Vafa-Witten invariants of projective surfaces

— overview

S : smooth projective surface$/\mathbb{C}$, $H_1(S, \mathbb{Z}) = 0$

H : very ample divisor on S

'94 : VW's S-duality

$$Z^{SU(r)}\left(-\frac{1}{\tau}\right) = \pm \left(\frac{r\tau}{\sqrt{-1}}\right)^{-\frac{e(S)}{2}} Z^{{}^L SU(r)}(\tau)$$

Goal : understand within algebraic geometry

SU(r) side

Tanaka-Thomas '17

moduli space:

$N := N_S^H(r, c_1, c_2)$: H-stable Higgs pairs (E, ϕ)

E torsion free sheaf on S
* $\mathrm{rk}(E) = r > 0$
* $c_1(E) = c_1 \in H^2(S, \mathbb{Z})$
* $c_2(E) = c_2 \in H^4(S, \mathbb{Z}) \cong \mathbb{Z}$

$\phi \in \mathrm{Hom}(E, E \otimes K_S)_0$!

N non-compact: $\mathbb{C}^* \curvearrowright N \quad t \cdot (E, \phi) = (E, t\phi)$

assume stable = semistable $\rightsquigarrow N^{\mathbb{C}^*}$ compact

$$N^{\mathbb{C}^*} = N^{hor} \sqcup N^{ver} \sqcup N^{rest}$$

N^{hor}: (E, ϕ) s.t. $\phi = 0$

$\rightsquigarrow$ Gieseker moduli space

$$M := M_S^H(r, c_1, c_2)$$

N^{ver}: (E, ϕ) s.t. $E = \bigoplus_{i=0}^{r-1} E_i \otimes t^{-i}$, $\mathrm{rk}\, E_i = 1$ $\forall i$

$E_i \cong I_{Z_i} \otimes L_i$ $\qquad N^{ver} \hookrightarrow \bigcup \prod_i \mathrm{Hilb}^{n_i}(S) \times \prod_i |\beta_i|$

$Z_i \subseteq S$ 0-dim.

$(n_0, \ldots, n_{r-1}) \in \mathbb{Z}^r$
$(\beta_1, \ldots, \beta_{r-1}) \in H^2(S, \mathbb{Z})^{r-1}$ eff.
s.t. $(\cdots)$

Thm. (Tanaka–Thomas)

cotangent cx. of N

$\exists$ symmetric perfect obstruction thy. $\mathbb{E} \to \mathbb{L}_N$

$\mathbb{E}^\vee[1] \cong \mathbb{E}$ $\qquad$ $[E^{-1} \to E^0]$

Note: $*$ for $[E] \in M$: $\quad \mathbb{E}^\vee|^{fix}_{(E,0)} \cong R\mathrm{Hom}(E,E)_0[1]$

$*$ $Ob_M|_E \cong \mathrm{Ext}^2(E,E)_0 \cong \mathrm{Hom}(E, E\otimes K_S)_0^*$

$M \leftarrow C(Ob_M) \subseteq N$

assume $h^{2,0}(S) > 0$ $\rightsquigarrow$ usually
- $*$ M singular
- $*$ $N^{ver} \neq \emptyset$
- $*$ $C(Ob_M) \neq N$

Invariant: $\int_{[N]^{vir}} 1 = \int_{[N^{\mathbb{C}^*}]^{vir}} \frac{1}{e(N^{vir})} \in \mathbb{Q}$

K-theoretic (Thomas): $\chi(N, \hat{\mathcal{O}}_N^{vir}) \in \mathbb{Q}(t^{1/2})$

← see Manschot's talk

$\hat{\mathcal{O}}_N^{vir} := \mathcal{O}_N^{vir} \otimes \sqrt{K_N^{vir}}$

Contribution N^{hor}:

$(\pm) e^{vir}(M) = \int_{[M]^{vir}} c_{vd}(\mathbb{E}^\vee|_M^{fix}), \quad vd := 2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_S)$

Contribution N^{ver}:

(Gholampour–Thomas)

$\int_{\prod_i \mathrm{Hilb}^{n_i}(S) \times \prod_i |\beta_i|} (\cdots)$

Contribution N^{rest}:

assume r prime $\rightsquigarrow$ 0   Thomas, using cosections

$$Ob_{N^{rest}} \twoheadrightarrow \mathcal{O}_{N^{rest}} \otimes h^{2,0}(S)$$

$$Z^{SU(r)}_{S,H,c_1}(q) := cst \cdot \sum_{c_2} \pm q^{c_2} \cdot \int_{[N^H_S(r,c_1,c_2)]^{vir}} 1$$

$$\in q^{\bullet}\,\mathbb{Q}((q))$$

$SU(r)/\mu_r$ side

Jiang-K

goal: to define

$$Z^{SU(r)/\mu_r}_{S,H,c_1}(q) = \sum_{w \in H^2(S,\mu_r)} e^{\frac{2\pi\sqrt{-1}\, c_1 \cdot w}{r}} \cdot Z_{S,H,w}$$

$$\begin{tikzcd}
1 \arrow[r] & \mu_r \arrow[r] \arrow[d, phantom, "\cap"] & SL_r \arrow[r] \arrow[d, phantom, "\cap"] & PGL_r \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathcal{O}_S^\times \arrow[r] & GL_r \arrow[r] & PGL_r \arrow[r] & 1
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A} \arrow[d, phantom, "\cap"] \arrow[rr] & & w \arrow[d, phantom, "\in"] \arrow[ddr, bend left] & \\
H^1(S,PGL_r) \arrow[rr, two heads, "\circledast"] \arrow[dr] & & H^2(S,\mu_r) \arrow[dl] & \\
& H^2(S,\mathcal{O}_S^\times) & \ni & \alpha
\end{tikzcd}$$

$\mathcal{A}$: Azumaya alg.

α : Brauer class

$\circledast$: "$\twoheadrightarrow$" from period-index thm. de Jong, Lieblich pick $\xi \in H^2(S,\mathbb{Z})$ s.t. $w = [\xi] \in H^2(S,\mu_r)$

may not be algebraic !

moduli space :

$M := M^H_{S,\mathcal{A}}(r,\xi,c_2)$: H-stable torsion free sheaves E of $\mathcal{A}$-modules

s.t./ $e^{\xi/r} \dfrac{ch(E)}{\sqrt{ch(\mathcal{A})}} = (r, \xi, \frac{1}{2}\xi^2 - c_2)$

B-field !

$\in \mathbb{Z}$!

r prime and $\alpha \neq 0 \Rightarrow$ * stability automatic

* $C(Ob_M) = N$, $N^{\mathbb{C}^*} = M$

$$Z_{S,H,w}(q) := \mathrm{cst} \cdot \sum_{c_2} \pm q^{c_2} e^{\mathrm{vir}}\left(M^H_{S,\mathcal{A}}(r,\xi,c_2)\right)$$

Note: * indep. of choices $\mathcal{A}$, ξ, H

* varying cx. structure S:

after applying elementary transf. to $\mathcal{A}$,

$\mathcal{A}$ is unobstructed (de Jong)

$\rightsquigarrow$ $Z_{S,H,w}$ defo. inv.

often can deform to cx. str. s.t. $\alpha = 0$

"PGL_r to GL_r red." v. Bree-Gholampour-Jiang-K

* now S-duality math. conj.

$q = e^{2\pi\sqrt{-1}\tau}$

- Back to $SU(r)$ partition fn... for $a, b \in H^2(S, \mathbb{Z})$

define: $\delta_{ab} := \begin{cases} 1 & \text{if } a \equiv b \mod r \cdot H^2(S, \mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$

Thm. (Laarakker)

w/ $\mathbb{Q}$-coeff.'s

$\forall_{r>0}$ $\exists$ power series $A(q), B(q), \{C_{ij}(q)\}_{1 \leq i \leq j \leq r-1}$:

$\forall$ (S, H), c_1 w/ previous assumptions:

$$Z^{SU(r)}_{S, H, c_1}(q)_{ver} = A^{\chi(\mathcal{O}_S)} B^{K_S^2} \sum_{(\beta_1, \dots, \beta_{r-1})} \delta_{c_1, \sum_i i\beta_i} \prod_i SW(\beta_i) \prod_{i \leq j} C_{ij}^{\beta_i \beta_j}$$

!

Conj (Göttsche-K-Laarakker)

$\forall r > 0$ $\exists$ power series $\tilde{A}(q), \tilde{B}(q), \{\tilde{C}_{ij}(q)\}_{1 \le i \le j \le r-1}$: w/ $\overline{\mathbb{Q}}$-coeff.'s

$\forall (S, H), c_1, c_2$ w/ previous assumpt. :

$e^{vir}(M_S^H(r, c_1, c_2))$ given by coeff. $q^{vd/2r}$ of

$$r^2 \cdot \tilde{A}^{\chi(\mathcal{O}_S)} \tilde{B}^{K_S^2} \sum_{(\beta_1, \dots, \beta_{r-1})} \epsilon_r^{\sum_i i\beta_i c_1} \prod_i SW(\beta_i) \prod_{i \le j} \tilde{C}_{ij}^{\beta_i \beta_j}$$

$\epsilon_r := e^{2\pi\sqrt{-1}/r}$

* Evidence small r:

Göttsche-K using Mochizuki's formula

$$\int_{[M]^{vir}} (\dots) \longleftrightarrow \int_{\prod_i \mathrm{Hilb}^{n_i}(S) \times \prod_i |\beta_i|} (\dots)$$

* $\boxed{vd = 0}$: $\int_{[M]^{vir}} 1$ Donaldson inv., consistent w/ Mariño-Moore conj

explicit conj. formula $\forall r$: Göttsche

* New approach: Joyce's vertex algebra wall-crossing

Upshot: for r prime: all VW inv. determined by

$A(q)$, $B(q)$, $C_{ij}(q)$ / $\tilde{A}(q)$, $\tilde{B}(q)$, $\tilde{C}_{ij}(q)$

swapped by S-duality

$S = K3$

* M smooth, $Ob_M = 0$
* M defo. equiv. to $\mathrm{Hilb}^{vd/2}(S)$ 'O Grady, ...

$$\xRightarrow{\text{Göttsche}} \tilde{A}(q) = 1/r\, \Delta(q^{\frac{1}{r}})^{\frac{1}{2}}, \quad \Delta(q) := q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

* $A(q) = (-1)^{r-1}/r\, \Delta(q^r)^{\frac{1}{2}}$ Tanaka-Thomas, Laarakker

Blow-up in point: $\hat{S} \xrightarrow{\pi} S$, $\hat{c}_1 := \pi^* c_1 - \ell E$ exc. $\mathbb{P}^1$

$\hat{H} := \pi^* H - \epsilon E$

$0 < \epsilon \ll 1$

Thm. (Kuhn-Leigh-Tanaka)

$$q^{\bullet} \sum_{c_2} e^{vir}\left(M_{\hat{S}}^{\hat{H}}(r, \hat{c}_1, c_2)\right) q^{c_2} = \frac{\Theta_{A_{r-1}^{\vee}, \ell}(q)}{\eta(q)^r} \sum_{c_2} e^{vir}\left(M_S^H(r, c_1, c_2)\right) q^{c_2}$$

twisted Θ-fn. $A_{r-1}^{\vee}$ lattice

Dedekind eta

Note * same shape as blow-up formula $e(\cdots)$

* Kuhn-Tanaka's blow-up algorithm; similar to Nakajima-Yoshioka's for framed sheaves $\widehat{\mathbb{P}^2} \to \mathbb{P}^2$

Define $\forall I \subseteq [r-1] := \{1, \dots, r-1\}$: $C_I := B \prod_{i \leq j \in I} C_{ij}$

Thm. (Arbesfeld - K - Laarakker) ← in progress, using KLT

$$C_{ij} = C_{r-j,\, r-i}\,, \quad \sum_{I \subseteq [r-1]} \epsilon_r^{\ell \sum_{i \in I} i}\, C_I^{-1} = \frac{\Theta_{A_{r-1}^\vee,\, \ell}(q)}{\eta(q)^r}$$

$\Longrightarrow$ $r = 2$, A, B, C_{ij} all determined ✓

$r = 3$, — " — except for ①

$r = 5$, — " — except for ④

previously derived using cosmic strings (VW '94)

Define $\Phi_{r,S,c} := \left(\frac{(-1)^{r-1}}{r\Delta(q^r)}\right)^{-\chi(\mathcal{O}_S)} \left(\frac{\Theta_{A_{r-1},0}(q)}{\eta(q)^r}\right)^{K_S^2} \cdot Z^{SU(r)}_{S,H,c}(q)_{ver}$!

suppose S minimal of general type $\leftsquigarrow$ $SW(0) = 1$

$SW(K_S) = (-1)^{\chi(\mathcal{O}_S)}$

define $t_{A_r,l} := \Theta_{A_r,0} / \Theta_{A_r,l}$

$$\Theta_{A_r,l}(q) := \sum_{v \in \mathbb{Z}^r} q^{\frac{1}{2}\langle v - l\lambda, v - l\lambda \rangle_{A_r}}$$

$$\lambda := \frac{1}{r+1}(r, r-1, \dots, 1)$$

Thm. $\Phi_{2,S,c_1} = \delta_{c_1,0} + \delta_{c_1,K_S} \cdot (-1)^{\chi(\mathcal{O}_S)} \cdot t_{A_1,1}^{K_S^2}$

Let $X_\pm$ be roots of $X^2 - 4 \cdot t_{A_2,1}^2 X + 4 t_{A_2,1} = 0$

Conj (Göttsche-K)

$$\Phi_{3,S,c_1} = \delta_{c_1,0} \cdot t_{A_2,1}^{K_S^2} \cdot \left(X_+^{K_S^2} + X_-^{K_S^2}\right) + \left(\delta_{c_1,K_S} + \delta_{c_1,-K_S}\right) \cdot (-1)^{\chi(\mathcal{O}_S)} \cdot t_{A_2,1}^{K_S^2}$$

Note: * Proved mod q^{11} (Laarakker)

* Corrects Labastida-Lozano

$$R := q^{\frac{1}{5}} / 1 + q / 1 + q^2 / 1 + q^3 / 1 + \cdots$$ Rogers-Ramanujan

$$R^{-5} - 11 - R^5 = \eta(q)^6 / \eta(q^5)^6$$ Hauptmodul $\Gamma_0(5)$

define:

$$\beta_1 := \frac{t_{A_{4,1}}}{25}\left(3R^{-5} + 2 - 8R^5\right)$$

$$\beta_2 := \frac{t_{A_{4,2}}}{25}\left(8R^{-5} + 2 - 3R^5\right)$$

Define: $\underline{X_\pm}$ sol. of:

$$X^2 - \frac{4}{5}\beta_1\left(\beta_1 t_{A_4,1}^{-1} - 1\right)\left(3R^{-5}+1\right)X + \frac{4}{5}\beta_1^2\left(3R^{-5}+1\right) = 0$$

Define $\underline{Y_\pm}$ sol. of

$$Y^2 - \frac{4}{5}\beta_2\left(\beta_2 t_{A_4,2}^{-1} - 1\right)\left(1-3R^5\right)Y + \frac{4}{5}\beta_2^2\left(1-3R^5\right) = 0$$

Define $\underline{Z}$ sol. of:

$$Z - \frac{6}{25}\left(8R^{-5} - 13 - 8R^5\right) + Z^{-1} = 0$$

$\rightsquigarrow$ $t_{A_4,i}$, R, β_1, β_2, $X_\pm$, $Y_\pm$, Z

Conj. (GKL)

$$\Phi_{\overline{5},s_1,c_1} = \delta_{c_1,0}\left\{\left(\frac{Z X_+^2 Y_+^2}{\beta_1\beta_2}\right)^{\frac{k_s^2}{2}} + \left(\frac{X_+^2 Y_-^2}{Z\beta_1\beta_2}\right)^{\frac{k_s^2}{2}} + \left(\frac{X_-^2 Y_+^2}{Z\beta_1\beta_2}\right)^{\frac{k_s^2}{2}} + \left(\frac{Z X_-^2 Y_-^2}{\beta_1\beta_2}\right)^{\frac{k_s^2}{2}}\right\}$$

$$+ \left(\delta_{c_1,k_s} + \delta_{c_1,-k_s}\right)\cdot\left\{\beta_1^{k_s^2} + (-1)^{\chi(\mathcal{O}_s)}\cdot\left(X_+^{k_s^2} + X_-^{k_s^2}\right)\right\}$$

$$+ \left(\delta_{c_1,2k_s} + \delta_{c_1,-2k_s}\right)\cdot\left\{\beta_2^{k_s^2} + (-1)^{\chi(\mathcal{O}_s)}\cdot\left(Y_+^{k_s^2} + Y_-^{k_s^2}\right)\right\}$$

Note: * proved mod q^{13}

Conj. ("horizontal/vertical duality", GKL)

$$B\left(-\frac{1}{\tau}\right) = \left(\frac{r\tau}{\sqrt{-1}}\right)^{\frac{1}{2}} \tilde{B}(\tau) \qquad C_{ij}\left(-\frac{1}{\tau}\right) = \tilde{C}_{ij}(\tau)$$

$\Longrightarrow$ closed expressions full SU(2), SU(3), SU(5) partition functions

checks: * $Z^{SU(r)}_{S,H,c_1}(q)_{\text{hor}}$ must be $\mathbb{Z}$-valued

(i.p. "Galois invariance") $\qquad \int_{[M]^{\mathrm{vir}}} 1$

* leading term produces "correct" Donaldson inv.

* closed expressions satisfy VW's S-duality.