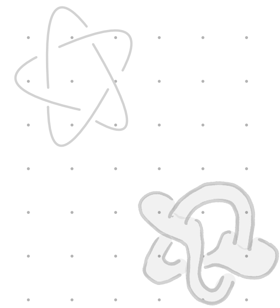

Homology concordance and knot Floer homology

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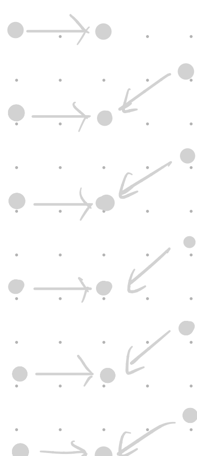
joint work with I. Dai, J. Hom, and M. Stoffregen



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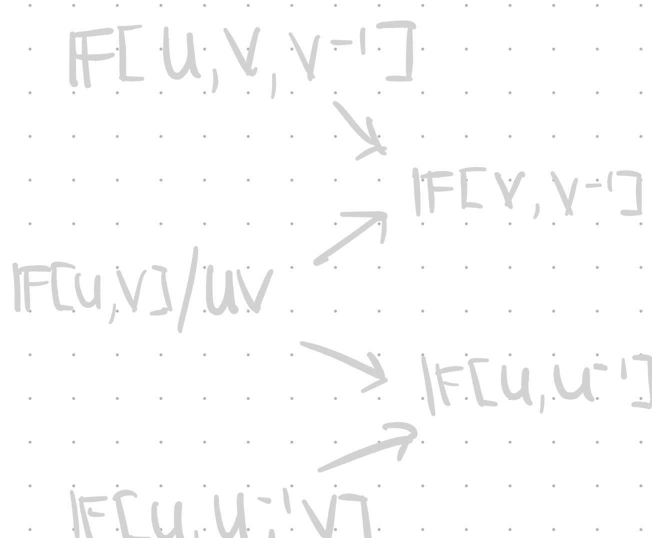
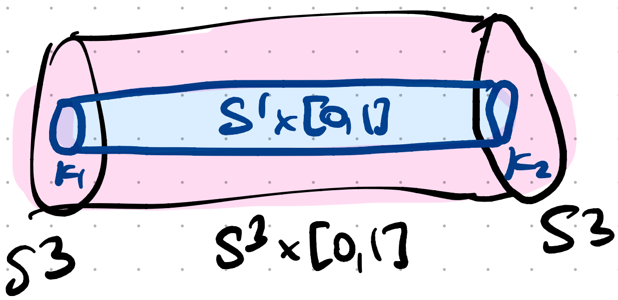


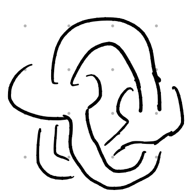
Knot concordance

<u>Def.</u> K_1, K_2 knots in S^3 are <u>smoothly concordant</u> if there exists a smoothly embedded cylinder $S^1 \times [0,1] \hookrightarrow S^3 \times [0,1]$ s.t. $\partial(S^1 \times [0,1]) = K_1 \sqcup -K_2$

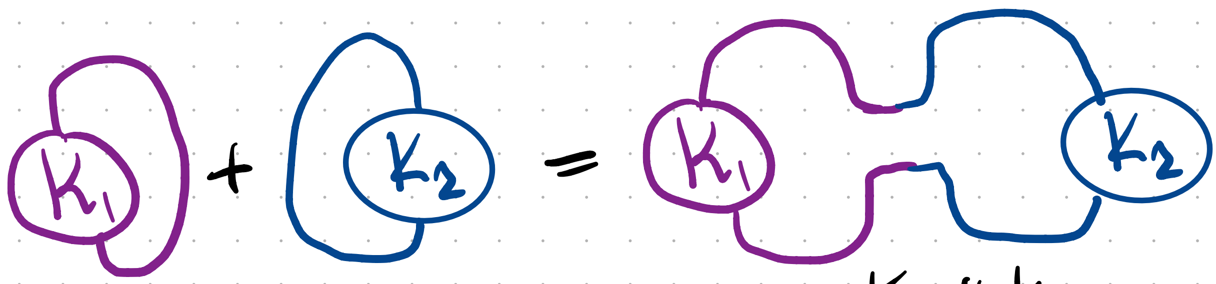
reverse orientation (K_2, S^3)



<u>ex.</u> isotopic knots are concordant

<u>ex.</u>  is concordant to an unknot

* knots in S^3



$K_1 \# K_2$

Def. The <u>concordance group</u>

$$\mathcal{C} = \{\text{knots in } S^3\} / \sim_{\text{concordance}}$$

- $\mathcal{C}$ is an abelian group, with operation

  $$[K_1] + [K_2] = [K_1 \# K_2]$$

- identity: [unknot]
- inverse of K: $-[K] = [-K]$

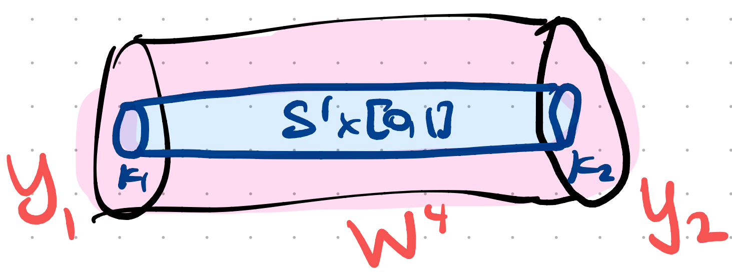
Q: What is the structure of $\mathcal{C}$?

A more general notion

Def. K_1, K_2 knots in Y_1, Y_2 $\mathbb{Z}HS^3$ are homology concordant if there exists a smoothly embedded cylinder $S^1 \times [0,1] \hookrightarrow W^4$ s.t. $\partial(W, S^1 \times [0,1]) = (Y_1, K_1) \sqcup (-Y_2, -K_2)$

- W compact oriented 4-mfd
- $\partial W = Y_1 \sqcup -Y_2$
- $H_*(Y_i; \mathbb{Z}) \xrightarrow{\cong} H_*(W; \mathbb{Z})$

W is a homology cobordism b/w Y_1 + Y_2



Def. $\hat{\mathcal{C}}_{\mathbb{Z}} = \{ (Y, K) \quad K \subset Y \ \mathbb{Z}HS^3 \} / \sim$ ← homology concordance group
$Y \sim S^3$

$\hat{\mathcal{C}}_{\mathbb{Z}}$ is an abelian group

- $(Y_1, K_1) + (Y_2, K_2) = (Y_1 \# Y_2, K_1 \# K_2)$
- inverse of (Y, K) is $(-Y, -K)$.

Thm. (Levine) The natural inclusion

$$\frac{\text{knots in } S^3}{\text{homology concordance}} = \mathcal{C}_{\mathbb{Z}} \hookrightarrow \hat{\mathcal{C}}_{\mathbb{Z}}$$

is not surjective.

Q: What is known about $\hat{\mathcal{C}}_{\mathbb{Z}} / \mathcal{C}_{\mathbb{Z}}$?

Thm (Hom-Levine-Lidman)

(1) $\hat{\mathcal{C}}_{\mathbb{Z}} / \mathcal{C}_{\mathbb{Z}}$ is infinitely generated.

(2) $\hat{\mathcal{C}}_{\mathbb{Z}} / \mathcal{C}_{\mathbb{Z}}$ has a $\mathbb{Z}$ subgroup.

Thm (Zhou) $\hat{\mathcal{C}}_{\mathbb{Z}} / \mathcal{C}_{\mathbb{Z}}$ has a $\mathbb{Z}^{\infty}$ subgroup.

<u>Thm</u> (Dai-Hom-Stoffregen-T.)

$$\hat{\mathcal{C}}_{\mathbb{Z}} / \mathcal{C}_{\mathbb{Z}} \text{ has a } \mathbb{Z}^{\infty} \text{ summand.}$$

<u>Thm</u> (DHST) For $(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$,

$\exists$ a homomorphism $\varphi_{i,j} : \hat{\mathcal{C}}_{\mathbb{Z}} \to \mathbb{Z}$ s.t.

(1) For any knot K in S^3,

$$\varphi_{i,0}(K) = \varphi_i(K) \leftarrow \text{from earlier work [DHST19]}$$

(2) For any knot in S^3, $j \neq 0$,

$$\varphi_{i,j}(K) = 0.$$

Thus, $\varphi_{i,j}$ descends to homomorphisms

$(j \neq 0)$ $\quad \varphi_{i,j} : \hat{\mathcal{C}}_{\mathbb{Z}} / \mathcal{C}_{\mathbb{Z}} \to \mathbb{Z}$.

(3) $\displaystyle\bigoplus_{n=1}^{\infty} \varphi_{n,n-1} : \hat{\mathcal{C}}_{\mathbb{Z}} / \mathcal{C}_{\mathbb{Z}} \to \mathbb{Z}^{\infty}$

is surjective.

Applications

$K \subset Y$ $\mathbb{Z}HS^3$, $Y \sim S^3$

(1) $\tau(Y,K) = \sum_{(i,j)} (i-j)\, \varphi_{i,j}(Y,K)$

(2) If $\exists j > 0$ s.t. $\varphi_{i,j}(K,Y) > 0$ (< 0) then $K \subset Y$ is not homology concordant to any knot in a negative (positive) Seifert fibered space.

Cor. There exist knots (Y,K) not homology concordant to any knot in any Seifert fibered space.

Key inputs

$K \subset S^3 \rightsquigarrow CFK_{\mathbb{F}[U,V]}(K)$ knot Floer complex

Ozsváth–Szabó and Rasmussen

φ_i is defined by first classifying knotlike complexes over $R = \frac{\mathbb{F}[U] \otimes \mathbb{F}[V]}{U \otimes V}$ up to local equivalence

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$K \subset Y$ $\mathbb{Z}HS^3 \rightsquigarrow CFK_{\mathbb{F}[U,V]}(Y, K)$

$\varphi_{i,j}$ is defined by first classifying knotlikes complexes over $\mathbb{X}$ up to local equivalence.

$$\mathbb{X} = \frac{\mathbb{F}[U_B, \{W_{B,i}\}_{i\in\mathbb{Z}}, V_T, \{W_{T,i}\}_{i\in\mathbb{Z}}]}{U_B V_T,\ U_B \cdot W_{B,i} = W_{B,i+1},\ V_T W_{T,i} = W_{T,i+1}}$$

$$\mathbb{X} \cong R_U \otimes_{\mathbb{F}} R_V / U_B \cdot V_T$$

To study knots in S^3:

Def. A **knot like complex** C is a freely, finitely generated $\mathbb{Z}\oplus\mathbb{Z}$ chain complex over R, bigraded, s.t.

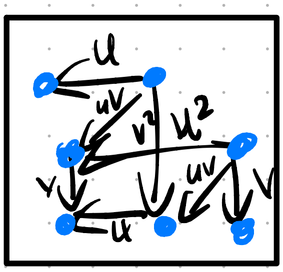
(1) $H_*(C/U)/(V\text{-torsion submod}) \cong \mathbb{F}[V]$

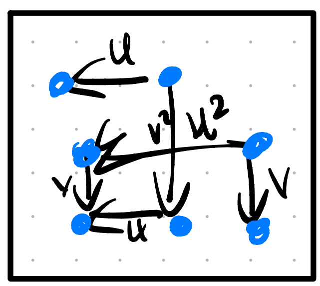
(2) $H_*(C/V)/(U\text{-torsion submod}) \cong \mathbb{F}[U]$

(3) grading conditions: (1) abs. gr_1-graded
(2) abs gr_2-graded

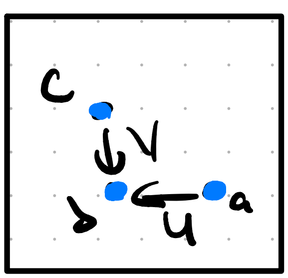
$\deg \partial = (-1,-1)$

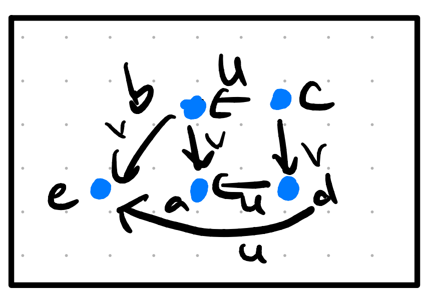
$\deg U = (-2,0)$, $\deg V = (0,-2)$

ex. $CFK_{\mathbb{F}[U,V]}(K)$ 

⇝ 

$CFK_R(K)$

ex. $CFK_R(-T_{2,3})$ 

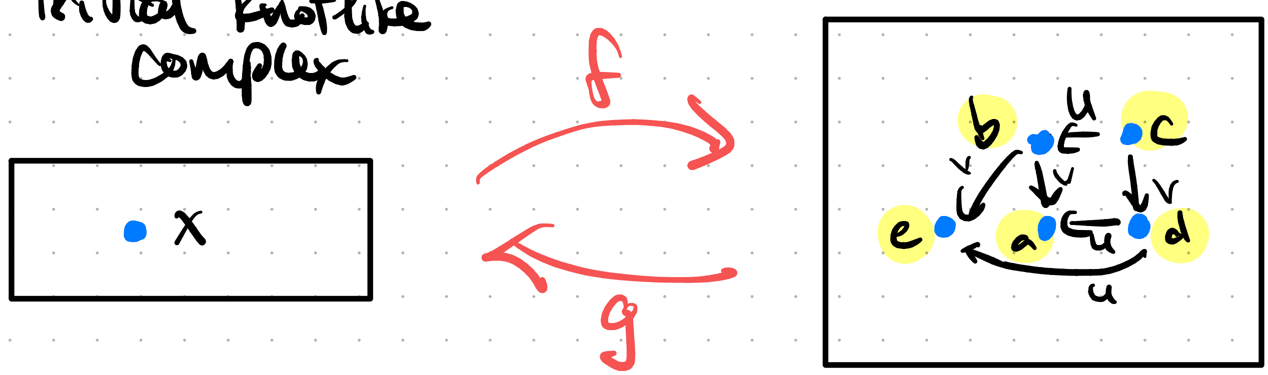
ex. $CFK_R(4_1)$ 

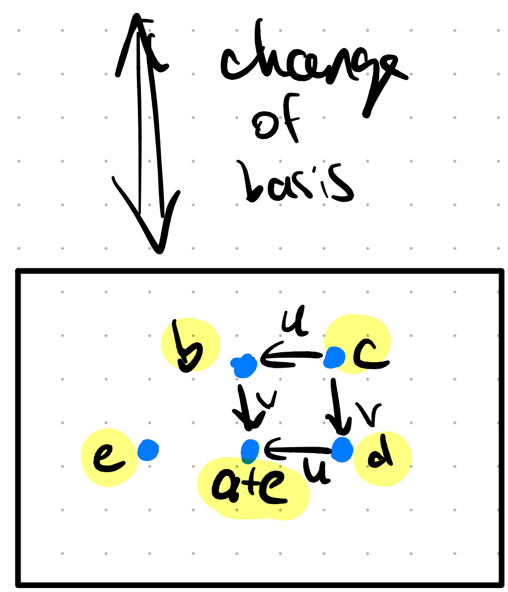
* knots in S^3

Def. Knot-like complexes C_1 and C_2 are locally equivalent if there exist maps $C_1 \underset{g}{\overset{f}{\rightleftarrows}} C_2$ s.t. f_*, g_* induces $\cong$ on $H_*(C/U)/V\text{-tors}$.

ex. $CFK_R(\text{unknot})$ trivial knotlike complex

$CFK_R(4_1)$





$f: x \mapsto e$

$g: a+e \mapsto 0$
$b \mapsto 0$
$c \mapsto 0$
$d \mapsto 0$
$e \mapsto x$

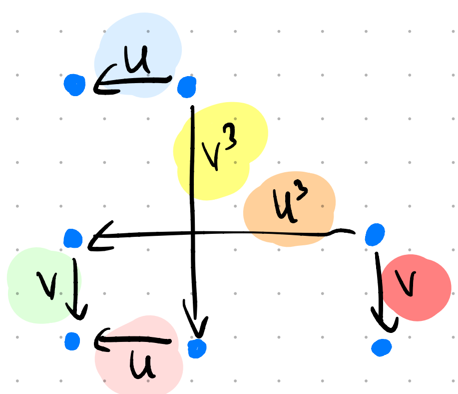
Thm. (Zemke, Hom) concordant knots in S^3 $\Rightarrow$ $CFK_R(K)$ locally equivalent

* knots in S^3

Thm. Every knotlike complex is locally equiv to a unique standard complex $\mathcal{C}(a_1, \dots, a_n)$ $a_i \in \mathbb{Z}$.

ex. $CFK_R(T_{2,3;2,5} \# -T_{2,3}) \sim$ locally equivalent

$$\mathcal{C}(1, -3, -1, 1, 3, -1)$$



$$\varphi_j = \begin{cases} 1 & j = 3 \\ 0 & \text{else} \end{cases}$$

Thm. (DHST) For each $j \in \mathbb{N}$,

$$\varphi_j(\mathcal{C}(a_1, \dots, a_n)) = \#\{a_{2k+1} = j\} - \#\{a_{2k+1} = -j\}$$

is a concordance homomorphism

$$\varphi_j : \mathcal{C} \to \mathbb{Z}.$$

The strategy of working w knot like complexes over $R = \frac{\mathbb{F}[U] \otimes \mathbb{F}[V]}{UV}$

fails for knots $K \subset Y \neq S^3$.

Why? For $K \subset Y$ $\mathbb{Z}HS^3$

$$H_*\left(CFK_R(Y,K) \otimes_{\mathbb{F}[V]} \mathbb{F}[U,U^{-1}]\right)$$

$$\cong \widehat{HF}(Y) \otimes_{\mathbb{F}} \mathbb{F}[U,U^{-1}]$$

$$\cong \bigoplus_{1}^{\mathrm{rk}\,\widehat{HF}(Y)} \mathbb{F}[U,U^{-1}]$$ For S^3 $\mathrm{rk}\,\widehat{HF}(S^3) = 1$

$CFK_R(Y,K)$ is not nec. a knotlike complex.

Key Fix. $CFK_{\mathcal{X}}(Y,K)$ knotlike complex over $\mathcal{X}$.

$$H_*\left(CFK_{\mathcal{X}}(Y,K) \otimes_{R_U} \mathbb{F}[U_B, U_B^{-1}, W_{B,0}, W_{B,0}^{-1}]\right)$$

$$\cong \mathbb{F}[U_B, U_B^{-1}, W_{B,0}, W_{B,0}^{-1}]$$

Define $CFK_{\mathbb{X}}(Y,K) = CFK_{\mathbb{F}[U,V]}(Y,K) \otimes_{\mathbb{F}[U,V]} \mathbb{X}$

$U \mapsto U_B + W_{T,0}$

$V \mapsto Y_T + W_{B,0}$

an $\mathbb{X}$-knotlike complex

Def. An $\mathbb{X}$-knotlike complex C is a chain cmplx over $\mathbb{X}$, freely, fin gen, bigraded s.t.

(1) $H_*(C \otimes_{\mathbb{X}} R_U) / (R_U\text{-torsion}) \cong R_U$

(2) $H_*(C \otimes_{\mathbb{X}} R_V) / (R_V\text{-torsion}) \cong R_V$

(3) grading conditions...

Def. $\mathbb{X}$-local equivalence: $\exists$ f, g bigraded $\mathbb{X}$-module chain maps

$$\begin{tikzcd} C_1 \arrow[r, bend left, "f"] & C_2 \arrow[l, bend left, "g"] \end{tikzcd}$$

s.t. f_*, g_* isomorphisms on $H_*(C \otimes_{\mathbb{X}} R_V) / (R_V\text{-torsion})$.

Thm. (DHST) Every $\mathfrak{X}$-knotlike complex is $\mathfrak{X}$-locally equivalent to a unique standard complex $C(b_1, \dots, b_n)$

$$b_{2k+1} \in \{\pm (U_B^i W_{B,0}^j)\} \Leftrightarrow (i,j) \in \mathbb{Z}\times\mathbb{Z}$$

$$b_{2k} \in \{\pm (V_T^i W_{T,0}^j)\} \Leftrightarrow (i,j) \in \mathbb{Z}\times\mathbb{Z}$$

Thm. Let $(i,j) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$. Define

$$e_{i,j}(C(b_1, \dots, b_n)) = \#\{b_{2k+1} = (i,j)\} - \#\{b_{2k+1} = -(i,j)\}$$

$e_{i,j}$ is a homology concordance homomorphism $e_{i,j} : \hat{\mathcal{C}}_{\mathbb{Z}} \to \mathbb{Z}$.

ex. $M_n = S^3_{+1}(T_{2,4n-1}) \# -S^3_{+1}(T_{2,4n-1})$

$K_n =$ core of surgery in $\uparrow$ $\#$ unknot

$CFK_{\mathbb{X}}(M_n, K_n)$ is locally equiv to

$$\begin{tikzcd} \bullet \arrow[d, "V_T^n W_{T,0}^{n-1}"'] & \\ \bullet & \bullet \arrow[l, "U_B^n W_{B,0}^{n-1}"] \end{tikzcd}$$

standard $\mathbb{X}$-knotlike complex

$\varphi_{n,n-1}(M_n, K_n) = 1$

$\varphi_{i,j}(M_n, K_n) = 0 \quad (i,j) \neq (n, n-1)$

$\rightsquigarrow$ $\mathbb{Z}^\infty$ subgroup of Zhou forms a $\mathbb{Z}^\infty$ summand of $\hat{\mathcal{C}}_{\mathbb{Z}} / \mathcal{C}_{\mathbb{Z}}$.