gan-Gross-Savin

These modular forms have nice properties

Theorem 1

The modular forms of weight $\ell \geq 1$ on G have a robust Fourier expansion, normalized over the integers, that is compatible with pullbacks between groups G above.

The theorem means:

1. For λ varying in a certain lattice Λ, there are explicit functions $W_\lambda : G \to Sym^{2\ell}(\mathbf{C}^2)$
2. such that if φ is a modular form of weight ℓ then
3. $\varphi(g)\text{``}=\text{''} \sum_{\lambda \in \Lambda} c_\varphi(\lambda) W_\lambda(g)$ for certain complex numbers $c_\varphi(\lambda)$.

The numbers $c_\varphi(\lambda)$ are (by definition) the **Fourier coefficients** of φ.

Fourier coefficients

- Given a modular φ form of weight ℓ, one can ask the question "Are all of φ's Fourier coefficients in some ring $R \subseteq \mathbf{C}$?"
- If $\iota : G_1 \subseteq G_2$ in the above sequence of groups, and if φ is modular form on G_2 of weight ℓ, then the pullback $\iota^*(\varphi)$ on G_1 is a modular form of weight ℓ.
- Moreover, the Fourier coefficients of $\iota^*\varphi$ are **finite sums** of the Fourier coefficients of φ

Fourier coefficients

If $R \subseteq \mathbf{C}$ is a subring, one says that φ has Fourier coefficients in R if all the values $c_\varphi(\lambda)$ are in fact valued in R.

- If λ is non-degenerate in a certain sense, these Fourier coefficients were defined by Gan-Gross-Savin, using a multiplicity one result of Wallach.
- There is no *a priori* reason to expect any modular form to have Fourier coefficients in a small ring (e.g., $\mathbf{Z}, \mathbf{Q}, \overline{\mathbf{Q}}$)

Modular forms with algebraic Fourier coefficients

Theorem 2

There are examples of modular forms with Fourier coefficients in small rings:

1. *On $E_{8,4}$, the minimal and next-to-minimal modular forms (weight 4 and weight 8) have Fourier coefficients in $\mathbf{Q}$. These modular forms have many Fourier coefficients equal to 0. Uses key input from work of W.T. Gan and G. Savin.*
2. *On $E_{6,4}$, there is a weight 4 modular form with all Fourier coefficients in $\mathbf{Z}$. This example is "distinguished" but not "singular", and is closely connected to "arithmetic invariant theory".*
3. *On G_2, there are nonzero cusp forms of arbitrarily large weight with all Fourier coefficients in $\overline{\mathbf{Q}}$. Examples constructed using the theta correspondence $\mathrm{SO}(4,4) \leftrightarrow \mathrm{Sp}_4$.*

- The Theorem says that some modular forms on exceptional groups possess "surprising" arithmeticity.

A weight 1/2 modular form

- Suppose R is a cubic ring, with $R \otimes \mathbf{Q}$ a totally real field.
- Let Q_R be the number of square roots of the inverse different in the narrow class group of R
- Call a cubic ring R **even monogenic** if $R = \mathbf{Z}[x]/(x^3 + bx^2 + cx + d)$ with b, c, d even integers.

Theorem 3 (Leslie-P.)

There is a weight 1/2 modular form on G_2 whose Fourier coefficients are the numbers Q_R for R even monogenic.

More precisely

Let $\mathfrak{d}^{-1}$ be the inverse different of R. One says a pair (I, μ) of a fractional R-ideals and a totally positive unit $\mu \in (R \otimes \mathbf{Q})^\times$ is a square root of $\mathfrak{d}^{-1}$ if

1. $\mu I^2 \subseteq \mathfrak{d}^{-1}$ and
2. $N(\mu)N(I)^2 d_R = 1$

Thank you

Thank you for your attention!