Floer homology and non-fibered knots

Steven Sivek (w/ John Baldwin)

Max Planck Institute for Mathematics / Imperial College London

5_2 $15n_{43522}$ $\mathrm{Wh}^-(T_{2,3}, 2)$ $\mathrm{Wh}^+(T_{2,3}, 2)$ $P(-3, 3, 2n+1)$

Q: given a knot invariant, which knots can it detect?

Ex knot Floer homology $\widehat{HFK}(K) = \bigoplus_{m,a\in\mathbb{Z}} \widehat{HFK}_m(K,a)$

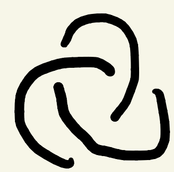
detects the Seifert genus $g(K)$, by

$g(K) = \max\{a \mid \widehat{HFK}(K,a) \neq 0\}$ (Ozsváth–Szabó '03)

$\rightsquigarrow$ $\widehat{HFK}$ detects the unknot ;

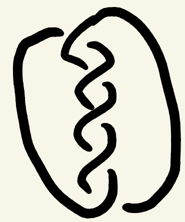
and K is fibered iff $\dim \widehat{HFK}(K, g(K)) = 1$

(Ghiggini, Ni '06)

$\rightsquigarrow$ $\widehat{HFK}$ detects the trefoils 

and figure eight.  (Ghiggini '06)

$\widehat{HFK}$ is only known to detect one other knot (up to mirroring), the cinquefoil $T(2,5)$.



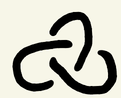
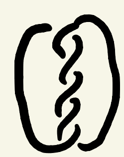
(Farber-Reinoso-Wang '22)

Ex **Khovanov homology** $\overline{Kh}(K) = \bigoplus_{h,q} \overline{Kh}^{h,q}(K)$

admits spectral sequences

- $\overline{Kh}(K) \Rightarrow KHI(\bar{K})$ (Kronheimer-Mrowka '10)
- $\overline{Kh}(K) \Rightarrow \widehat{HFK}(\bar{K})$ (Dowlin '18)

so it detects

- the unknot  (Kronheimer-Mrowka '10)
- the trefoils  (Baldwin-S. '18)
- the figure eight  (Baldwin-Dowlin-Levine-Lidman-Sazdanović '20)
- the cinquefoils.  (Baldwin-Hu-S. '21)

Common feature: all of these rely on K being fibered, i.e. $\dim \widehat{HFK}(K, g(K)) = 1$.

Goal: say something about non-fibered knots too!

Def. A knot $K \subset S^3$ is **nearly fibered** if

$$\dim \widehat{HFK}(K, g(K)) = 2.$$

Thm (Baldwin–S '22)

A genus-1 knot is nearly fibered iff it is one of:

5_2 $15n_{43522}$ $\mathrm{Wh}^-(T_{2,3}, 2)$ $\mathrm{Wh}^+(T_{2,3}, 2)$ $P(-3, 3, 2n+1)$

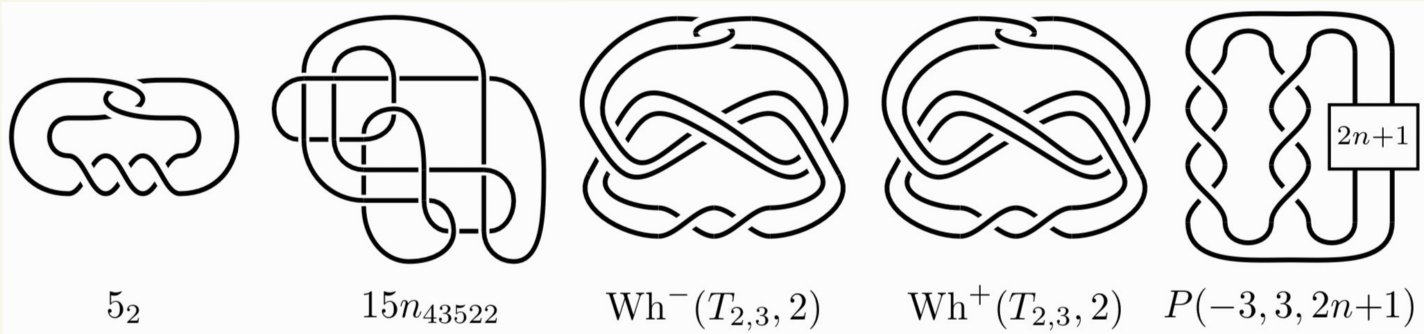
5_2 $\quad$ $15n_{43522}$ $\quad$ $\mathrm{Wh}^-(T_{2,3},2)$ $\quad$ $\mathrm{Wh}^+(T_{2,3},2)$ $\quad$ $P(-3,3,2n+1)$

Cor. • $\widehat{HFK}$ detects 5_2 and $\mathrm{Wh}^+(T_{23},2)$.

• $\widehat{HFK}$ detects membership in $\{15n_{43522}, \mathrm{Wh}^-(T_{23},2)\}$

and in $\{P(-3,3,2n+1) \mid n \in \mathbb{Z}\}$.

K	$\widehat{HFK}(K,1;\mathbb{Q})$	$\widehat{HFK}(K,0;\mathbb{Q})$	$\widehat{HFK}(K,-1;\mathbb{Q})$
5_2	$\mathbb{Q}^2_{(2)}$	$\mathbb{Q}^3_{(1)}$	$\mathbb{Q}^2_{(0)}$
$15n_{43522}$	$\mathbb{Q}^2_{(0)}$	$\mathbb{Q}^4_{(-1)} \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}^2_{(-2)}$
$\mathrm{Wh}^-(T_{2,3},2)$	$\mathbb{Q}^2_{(0)}$	$\mathbb{Q}^4_{(-1)} \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}^2_{(-2)}$
$P(-3,3,2n+1)$	$\mathbb{Q}^2_{(1)}$	$\mathbb{Q}^5_{(0)}$	$\mathbb{Q}^2_{(-1)}$
$\mathrm{Wh}^+(T_{2,3},2)$	$\mathbb{Q}^2_{(-1)}$	$\mathbb{Q}^4_{(-2)} \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}^2_{(-3)}$



<u>Cor.</u>

- $\overline{Kh}$ detects 5_2.
- $\overline{Kh}(K)$ + $\Delta_K(t)$ detect each $P(-3,3,2n+1)$.
- HOMFLY homology detects each $P(-3,3,2n+1)$.

K	$\widehat{HFK}(K,1;\mathbb{Q})$	$\widehat{HFK}(K,0;\mathbb{Q})$	$\widehat{HFK}(K,-1;\mathbb{Q})$
5_2	$\mathbb{Q}^2_{(2)}$	$\mathbb{Q}^3_{(1)}$	$\mathbb{Q}^2_{(0)}$
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$P(-3,3,2n+1)$	$\mathbb{Q}^2_{(1)}$	$\mathbb{Q}^5_{(0)}$	$\mathbb{Q}^2_{(-1)}$
$\mathrm{Wh}^+(T_{2,3},2)$	$\mathbb{Q}^2_{(-1)}$	$\mathbb{Q}^4_{(-2)} \oplus \mathbb{Q}_{(0)}$	$\mathbb{Q}^2_{(-3)}$

Applications to Dehn surgery

Def. A slope $r \in \mathbb{Q}$ is **characterizing** for $K \subset S^3$ if

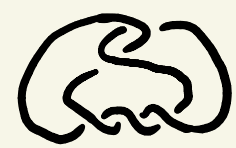
$$S^3_r(J) \cong S^3_r(K) \text{ implies } J \cong K.$$

Ex. All slopes are characterizing for

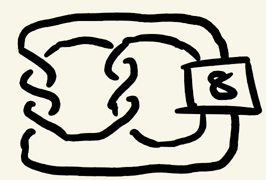
- the unknot (Kronheimer-Mrowka-Ozsváth-Szabó '03)
- the trefoils and figure eight. (Ozsváth-Szabó '06)

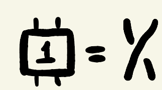
Thm (Baldwin-S. '22) arXiv: 2209.09805

Every $r \in \mathbb{Q}\setminus\mathbb{Z}_{>0}$ is characterizing for 5_2.



Note $S^3_1(5_2) \cong S^3_1(P(-3,3,8))$.





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Def. A slope $r \in \mathbb{Q}$ is characterizing for $K \subset S^3$ if

$$S^3_r(J) \cong S^3_r(K) \text{ implies } J \cong K.$$

Thm 0-surgery characterizes the unknot, trefoils, figure eight.

(Gabai '87)

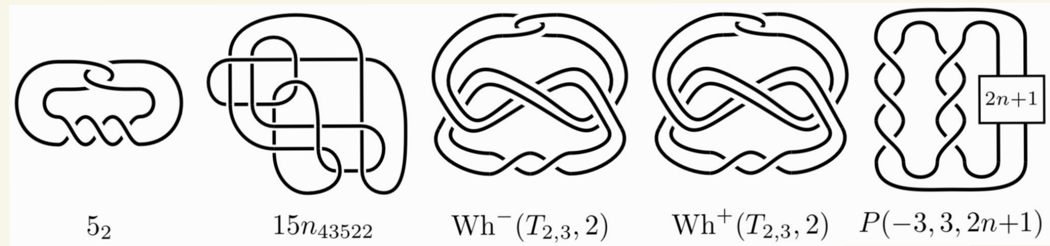
Thm (Baldwin-S. '22) arXiv: 2211.04280

0-surgery characterizes each of the following:

5_2 $15n_{43522}$ $\mathrm{Wh}^-(T_{2,3},2)$ $\mathrm{Wh}^+(T_{2,3},2)$ $P(-3,3,2n+1)$

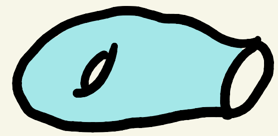
Thm (Cheetham-West '22) $\pi_1(S^3 \setminus K)$ is determined by its finite quotients for $K = 5_2, 15n_{43522}, P(-3,3,2n+1)$.

Proof outline, part 1



Let K have genus 1, $\dim \widehat{HFK}(K,1) = 2$.

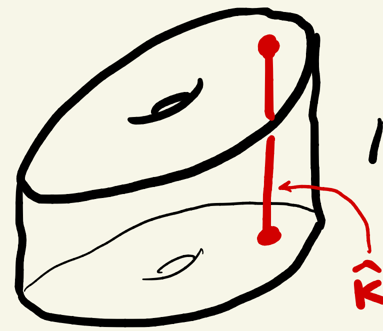
F = genus-1 Seifert surface



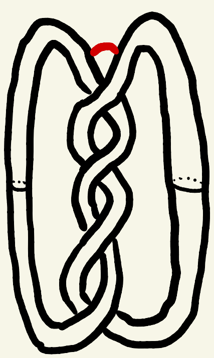
F

$\rightsquigarrow$ incompressible torus $\hat{F} \subset S^3_0(K)$.

Cut open: $M_F := S^3_0(K) \setminus N(\hat{F})$.



M_F



Thm (Cantwell–Conlon) if $K = 5_2$ or $P(-3,3,2n+1)$

then M_F is the complement of a $(2,4)$ torus link.

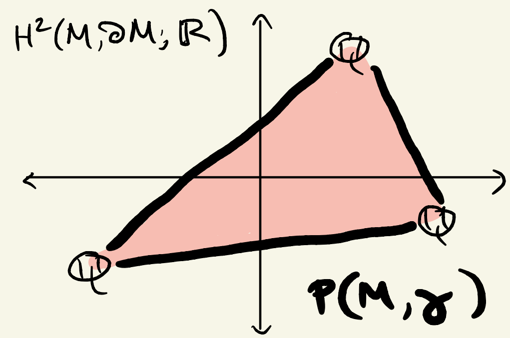
Idea: use $\dim \widehat{HFK}(K,1) = 2$ to determine M_F.

$S^3(F)$

$:= (S^3 \setminus N(F), \lambda_K)$

Thm (Juhász) $SFH(S^3(F)) \cong \widehat{HFK}(K,1) \cong \mathbb{Q}^2$.

Idea: SFH detects the sutured Thurston norm via the width of the sutured Floer polytope.

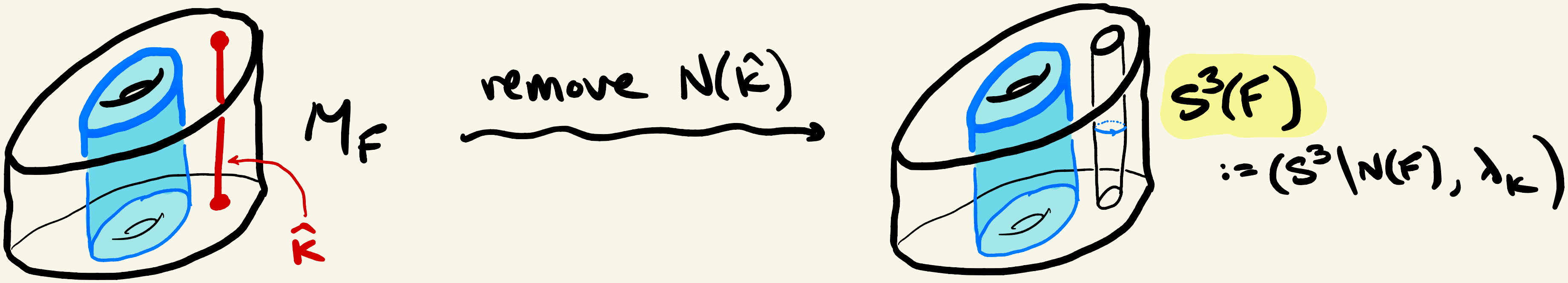


So width 0 in some direction $\rightsquigarrow$ annuli in (M,γ):

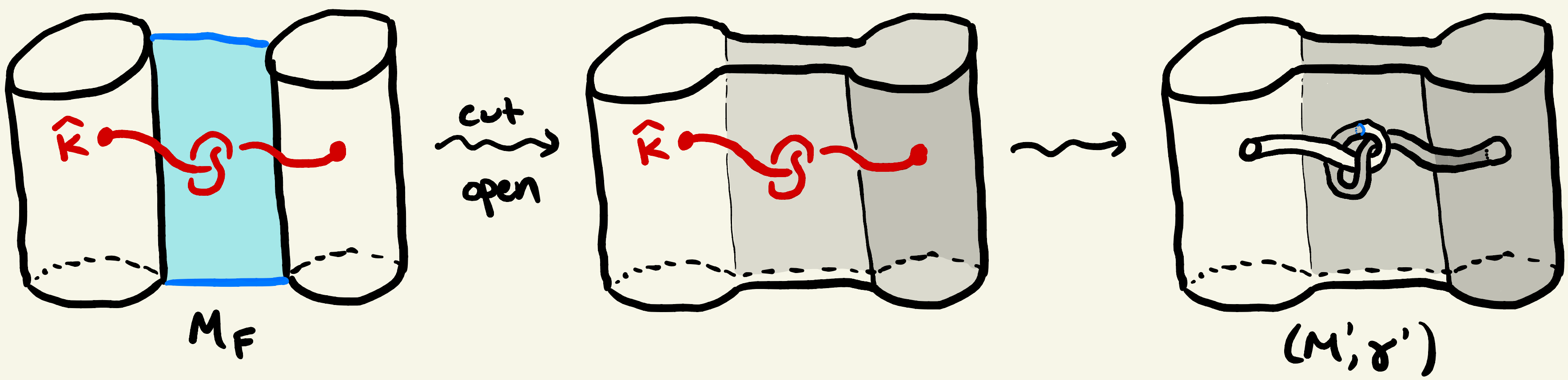
Thm (Juhász) $H_2(M)=0$, (M,γ) taut, horizontally prime, reduced

$$\Rightarrow \quad \underbrace{\dim SFH(M,\gamma)}_{=2 \text{ for } S^3(F)} > b_1(M) = \underbrace{\tfrac{1}{2} b_1(\partial M)}_{=2 \text{ for } S^3(F)}.$$

Cor.: we find an essential annulus in $S^3(F)$, hence in M_F.



Note: the annulus identifies M_F as the complement of a cable of $J \subset Y$.



Then $SFH(M',\gamma') \cong SFH(S^3(F)) \cong \mathbb{Q}^2$, so

we get a (separating) essential annulus in (M',γ')!

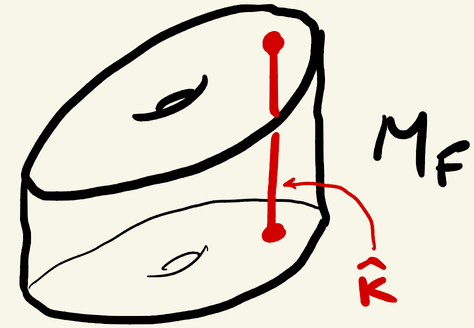
One piece of complement has $\dim SFH = 1 \Rightarrow$ it's a product.

Upshot: up to isotopy, $\hat{K}$ lies in the cabling annulus.

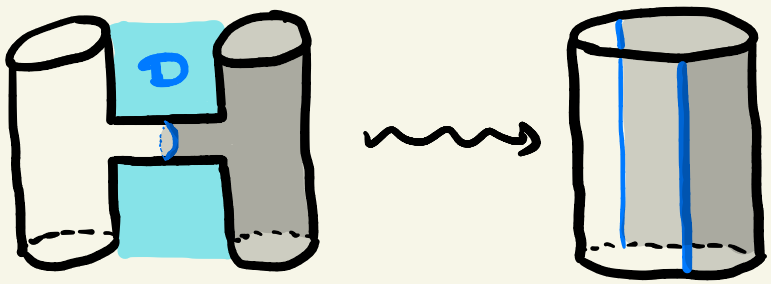
So far: $M_F := S^3_0(K) \setminus N(\hat{F})$

is the complement of a cable of $J \subset Y$,

and $\hat{K}$ lies in the cabling annulus.

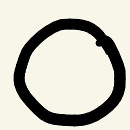
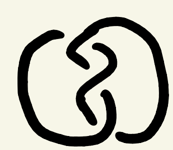


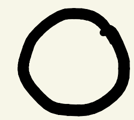
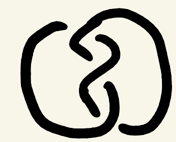
In $S^3(F)$, remove $D = \text{annulus} \setminus N(\hat{K})$



$S^3(F)$ $\quad$ $(Y \setminus N(J), \lambda_{cable})$

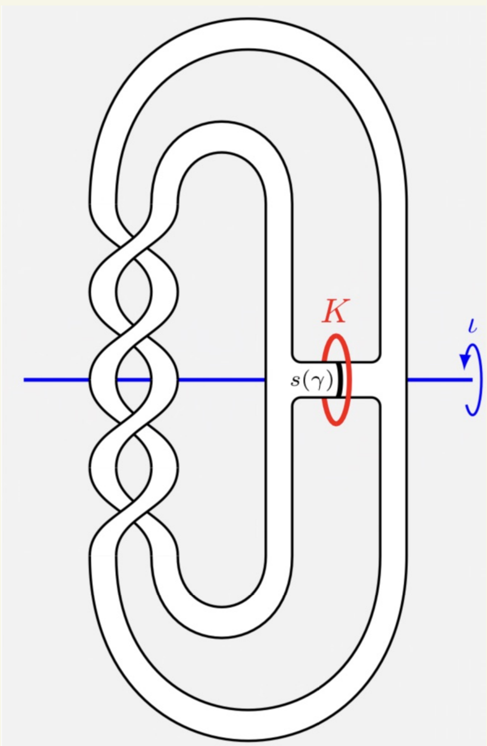
so
- $Y \setminus N(J)$ embeds in S^3;
- $SFH(Y \setminus N(J), \lambda_{cable}) \cong SFH(S^3(F)) \cong \mathbb{Q}^2$.

Thm If K is nearly fibered of genus 1, then M_F is the complement of the $(2,4)$-cable of  or .

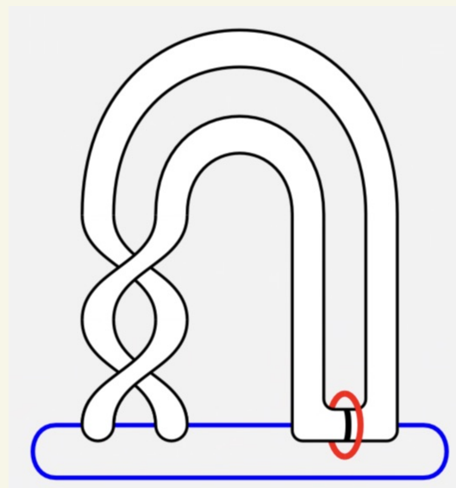
Thm If K is nearly fibered of genus 1, then M_F is the complement of the (2,4)-cable of  or .

Proof outline, part 2: classify all such K.

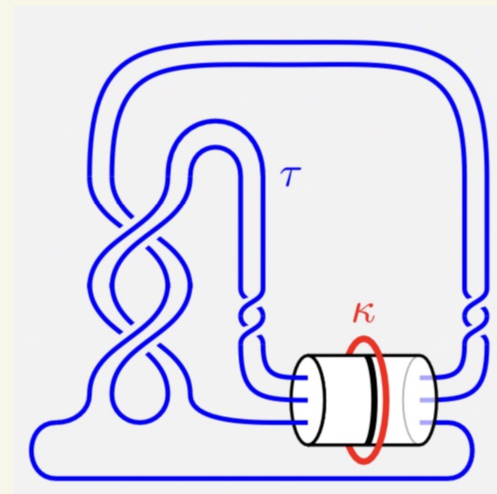
$S^3(F)$ admits an involution ι:



$\xrightarrow{\text{quotient}}$



$\cong$



On $\partial N(F) \cong \partial(F \times [-1,1])$, ι is

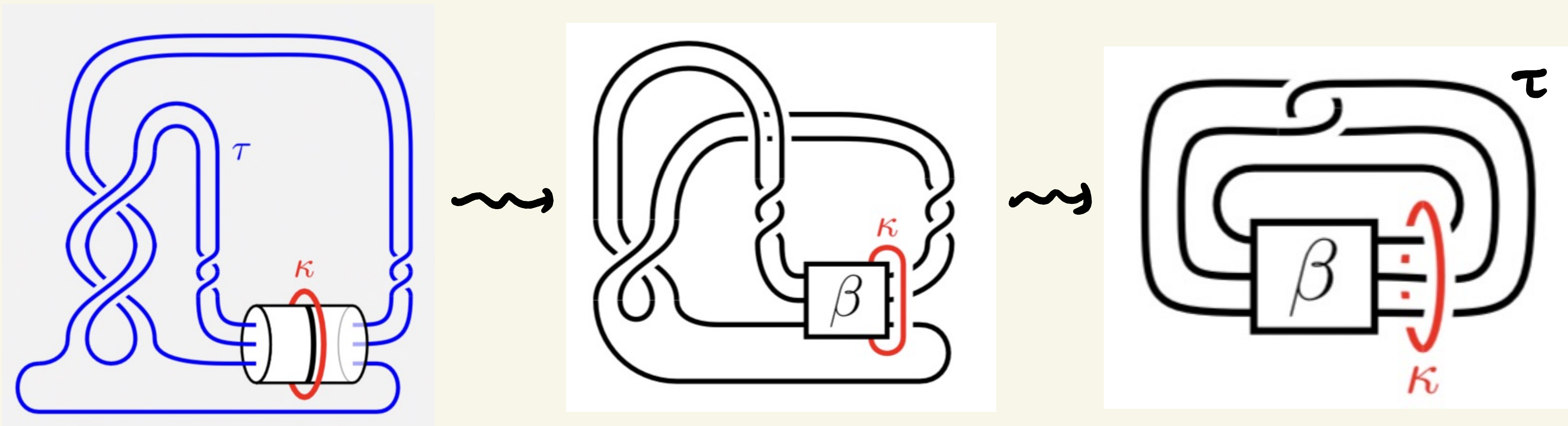
- hyperelliptic involution on $F \times \{\pm 1\}$
- fixes each $\partial F \times \{t\}$ setwise.

So ι extends over $F \times [-1,1]$, with quotient $D^2 \times [-1,1]$.

In other words: $S^3 \cong (S^3 \setminus N(F)) \cup N(F)$

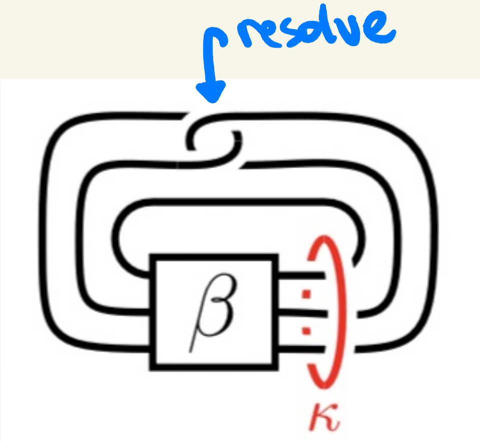
is the branched double cover of $\tau \cup \beta$

for some 3-braid $\beta \subset D^2 \times I$.



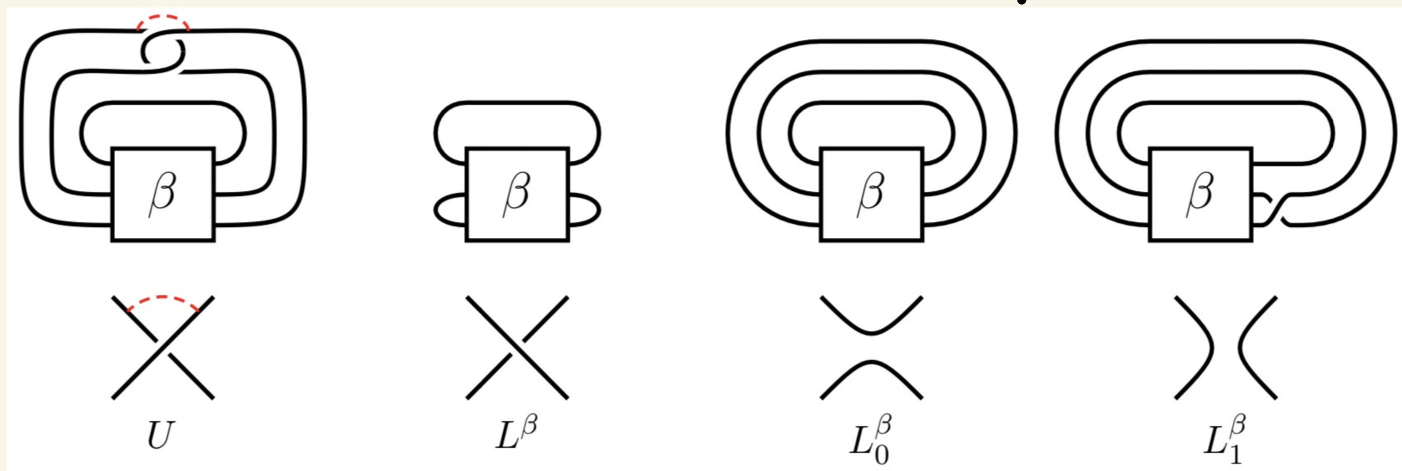
We need to

- find all β such that $\tau \cup \beta$ is unknotted;
- lift κ to $\Sigma_2(\tau \cup \beta) \cong S^3$ to find the corresponding knot K.



How to find all β such that $\tau \cup \beta = O$?

Resolve a crossing in several ways:



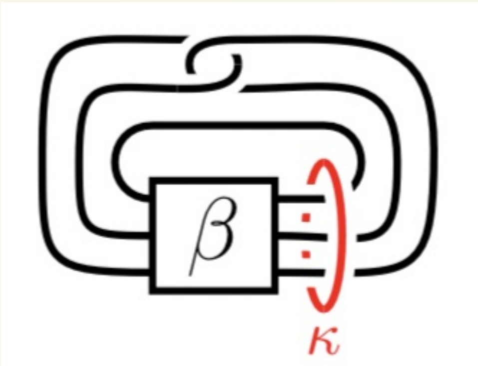
Branched double covers: S^3 $\quad S^3_{\frac{2n+1}{2}}(\gamma)$ $\quad S^3_n(\gamma)$ $\quad S^3_{n+1}(\gamma)$.

Then L^β is 2-bridge $\rightsquigarrow$ $S^3_{\frac{2n+1}{2}}(\gamma)$ is a lens space.

Cyclic surgery thm $\Rightarrow$ γ is a torus knot $T_{a,b}$, $\frac{2n+1}{2} = \frac{2ab \pm 1}{2}$;
or unknot, $n \in \mathbb{Z}$.

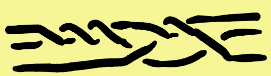
Now $\Sigma(L_0^\beta) \cong S^3_n(\gamma)$ is a lens space or $L(a,b) \# L(b,a)$, so

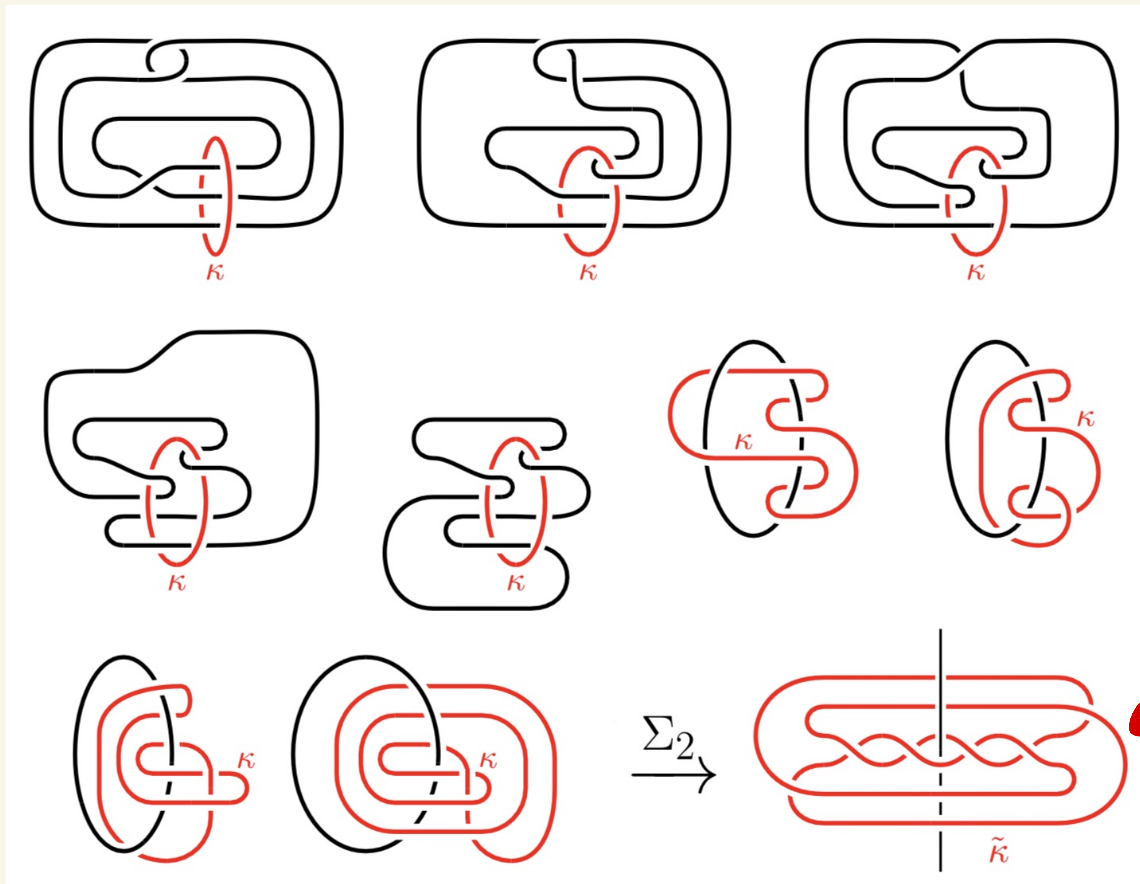
L_0^β is 2-bridge (or sum of 2-bridge) with braid index ≤ 3. $\Rightarrow$ Murasugi)



This leads to a complete list of β.

To recover K, e.g. when $\beta =$  :

β	K
	S_2
	$15n_{43522}$
	$P(-3,3,2n+1)$



S_2

Similar analysis when $M_F \cong S^3 \setminus N(C_{2,4}(T_{2,3}))$ gives

$K = Wh^+(T_{23}, 2)$ or $Wh^-(T_{23}, 2)$. $\blacksquare$