Hyperkähler Mirror Symmetry.

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Unhelpful remark: "right way" to understand these dual constructions arguably 3d mirror symmetry

(Coulomb branch, Higgs branch, ...).

But the story is so rich, also exists an interpretation via 2d (usual) mirror symmetry, which many mathematicians (e.g., me) are more familiar with.

Recall. "Classical" mirror symmetry:

e.g., mirror CY3s [Cox–Katz]

$X = \{$(smooth) quintic hypersurface $\subset \mathbb{P}^4\}$	$Y_\psi = \{[x_0 : x_1 : x_2 : x_3 : x_4] \subset \mathbb{P}^4 \mid \sum_{i=0}^{4} x_i^5 - 5\psi \prod_{j=0}^{4} x_j = 0\}$,
$H^2(X;\mathbb{Z}) \simeq \mathbb{Z}$	quotient by $(\mathbb{Z}/5)^3$, crepantly

$[dH] \longleftrightarrow d$

resolve

A - SIDE

(combinatorial)

Generating function of

N_d ← (virtual) counts of rational curves $\subset X$ of class $d[H]$

$\xleftrightarrow{\text{[mirror map]}}$

Properties: really an expansion in all this enumerative information

exponentially-quickly converging!

Typically, flow of information is

B - SIDE

(analytic)

Periods of Y_z

(i.e. $\int \Omega^{3,0}_{Y_z}$ [monodromy-invariant 3-cycle])

expanded about $z = \infty$

(in canonical coordinates).

Properties: solves a differential equation [Picard-Fuchs],

exact (e.g., belongs to some finite-dimensional vector space of "modular forms"?), ...

B-side

A - side ← [predictions of counts of rational curves!]

Remark. Extension of this story to

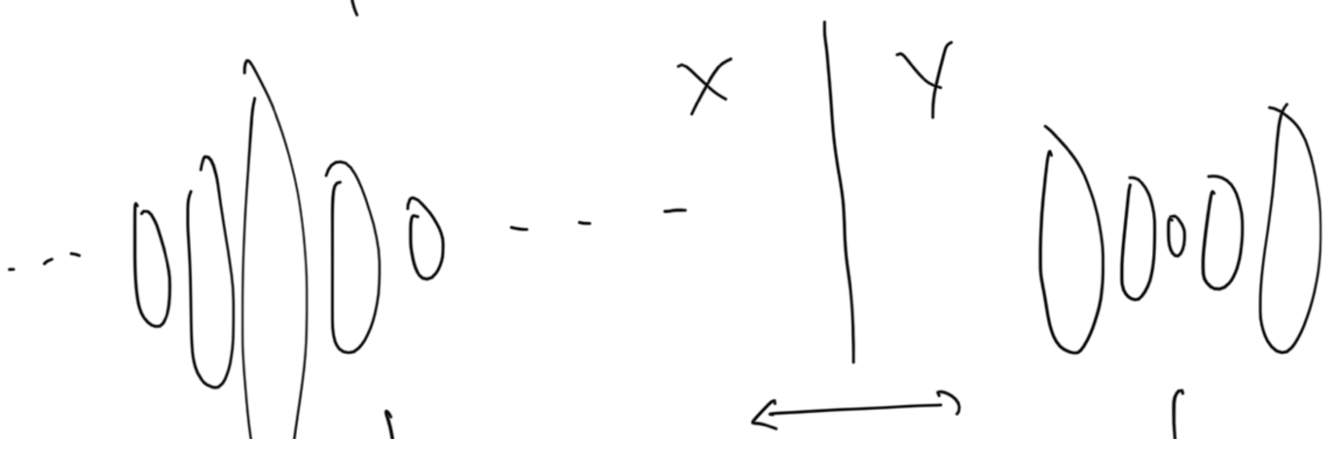
(VFC) counts of higher-genus curves $\longleftrightarrow$ BCOV real-analytic expressions satisfying holomorphic anomaly, ...

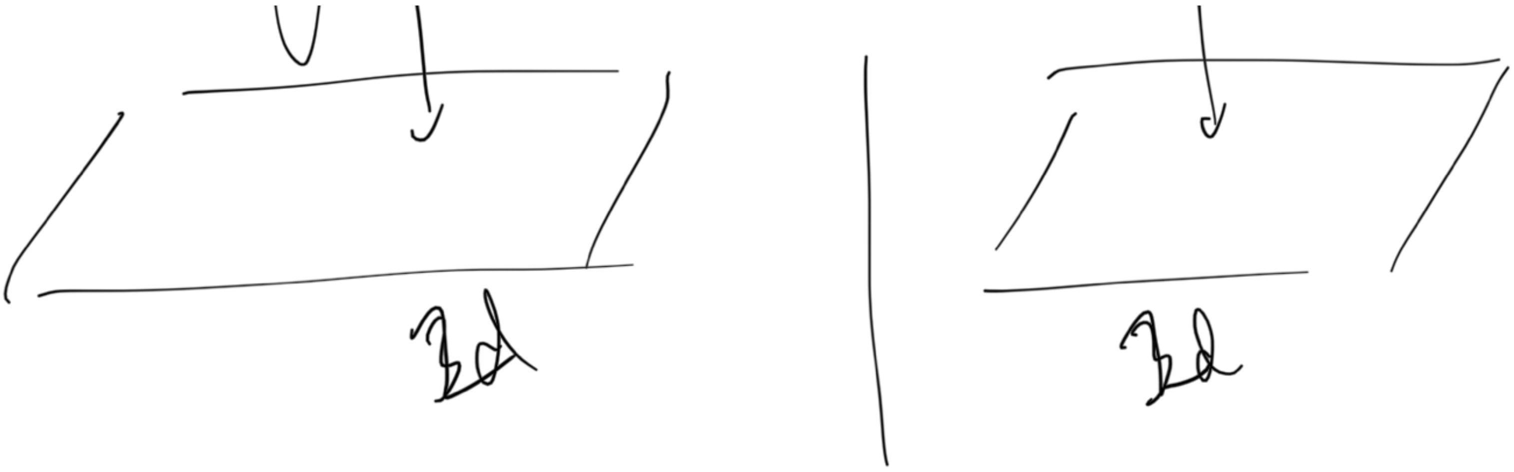
I will refuse to talk about this extension in our context! (Yes, it exists...)

What's next? Open string / homological mirror symmetry:

[Strominger-Yau-Zaslow, Gross-Siebert
(K3) Kontsevich-Soibelman, Fukaya]

(formally about $\psi = \infty, \ldots$), one expects X & Y to be dual (special) Lagrangian torus fibrations, i.e.





(real, fiberwise "Fourier-Mukai" in 3 of the 6 dimensions)

counts of holomorphic discs (w/ ∂ on the sLAG fibers) $\longleftrightarrow$ complex structure of Y_4 (e.g., structure coefficients for ring of regular functions in some canonical form)

$$\Omega(\underbrace{\gamma}_{\text{class in relative homology}}, \underbrace{u}_{\text{point in the base of the fibration}})$$

My understanding of Gross-Siebert mirror symmetry for K3s: one may <u>define</u> (via log GW, or nonarchimedean, or tropical, or A_∞-categorical, ... [Y. S. Lin, ...]) invariants that wall-cross appropriately, so as to yield a well-defined mirror K3 (formally).

Q: What are these wall-crossing invariants $\Omega(\gamma, u)$?

Q: What are the ...

Ans: Hilariously intractable!

But! We're doing hyperkähler geometry!

Metric mirror symmetry:

$$\boxed{\Omega(\gamma, u)} \longleftrightarrow \boxed{\text{Hyperkähler metric}}$$

(similarly, metric derived McKay, ... [§ 2.13, 2203.13730])

Okay, great. How? [Gaiotto-Moore-Neitzke]:

$\mathcal{X}_\gamma(\zeta, u, \theta)$ satisfying certain wall-crossing (& monodromy) properties

[i.e., solving an irregular Riemann-Hilbert problem]

$$\mathcal{X}_\gamma(\zeta, \theta) = \mathcal{X}^{sf}_\gamma(\zeta, \theta) \exp\left[\sum \Omega(\gamma', u) \langle \gamma, \gamma' \rangle \int \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \mathcal{X}(\zeta')) \right.$$

- $\mathcal{X}_\gamma(\zeta,u,\theta) = \mathcal{X}_\gamma^{sf}(\zeta,u,\theta)\exp\left(\frac{1}{4\pi i}\sum_{\gamma'}\Omega(\gamma',u)\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'}(u)}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log(1-\mathcal{X}_{\gamma'}(\zeta'))\right)$

- $\varpi(\zeta) = \frac{1}{8\pi}\langle d_{u,\theta}\log\mathcal{X}_\bullet(\zeta,u,\theta), d_{u,\theta}\log\mathcal{X}_\bullet(\zeta,u,\theta)\rangle$

  ?

(the "metric mirror map").

But! Flow of information is usually

$\boxed{\text{Hyperkähler metric}}$ $\longrightarrow$ $\boxed{\Omega(\gamma,u)}$

So! Reduced to the much easier problem of

<u>explicitly compute a Ricci-flat metric on a compact manifold.</u>

Great.

... Eventually.

'But... we're thinking about [illegible]'
started looking at

$$\hat{T}^2 \times T^2/\mathbb{Z}_2 \quad \Big| \quad T^4/\mathbb{Z}_2 \simeq T^2 \times T^2/\mathbb{Z}_2$$

This story (or more generally T^4/Γ, or

even $\mathbb{R}^4/\hat{\Gamma}$ with $0 \to \mathbb{Z}^r \to \hat{\Gamma} \to \Gamma \to 1$)

$0 \leq r \leq 4$

is ∞-dim'l Kronheimer.

By the way!

$\{$Literal ∞-dim'l Kronheimer$\}$ $\longleftrightarrow$ $\{$Gauge theory set-up$\}$

is a Fourier-duality in all 4 circles.

(So, to get to the A-side, just Poisson-resum in 2 of the 4 directions — lots of

Hankel transforms & Bessel functions start showing up!)

Recall. In Kronheimer's story, for $\Gamma \subset \mathbb{C}^2$, the group is by definition $\underline{G = U(R_\Gamma)^\Gamma}$, where

$R_\Gamma := \mathrm{Maps}(\Gamma, \mathbb{C})$ is the regular representation, (w/ standard $\langle , \rangle$ Hermitian)

$U(R_\Gamma) :=$ unitary operators on R_Γ,

$U(R_\Gamma)^\Gamma := \Gamma$-equivariant unitary operators on R_Γ.

Eventually we want, e.g., $\Gamma = \mathbb{Z}^4 \rtimes \mathbb{Z}_2$, but main idea comes across just with $\Gamma = \mathbb{Z}$:

$$R_\Gamma \rightsquigarrow \ell^2(\mathbb{Z})$$

& $\mathcal{G} := \left\{ (g_{mn})_{m,n \in \mathbb{Z}} \;\middle|\; \begin{array}{l} \text{columns are (square-} \\ \text{summable \&)} \\ \text{orthonormal} \end{array} \right.$

i.e. $g_{m,n} = g_{m+1,n+1} \quad \forall m,n \}$

So, a typical element looks like

$$\begin{pmatrix} & \vdots & \vdots & \\ & g_{-1 0} & \vdots & \\ & g_{00} & g_{-1 0} & \\ & g_{10} & g_{00} & \\ & \vdots & \vdots & \end{pmatrix}$$

$\Longrightarrow$ all the information just contained in this column.

Fourier transform $\Longrightarrow$ $\hat{g} \in L^2(S^1)$

$$\text{s.t.} \int_{S^1} \langle \hat{g}, e^{ik\theta} \hat{g} \rangle = \delta_{0,k}.$$

$$\Longrightarrow \quad |\hat{g}|^2 = 1 \quad (\text{a.e.})$$

$$\Longrightarrow \quad \hat{g} \in L^2(S^1, U(1)).$$

(Check group multiplication operator, ...)

Similarly, Kronheimer's $\mathcal{A}_\Gamma := (u(R_\Gamma) \otimes \mathbb{C}^2)^\Gamma$

$\rightsquigarrow$ connections (+ scalar fields)

on dual torus.

Okay! Hence, the events of MZ's talk.

[How does Fourier duality work for Γ acting by <u>affine</u> linear maps? §2.A of 2203.13730

... asymmetric orbifolds.]

Upshot! We want to solve, schematically, "$\underline{F^+ = \delta\xi}$"

∴ the point of MZ's talk is to find appropriate function spaces between which

we may explicitly invert the linearization of $A \mapsto F^+ (\oplus$ gauge-fixing$)$

So... iteratively apply your parametrix to obtain an expansion in ξ!

Examples. [2010.12581] [all "conjectural"]

Let's study an intermediate case of $r=2$, i.e., trying to study deformations of flat orbifolds $(\mathbb{R}^2 \times \hat{T}^2)/\Gamma$; here $\Gamma = \mathbb{Z}_q$ for $q = 2, 3, 4, 6$. (Eisenstein or Gaussian lattice L.)

Actually more fun to start with $r=4$; $\underline{(\hat{T}^2 \times \hat{T}^2)/\Gamma}$

What's the basic datum? We're trying to solve for a (gauge-fixed) <u>connection</u> (on a trivial bundle, so just an <u>$su(q)$-valued 1-form</u>) (or just 4 $su(q)$-valued functions) <u>(or just 2 $sl(q,\mathbb{C})$-valued functions U, V)</u>.

depending on
- ξ, as we want to solve "$F^+ = \xi\, S$"
- functional dependence on $\hat{T}^2 \times \hat{T}^2$

  $\longrightarrow$ Fourier transform to dependence on $L \times L$, i.e., $n^u, n^v \in L$.
- original flat connection parameters $u, v \in T^2 \times T^2$, i.e. $u, v \in \mathbb{C}$ (mod periodicities).

So! We'll write $U_n, V_n \in sl(q, \mathbb{C})$ for $n = (n^u, n^v) \in L \times L$ as <u>power series in ξ</u> ; <u>functions of u, v</u>.

Zeroeth order in ξ:

$$U_0 = u \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & \end{pmatrix}, \quad V_0 = v \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}$$

$$(U_n = V_n = 0 \text{ for } n \neq 0).$$

First order in ζ: (explicit coordinates for a K3 "over $\mathbb{R}[\varepsilon]/\varepsilon^2$"!)

$$U_n = \frac{1}{2q} \sum_{j=0}^{q-1} \sum_{i=1}^{q-1} \frac{2\zeta_{n,i,\mathbb{C}} \overline{N^v_{i,j}} + \zeta_{n,i,\mathbb{R}} N^u_{i,j}}{D_{i,j}} S_{i+j} S_j^\dagger$$

$$V_n = \frac{1}{2q} \sum_{j=0}^{q-1} \sum_{i=1}^{q-1} \frac{-2\zeta_{n,i,\mathbb{C}} \overline{N^u_{i,j}} + \zeta_{n,i,\mathbb{R}} N^v_{i,j}}{D_{i,j}} S_{i+j} S_j^\dagger$$

for $\kappa_q = e^{2\pi i/q}$, $S_j = \begin{pmatrix} 1 \\ \kappa_q^{j} \\ \kappa_q^{2j} \\ \vdots \\ \kappa_q^{(q-1)j} \end{pmatrix}$, $\quad N^u_{i,j} = n^u + (1-\kappa_q^{i})\kappa_q^{j} u$, $\quad N^v_{i,j} = n^v + (1-\kappa_q^{-i})\kappa_q^{-j} v$,

$$D_{i,j} = |N^u_{i,j}|^2 + |N^v_{i,j}|^2.$$

(higher-order in ζ expressions...)

okay! specialize to $n^v = 0$ now if you like & let's get to the comparison – what's the "mirror map"? (only turn on $\zeta_{\mathbb{R}}$)

In practice: more pleasant to compare $\omega(\tau)$ but need to compare (a, z) coordinates (A-side) w/ (u, v) coordinates (B-side).

[Right now, we "basically guess", ...]

Other comparisons to do as well! On the Coulomb/A-side, what should the mass parameters be to match the flat orbifold point?

(e.g. for $SU(2)$ $N_f = 4$ $(q=2)$, one awful singularity of the (Kodaira) elliptic fibration vs. $4\ A_1$ singularities in same fiber)

$\rightsquigarrow$ turn on appropriate real masses. (Magic point!)

Essentially a "topological" computation; we found via rep theory (Borel-de Siebenthal).

... lots of computation

[Poisson resummation]

$q = 3$, so spectrum of MN_6; Kodaira type II^* (or II at ∞) fiber

$$\sum \sum m^2 \Omega(m, p, q, R)\, \phi_R(n\Theta/m)$$

$m|n$ ${}'K$ $\uparrow$ # of copies of R $\uparrow$ characters of ${}'K$ expanded around multiples of the magic pt

reps of E_6 in the BPS state space w/ charges $m(p,q) \in H^1(T^2)$

$$= n^2 (-1)^n \sum_{\lambda \in \mathbb{Z}_3} K_3^{\,n\lambda(p+q)} \left(-\frac{4\pi^4 R^2}{3} \left| \xi_{n_B'\lambda R} \right. \right)$$

$\Longrightarrow$ constrains $\Omega(\gamma)$ for "arbitrarily high values of γ" & also makes predictions!

Then (4.87),

<u>Conj.</u>

$$\Omega(1,1,4,\underline{1728}) = 1$$

$$\Omega(1,1,4,\underline{351'}) = 0$$

$$\Omega(1,1,4,\underline{351}) = 3$$

$$\Omega(1,1,4,\underline{27}) = 9.$$

(can make predictions mod 3, ...).

Derived McKay.

$$D^b_{Coh}(T^4/\mathbb{Z}_2) \simeq D^b_{Coh}(K3) .$$

$$\wr$$

$\left\{ \mathbb{Z}_2\text{-equiv stable sheaves of } T^4 \right\} \longleftrightarrow \left\{ \text{stable sheaves on } K3 \right\}.$

~~$\updownarrow$ DUY~~ $\quad$ $\Updownarrow$ DUY

instantons

$\left\{ \text{sing. equiv inst.} \right\} \longleftrightarrow$ metrically.