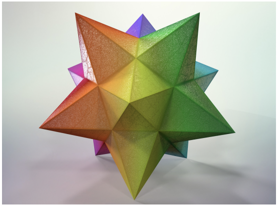
How to describe a domain in Euclidean space

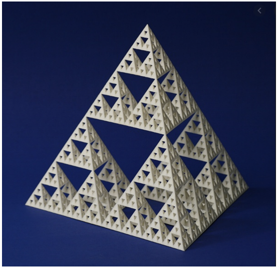
Tatiana Toro

University of Washington

2021 Blackwell - Tapia conference

A few domains





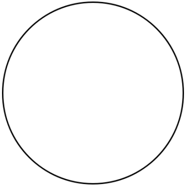
Motivation

How do we describe the features of a domain? What do we mean by features?

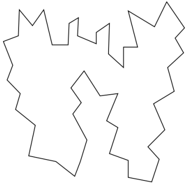
What mathematical tools come into play? Functional analysis, partial differential equations, harmonic analysis, potential theory and geometric measure theory.

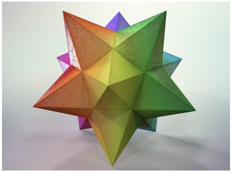
What is the relationship between the geometry of a domain, the *smoothness of its boundary* and the properties of the solutions to a differential operator on this domain?

Smooth domains

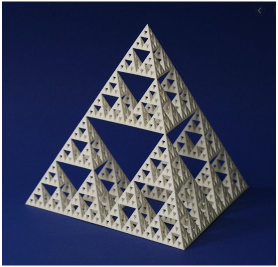


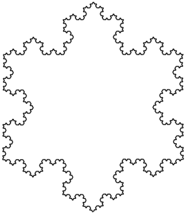
Lipschitz domains

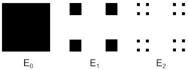




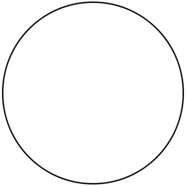
How about these?



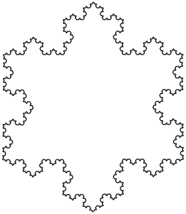




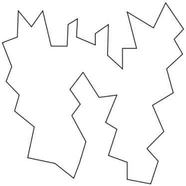
Model domains



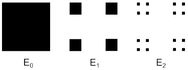
Smooth domain



Snowflake



Lipschitz
domain

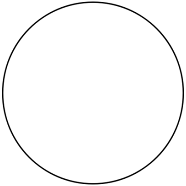


Complement of
the 4-corner
Cantor set – $\mathcal{C}^c$

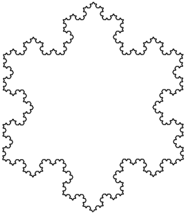
Lenses

- Geometric properties of the boundary of the domain
  - Behavior of the surface measure $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$: growth rate of the surface measure on balls centered at the boundary
  - Existence of tangents
- Geometric properties of the domain
  - (Quantitative) openess
  - (Quantitative) path-connectedness
- Boundary regularity of solutions to canonical partial differential equations on the domain

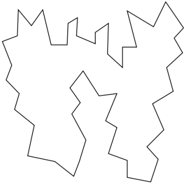
Model domains



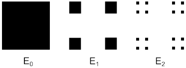
Smooth domain



Snowflake



Lipschitz domain



Complement of the 4-corner Cantor set – $\mathcal{C}^c$

Geometric properties of the boundary of the model domains in $\Omega \subset \mathbb{R}^n$

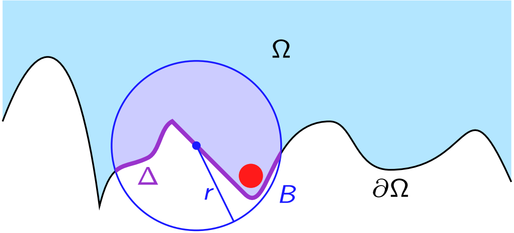
	Smooth	Lipschitz	Quasi-ball	$\Omega = \mathcal{C}^c$
Surface measure to $\partial\Omega$	finite	finite	infinite	finite
Tangents to $\partial\Omega$	everywhere	almost everywhere	nowhere	nowhere
$\sigma(B(q,r))$	$\sim r^{n-1}$	$\sim r^{n-1}$	∞	$\sim r^{n-1}$

- A domain $\Omega \subset \mathbb{R}^n$ is Ahlfors regular, if there exists $C > 1$ such that for all $q \in \partial\Omega$ and $r \in (0, \operatorname{diam} \partial\Omega)$

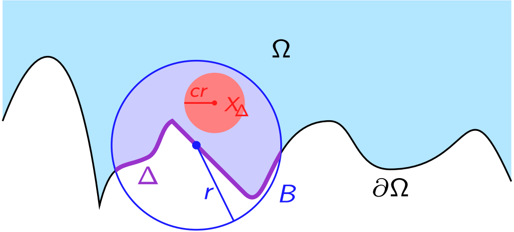
$$C^{-1} r^{n-1} \leq \sigma(B(q,r)) \leq C r^{n-1}$$

where $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$ denotes the surface measure to the boundary.

Openness and quantitative openness

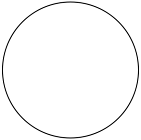


Open set

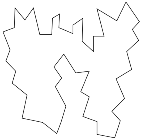


Interior Corkscrew

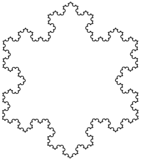
Geometric properties of the model domains in $\Omega \subset \mathbb{R}^n$



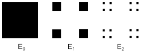
Smooth



Lipschitz



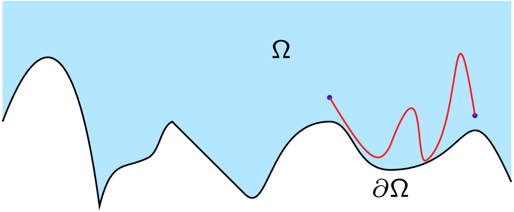
Quasi-ball



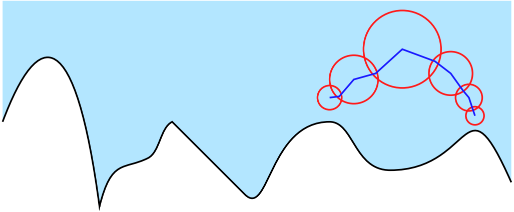
$\mathcal{C}^c$

	Smooth	Lipschitz	Quasi-ball	$\Omega = \mathcal{C}^c$
Interior corkscrew balls	yes	yes	yes	yes
Exterior corkscrew balls	yes	yes	yes	no

Path-connectedness and quantitative path-connectedness

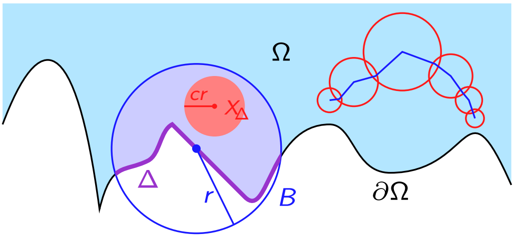


Connected set

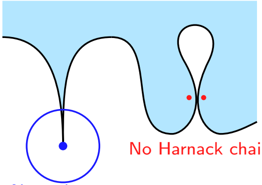


Harnack Chain

Uniform domains



Uniform Domain



Geometric properties of the model domains in $\Omega \subset \mathbb{R}^n$

	Smooth	Lipschitz	Quasi-ball	$\Omega = \mathcal{C}^c$
Ahlfors regular	yes	yes	no	yes
Tangents to $\partial\Omega$	everywhere	almost everywhere	no	no
Interior corkscrew balls	yes	yes	yes	yes
Harnack chains	yes	yes	yes	yes
Exterior corkscrew balls	yes	yes	yes	no

Dirichlet problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f \in C(\partial\Omega)$. Does there exist a function $u_f \in C(\overline{\Omega})$ such that

$$\begin{cases} Lu &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega \text{ ?} \end{cases} \tag{1}$$

Here $Lu = -\mathrm{div}\,(A(x)\nabla u)$ and $A(x) = (a_{ij}(x))_{ij}$ is an **uniformly elliptic** matrix with bounded measurable coefficients, i.e.

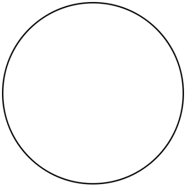
$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for } x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

- If $A = Id$, $L = -\Delta$ the Laplacian (i.e. $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$)
- $-\Delta$ is the Laplacian in homogeneous media, L is the Laplacian in inhomogeneous media (A might be discontinuous).
- For $L = -\Delta$, u in (1) describes the temperature in steady state when the boundary temperature is f.

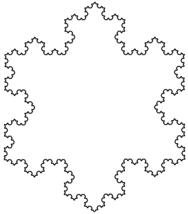
Some background

- The maximum principle holds for the solutions to these equations.
- De Giorgi-Nash-Moser (1960): the solution u is Hölder continuous in Ω.
- Additional smoothness of A implies additional smoothness of u.
- The question we are focusing on concerns the behavior of u at the boundary. The answer depends on the geometry of the domain.
- Ω is **regular** for L if for all $f \in C(\partial\Omega)$, $u_f = f \in C(\overline{\Omega})$.
- Wiener (1924): Characterization of regular domains for the Laplacian.
- Littman-Stampacchia-Weinberger (1963): Ω is regular for L if and only if Ω is regular for the Laplacian.

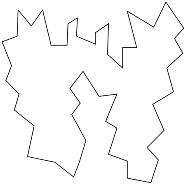
Example of (Wiener) regular domains



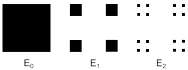
Smooth domain



Quasi-ball



Lipschitz domain



Complement of the 4-corner Cantor set

Elliptic measure

- If Ω is regular then by the Maximum Principle and the Riesz Representation Theorem there is a family of *probability* measures $\{\omega_L^x\}_{x\in\Omega}$ s.t.

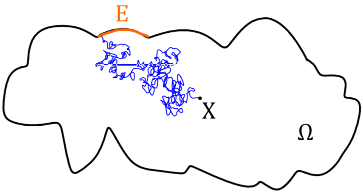
$$u(x) = \int_{\partial\Omega} f(q) d\omega_L^x(q).$$

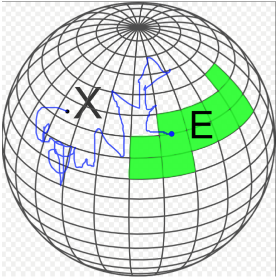
- ω_L^x is the $L-$elliptic measure of Ω with pole x. If L is the Laplacian $\omega_L = \omega$ is the harmonic measure.

- If Ω is regular and connected the Harnack principle implies that for $x, y \in \Omega$, ω_L^x and ω_L^y are mutually absolutely continuous.

- ω_L is the main building block for all the solutions to (1). It determines the behavior of u at the boundary.

Harmonic Measure





$\omega^x(E)$ denotes the probability that a Brownian motion starting at x will first hit the boundary at a point of $E \subset \partial\Omega$.

Can *harmonic/elliptic* measure distinguish between:

- Smooth domains
- Lipschitz domains
- Quasi-balls
- Complement of the 4-corner Cantor set or the complement of the Sierpinski Tetrahedron?

Can the relationship between the surface measure and *harmonic/elliptic* measure distinguish between:

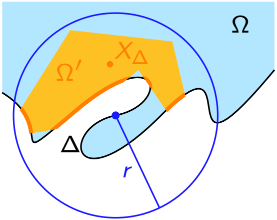
- Smooth domains
- Lipschitz domains
- Complement of the 4-corner Cantor set or the complement of the Sierpinski Tetrahedron?

Questions

- For Ahlfors regular uniform domains does the existence of exterior corkscrew balls affect the relationship between σ and ω_L?
- Can the relationship between σ and ω_L in an Ahlfors regular uniform domain predict the existence of exterior corkscrew balls?
- How does this depend on L?
- Why do we care?
- What type of relationship can we expect between two measures?

We care because ...

- Geometrically: David and Jerison (1990) proved that if $\Omega \subset \mathbb{R}^n$ is an Ahlfors regular uniform domain satisfying the exterior corkscrew condition then for $q \in \partial\Omega$, $r \in (0, \operatorname{diam} \partial\Omega)$, if $\Delta = \partial\Omega \cap B(q, r)$



This is our gold standard of smoothness!

  - there exists $\Omega' \subset \Omega \cap B(q, r)$ Lipschitz, $\operatorname{dist}(X_\Delta, \partial\Omega') \sim r$
  - ample contact $\sigma(\partial\Omega' \cap \Delta) \gtrsim r^{n-1}$

- Analytically: the relationship between σ and ω_L determines what classes of functions f the Dirichlet problem can be solved for.

Relationships between measures

Given $\Omega \subset \mathbb{R}^n$ and two measures σ and ω_L supported on $\partial\Omega$ we say that:

- ω_L is absolutely continuous w.r.t. σ if $\sigma(E) = 0$ implies that $\omega_L(E) = 0$.
- ω_L is *quantitatively absolutely continuous* w.r.t. σ if $\forall\epsilon > 0$, $\exists\delta > 0$ such that for $q \in \partial\Omega$ and $r \in (0, \operatorname{diam}\partial\Omega)$ if $\Delta = B(q, r) \cap \partial\Omega$, $E \subset \Delta$ and

$$\frac{\sigma(E)}{\sigma(\Delta)} < \delta \quad \text{then} \quad \frac{\omega_L(E)}{\omega_L(\Delta)} < \epsilon.$$

Notation: $\omega_L \in A_\infty(\sigma)$.

Questions

- If $L = -\Delta$ and $\omega = \omega_L$:
  - For what type of domains Ω is $\omega \in A_\infty(\sigma)$?
  - What does the fact that $\omega \in A_\infty(\sigma)$ imply about the geometry of Ω?

- If $L = -\mathrm{div}(A(X)\nabla\,)$ where A is bounded measurable and uniformly elliptic:
  - For what type of domains Ω is $\omega_L \in A_\infty(\sigma)$?
  - What does the fact that $\omega_L \in A_\infty(\sigma)$ imply about the geometry of Ω?

The case of the Laplacian

Results:

- Dahlberg (1977): If Ω is a Lipschitz domain then $\omega \in A_\infty(\sigma)$.
- David-Jerison & Semmes (1991): If Ω is an Ahlfors regular uniform domain satisfying the exterior corkscrew condition then $\omega \in A_\infty(\sigma)$.
- Ω Ahlfors regular uniform domain & $L = -\Delta$

  $\omega \in A_\infty(\sigma) \iff$ Exterior corkscrew balls exist

  - Hofmann - Martell (2014)
  - Hofmann - Martell - Uriarte Tuero (2014)
  - Azzam - Hofmann - Martell - Nyström -Toro (2015)

General operators L

- Caffarelli-Fabes-Kenig, Modica-Mortola, Modica-Mortola-Salsa (1981-2): There exist Lipschitz domains and operators L for which ω_L and σ are mutually singular, $\omega_L \perp \sigma$.

Questions

- *Characterize the operators L for which $\omega_L \in A_\infty(\sigma)$.*
- *To what extent does this characterization depend on the domain?*
- *Does the fact that $\omega_L \in A_\infty(\sigma)$ imply anything about the geometry of Ω?*

Oscillation based approach: DKP operators

$L = -\text{div}(A(X)\nabla)$ where A is bounded measurable and uniformly elliptic is called a Dahlberg-Kenig-Pipher (DKP) operator if

- $\big\| |\nabla A| \, \delta \big\|_{L^\infty(\Omega)} < \infty$,
- $|\nabla A|^2\delta$ satisfies the Carleson measure estimate, i.e.

$$\sup_{\substack{q\in\partial\Omega \\ 0<r<\text{diam}(\partial\Omega)}} \frac{1}{\sigma(B(q,r)\cap\partial\Omega)} \iint_{B(q,r)\cap\Omega} |\nabla A(x)|^2\delta(x)dx < \infty.$$

Here $\delta(x) = \text{dist}(x, \partial\Omega)$.

Results

- Kenig-Pipher (2001): Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, if L is a DKP operator then $\omega_L \in A_\infty(\sigma)$.
- Let $\Omega \subset \mathbb{R}^n$ be an Ahlfors regular uniform domain satisfying the exterior corkscrew condition, if L is a DKP operator then $\omega_L \in A_\infty(\sigma)$.

Do DKP operators predict the existence of corkscrew balls on Ahlfors regular uniform domains?

My collaborators: S. Hofmann, J.M. Martell, S. Mayboroda & Z. Zhao

Ω Ahlfors regular uniform domain & L a DKP operator

$$\omega_L \in A_\infty(\sigma) \quad \Longleftrightarrow \quad \text{Exterior corkscrew balls exist}$$

THANK YOU FOR YOUR ATTENTION!

Why do we like this result?

- DKP operators distinguish between Ahlfors regular uniform domains with exterior corkscrew balls (e.g. smooth or Lipschitz) and the complement of the 4-corner Cantor set or of the Sierpinski tetrahedron.
- Under the background geometric hypothesis it provides a complete classification of the geometry of the domain in terms of the behavior of the elliptic measure of DKP operators.
- The DKP condition is optimal (Modica - Mortola & Poggi).

The proof takes an unexpected route

- Small Carleson norm case (argument inspired by ideas in Geometric Measure Theory)., i.e

$$\sup_{\substack{q\in\partial\Omega \\ 0<r<\operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(q,r)\cap\partial\Omega)} \iint_{B(q,r)\cap\Omega} |\nabla A(x)|^2 \delta(x) dx \ll 1.$$

- Extrapolation argument: a general pathway to self-improvement of scale-invariant small constant estimates.

- Transfer mechanism of quantitative absolute continuity of elliptic measure between a domain and its subdomains.

The proof raises a question

- What can be said about the elliptic measure of an operator satisfying

$$\sup_{\substack{q\in\partial\Omega \\ 0<r<\mathrm{diam}(\partial\Omega)}} \frac{1}{\sigma(B(q,r)\cap\partial\Omega)} \iint_{B(q,r)\cap\Omega} |\nabla A(x)|^2 \delta(x)dx \ll 1?$$

  - Upper half plane & C^1 Dini domains: Bortz, Zhao & Toro.
  - What is the optimal domain for which such a result holds?

THANK YOU FOR YOUR ATTENTION!