The scientific work of Tatiana Toro

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It is an honor and a pleasure to present some of the scientific work of Tatiana Toro, which I hold in the highest regard. Tatiana is also a long-time collaborator, colleague and dear friend, which makes this a doubly pleasant occassion for me. Tatiana obtained her PhD in 1992, at Stanford, under the supervision of Leon Simon, one of the leading researchers in geometric measure theory (GMT) and geometric analysis, who instilled in Tatiana the love for these subjects that permeates all of her scientific work. Tatiana spent the period 1994-1996 as an Assistant Professor at Chicago, and in this period we were very fortunate to find a common direction of research, at the

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intersection of geometric measure theory, harmonic analysis, potential theory and free boundary theory. This turned out to be a very fruitful direction of research, which has influenced a lot of Tatiana's work since then. Tatiana spent the Fall of 1997 at MSRI, participating in a program on harmonic analysis and partial differential equations, during which she met Guy David, with whom she has collaborated closely for many years, also in topics in geometric measure theory and partial differential equations.

Tatiana Toro is a very deep researcher, who finds hidden connections between seemingly distant areas, and who is always extremely careful with intricate technical details.

Let me try to give the flavor of some of Tatiana Toro's wonderful works.

Toro's dissertation was a very surprising and deep piece of mathematics, which she then succeeded to put in a wider context in subsequent works.

Recall the Sobolev embedding in $\mathbb{R}^2$: if a function φ has k derivatives in L^p, and $\frac{1}{q} = \frac{1}{p} - \frac{k}{2}$, then $\varphi \in L^q$, provided $q < \infty$.

For instance, when $k=1$, $p=2$, consider $\varphi(x) = \log\log|x|^{-1}$, $|x|$ small, which is unbounded near 0, but $|\nabla\varphi(x)| \simeq \frac{1}{|x|\log 1/|x|} \in L^2_{loc}(\mathbb{R}^2)$.

Similar examples can be found showing that if φ has 2 derivatives in L^2, its gradient need not be bounded. Hence, the graph of φ in $\mathbb{R}^3 = \{(x,\varphi(x)), x\in\mathbb{R}^2\}$ is not a Lipschitz graph. Toro's result is that one can re-parametrize the graph, by $\mathbb{R}^2$, in a bi-Lipschitz way! Pretty amazing! Toro then extended this to show, more generally, that surfaces with generalized 2$^{\text{nd}}$ fundamental form in L^2 are Lipschitz manifolds (JDG 1994). Toro also gave, soon after, extensions to higher dimensions, finding geometric conditions that give existence of bi-Lipschitz parametrizations (Duke J., 1995). Toro did this by an extension and sharpening of a well known theorem in GMT, due to Reifenberg. A locally compact set $\Sigma \subset \mathbb{R}^{n+m}$, of (Hausdorff) dimension n is said to be (δ, R) Reifenberg flat, if for every $y_0 \in \Sigma \cap B(y_0, R)$, and every $0 < r < R$, there is an n-dimensional plane $L(y,r)$, such that the Hausdorff distance D between $\Sigma \cap B(y,r)$ and $L(y,r) \cap B(y,r)$ is smaller than δr. Reifenberg had proved that if δ is small enough, $\Sigma \cap B(y_0,R)$ admits a C^α, $\alpha = \alpha(\delta) < 1$ parametrization. Toro showed that a strengthened Reifenberg condition gives rise to bi-Lipschitz parametrizations and that graphs of φ on $\mathbb{R}^2$, with second derivatives in L^2 verify this condition. In the process, Toro also simplified greatly Reifenberg's original proof. These concepts and ideas were later generalized to the context of metric spaces in a beautiful work of Toro and G. David (Math Annalen, 1999) and later on in the monumental work of Toro and G. David

(Mem. AMS, 2012). I now turn to Toro's works at the intersection of geometric measure theory, potential theory and free boundary theory. One of the central questions in GMT is the extent to which the "regularity" of a measure determines the geometry of its support. For example, a very basic conjecture that stimulated the development of GMT says that if μ is a Borel measure on $\mathbb{R}^n$ and $\lim_{r \downarrow 0} \frac{\mu(B(x,r))}{\omega_k r^k} = 1$, $0 < k \leq n$, for a.e x $(d\mu)$, then μ is the k-dimensional Hausdorff measure concentrated on a countable union of k-submanifolds (Besicovitch $k=1, n=2$; David Preiss 1987, in generality). On the other hand, potential theory has as one of its central questions the extent to which the geometry of a domain influences the boundary regularity of harmonic functions. One way in which this has been studied is through harmonic measure, the (family) of measures $\{d\omega^x, x \in \Omega\}$ such that, for a given $f \in C(\partial\Omega)$, the solution of the Dirichlet problem (for the Laplacian) is given by $u(x) = \int_{\partial\Omega} f \, d\omega^x$.

This has been studied extensively,

in works of many authors, including Calderón (1950), Carleson (1962), Hunt-Wheeden (1968-70), Dahlberg (1977-1981), Jerison-K. (1982), etc. The question that Toro and I studied, in a number of works, was the extent to which "regularity" properties of the harmonic measure determines the geometry of the boundary of the domain.

For the case of the plane, this had been done using complex analysis and conformal mapping. We set out to understand this in higher dimensions. Toro showed here that GMT was an indispensable tool for this, and Toro's expertise in the subject was, as a consequence, indispensable for the success of our projects.

For example, the fact that a domain has a (δ, R) Reifenberg flat boundary, with $\delta \to 0$ as $R \to 0$, i.e. the domain is "Reifenberg vanishing" implies that the harmonic measure is "optimally doubling" (i.e $\lim_{r \to 0} \frac{\omega(B(Q,2r) \cap \partial\Omega)}{\omega(B(Q,r) \cap \partial\Omega)} = 2^{n-1}$ (K-Toro, Duke J. 1997)), while this last fact, for a sufficiently Reifenberg flat domain implies that the domain is in fact Reifenberg vanishing (K-Toro, Annals 1999).

Moreover, in case the domain, in addition to being Reifenberg vanishing, has a "good surface measure" (roughly speaking, the domain is "vanishing chord-arc" in the terminology of David-Semmes) then the Poisson kernel (the Radon-Nykodym derivative of harmonic measure with respect to surface measure) has a logarithm that has "vanishing mean oscillation" (an integral version of continuity) (K-Toro Duke J 1997), while the converse, which was dubbed "a free boundary problem for harmonic measure", that is vanishing mean oscillation of the logarithm of the Poisson kernel, implies, under sufficient Reifenberg flatness, that the domain is a vanishing chord-arc domain. (K-Toro, Annals 1999, Ann. Sc. ENS 2003).

In a related direction, K-Toro introduced a "2-phase free boundary problem for harmonic measure". Roughly speaking, consider a domain Ω and its complement ${}^c\Omega$. Do good "regularity properties" of the Radon-Nykodym derivative of the harmonic measure of Ω, with respect to the harmonic measure of ${}^c\Omega$, give good geometric properties of $\partial\Omega$?

This was inspired by work in the plane, due to Bishop, Carleson, Garnett and Jones, and a higher dimensional conjecture of Bishop. Preliminary results were obtained by K-Toro (Crelle J, 2006). A satisfactory higher dimensional theory was found by K-Preiss-Toro (JAMS, 2009). This used a fundamental insight of

Toro, relating the problem to a crucial step in Preiss' proof of the Besicovitch conjecture, showing that GMT can be used to replace complex analysis in higher dimensions.

In work with G. David (Calc. of Var. PDE, 2015) Toro initiated the study of almost minimizers for a classical free boundary problem introduced by Alt-Caffarelli and Alt-Caffarelli-Friedman in the early 80's. These authors considered minimizers of the functional $J(u) = \int_\Omega \left[|\nabla u|^2 + q_+^2(x) \chi_{\{u>0\}}(x) + q_-^2(x) \chi_{\{u<0\}}(x) \right] dx$, for given boundary conditions. They proved the optimal Lipschitz regularity of minimizers, and regularity of the free boundary

($\mu \equiv 0$) under a certain Reifenberg type flatness assumption. The works of Alt-Caffarelli and Alt-Caffarelli-Friedman have been extremely influential in the development of the theory of free boundary problems, and they remain pillars of the theory. In proving their results, the authors derive a pde for minimizers and prove a wonderful monotonicity formula for the solutions. A bit later, Caffarelli, in a series of important papers, developed a theory of "viscosity solutions" for these problems, expanding the theory in another direction. In the work with David, Toro considered almost minimizers of J:

u is an almost minimizer for J in Ω if $\forall$ ball $B(x,r)$, with $\overline{B(x,r)} \subset \Omega$, and every, with $v|_{\partial B(x,r)} = u|_{\partial B(x,r)}$, and with $J_{x,r} = J|_{B(x,r)}$, we have

$$J_{x,r}(u) \le (1 + \kappa r^{\alpha}) J_{x,r}(v), \ \alpha > 0$$

Almost minimizers need not verify a pde, or satisfy a monotonicity formula. They were first considered by Almgren in a geometric context, studying almost-area minimizing surfaces (GMT!).

with later applications to isoperimetric partitions. They are inherently useful in the study of stability issues, and for variants with additional "lower order terms".

In later work, with David and Engelstein (Advances, 2019), Toro proved that the free boundary is uniformly rectifiable (GMT!). With David, Engelstein and

Smit Vega Garcia (Math. Z, 2021), Toro treated corresponding variable coefficient problems for almost-minimizers. De Silva and Savin have simultaneously developed a viscosity theory for these problems.

In a very recent work of Toro (with Hofmann, Martell, Mayboroda and Zhao, GAFA 2021) Toro has completed a long-standing goal of the community of researchers working on elliptic operators L with rough coefficients on domains with minimal geometric conditions.

This goal was to give a geometric characterization of those domains Ω for which the elliptic measure associated with L (the L analog of the harmonic measure for Δ), verifies a scale invariant version of absolute continuity with respect to the surface measure on $\partial\Omega$, valid for a general class of operators L. The case when $L = \Delta$ had been solved by the cumulative effort of many people; the final step was taken in a recent paper by Azzam, Hofmann, Martell, Mourgoglou and Tolsa.

These works used crucially properties associated with Δ, like the monotonicity formula of Alt-Caffarelli-Friedman. Thus a new approach had to be devised to do this for a general class of operators L. The candidate class was the so-called DKP operators, and (you may have guessed by now) GMT came to the rescue.

Using tools and ideas from GMT, Toro and collaborators succeeded in doing this.

As I hope that this bird's-eye view of Tatiana Toro's works shows, Toro's work is deep, broad and influential, making fundamental contributions to geometry, analysis and partial differential equations, with the thread of GMT running through it.

Thank you!