The existence and non-existence results of $\mathbb{Z}_2$ harmonic 1-forms

Siqi He

Academy of Mathematics and System Science

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The singular $\mathbb{Z}_2$ harmonic 1-form

Let (M, g) be a smooth closed Riemannian manifold, Z be a codimensional 2 closed submanifold of M, let $\mathcal{I}$ be a flat line bundle over $M \setminus Z$ with monodromy -1 along small loop linking Z. A singular $\mathbb{Z}_2$ harmonic 1-forms is a section $v \in \Gamma(\mathcal{I})$ such that

(i) $dv = d \star v = 0$,

(ii) $v \in L^2$.

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(i) $dv = d \star v = 0$,

(ii) $v \in L^2$.

As $\mathcal{I}$ has monodromy -1 along Z, v could also be understood as a two-valued 1-form $\pm v$ defined over $M \setminus Z$ and we usually call Z the singular set.

The singular $\mathbb{Z}_2$ harmonic 1-form

Example

Let $M = \mathbb{C}$ with complex coordinate z, $Z = \{z = 0\}$, let $\mathcal{I}$ be the Möbius bundle over $\mathbb{C} \setminus \{0\}$, then $v = \Re(z^{-\frac{1}{2}} dz)$ is a section of $\mathcal{I}$. As v is a real part of a meromorphic form, v is harmonic. As $|v| \sim |z|^{-\frac{1}{2}}$ along 0, $v \in L^2$. Actually, $v = \Re(z^{-\frac{1}{2}+k} dz)$ for $k \geq 0$ are all $\mathbb{Z}_2$ harmonic 1-forms.

The singular $\mathbb{Z}_2$ harmonic 1-form

The flat bundle $\mathcal{I}$ defines an representation $\rho : \pi_1(M \setminus Z) \to \{\pm 1\}$ and using the kernel of ρ, we could define a double branched covering $p : \tilde{M} \to M$, togather with an involution $\sigma : \tilde{M} \to \tilde{M}$, such that $\sigma^2 = \mathrm{Id}$. The involution induces a decomposition of the cohomology

$$H^k(\tilde{M}; \mathbb{R}) = H^k_-(\tilde{M}) \oplus H^k_+(\tilde{M}),$$

where $H^k_+(\tilde{M}) \cong H^k(M)$.

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where $H^k_+(\tilde{M}) \cong H^k(M)$.
The pull-back $\tilde{v} := p^\star v$ is a 1-form on $\tilde{M}$ will be anti-invariant under the involution $\sigma^\star \tilde{v} = -\tilde{v}$. Moreover, it is harmonic w.r.t. the pull-back singular metric $p^\star g$

$$d\tilde{v} = d \star_{p^\star g} \tilde{v} = 0.$$

By the work of Teleman, also work of S.Wang, there is L^2 Hodge theorem for this singular metric $p^\star g$. Therefore, finding singular $\mathbb{Z}_2$ harmonic 1-forms is purely a topological problem, which could be identified with the space of $H^1_-(\tilde{M})$ for $\tilde{M}$ to be any double branched covering of M,

$$\begin{aligned}&\{\text{singular } \mathbb{Z}_2 \text{ harmonic one forms } (\mathrm{Z}, \mathcal{I}, \mathrm{v})\}\\ \cong&\{p : \tilde{M} \to M \text{ double branched covering, } H^1_-(\tilde{M}).\}\end{aligned}$$

Example

Let $K \subset S^3$ be a oriented knot or link, there is a double branched covering $\tilde{M}$ along K. As S^3 has trivial 1st homology, $H^1(\tilde{M}) = H^1_-(\tilde{M})$ are all anti-invariant under the involution. Therefore, the existence of singular $\mathbb{Z}_2$ harmonic 1-form is equivalent to the condition that $H^1(\tilde{M})$ is non-trivial, which is satisfied if and only if the Alexander polynomial $\Delta_K(-1) = 0$.

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However, if K only consists of one component, then $\Delta_K(-1)$ is an odd number. Therefore, the branched set of a singular $\mathbb{Z}_2$ harmonic 1-form over S^3 must have at least two components. (Haydys 20')

$\mathbb{Z}_2$ harmonic 1-forms

Given a singular $\mathbb{Z}_2$ harmonic 1-form v, $|v|$ is well-defined over $M \setminus Z$ and $|v|$ will blow-up at order $-\frac{1}{2}$ along the singular set Z, for example $\Re(z^{-\frac{1}{2}} dz)$.

Definition

A $\mathbb{Z}_2$ harmonic 1-form is a singular $\mathbb{Z}_2$ harmonic 1-form v such that $|v|$ extends continously to a Hölder function on M.

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Example

Let Σ be a Riemann surface of genus ≥ 2, given a quadratic differential $q \in H^0(K^2)$, then any square root of q will define a $\mathbb{Z}_2$ harmonic 1-form. Near a zero of q, we could write $q = z^k dz \otimes dz$, while $v := \Re(z^{\frac{k}{2}}dz)$ is a $\mathbb{Z}_2$ harmonic 1-form with monodromy -1 along odd zeros of q.

Analytic Aspects of Gauge Theory

One of the main motivation to study the $\mathbb{Z}_2$ harmonic 1-form is coming from analytic aspects of gauge theory equations with non-compact gauge group, which might lead to new 4-dimensional gauge theory invariants. For example, the $PSL(2,\mathbb{C})$ flat connections, the Kapustin-Witten equations, the Vafa-Witten equations and Haydys-Witten equations.

The non-compactness behavior of these new gauge theory equations have been widely studied.

- $PSL(2,\mathbb{C})$ flat connections, Kapustin-Witten equations, Vafa-Witten equations: Taubes.
- Multispinor Seiberg-Witten equations: Haydys-Walpuski, Taubes, Walpuski-Zhang.
- Hitchin equations: Mazzeo-Swoboda-Weiss-Witt, Fredrickson.

Non-compactness behavior of new gauge theory equations

Witten 09' introduce a gauge theory approach to Jones polynomial and Khovanov homology by studying the Kapustin-Witten equations and Haydys-Witten equations. Taubes in his series of papers studies the non-compactness behavior of the Kapustin-Witten equations and the $\mathbb{Z}_2$ harmonic 1-forms will be the ideal boundary of Taubes' compactification. Even we are not going to discuss the details of Taubes' compactification, we will give you some intuition.

Non-compactness behavior of new gauge theory equations

The flat $PSL(2,\mathbb{C})$ connections are a speical case of the Kapustin-Witten equations, which could be identified with the character variety. As $PSL(2,\mathbb{C})$ is a non-compact group, you expect that the moduli space of flat $PSL(2,\mathbb{C})$ connection, hence the KW moduli space, is also non-compact.

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In our previous example over Riemannian surface, $\mathbb{Z}_2$ harmonic 1-forms come from the square root of quadratic differentials $H^0(K^2)$. If we fixed a hyperbolic strucutre on a Riemannian surface Σ, then all discrete faithful representations $\pi_1(\Sigma) \to PSL(2;\mathbb{R})$ could be identified with the space of quadratic differential $q \in H^0(K^2)$. As $H^0(K^2)$ is a finite dimensional vector space, we could compactified the space of $H^0(K^2)$ by quadratic differentials with unit norm.

A singular $\mathbb{Z}_2$ harmonic 1-form is purely topological while you might regard a $\mathbb{Z}_2$ harmonic 1-form as a geometric object.
Over S^3, from previous examples we see that there exists a huge amount of singular $\mathbb{Z}_2$ harmonic 1-forms. Moreover, v will satisfies the following Weinzenböck identity

$$\Delta|v|^2 + |\nabla v|^2 + \mathrm{Ric}(v, v) = 0.$$

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For a $\mathbb{Z}_2$ harmonic 1-forms, we found that $\int_{S^3} \Delta|v|^2 = 0$, thus v has to be trivial over the sphere with round metric. From S^3, the existence of $\mathbb{Z}_2$ harmonic 1-forms is quite different comparing to the singular $\mathbb{Z}_2$ harmonic 1-forms. Currently, I don't know whether there exists a $\mathbb{Z}_2$ harmonic 1-form over S^3 with Ricci curvature non-positive metric.

Questions

In today's talk, we will have some discussions on the following two questions:

(1) could you find examples of $\mathbb{Z}_2$ harmonic 1-forms on a closed 3-manifold?

(2) is there any obstruction on the existence of $\mathbb{Z}_2$ harmonic 1-forms besides the non-trivial 1st Betti number or Ricci positive condition?

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Question (1): Could you find examples of $\mathbb{Z}_2$ harmonic 1-forms on a 3-manifold?

By the work of Taubes, you might think that $\mathbb{Z}_2$ harmonic 1-forms exist widely, but it is actually very hard to construct examples of them. Doan-Wapuski 17' construct examples of $\mathbb{Z}_2$ harmonic spinors associte to 2-spinors Seiberg-Witten equations over 3-manifold with $b_1 > 0$.

Today, we will introduce an extra $\mathbb{Z}_3$ symmetry to make a singular $\mathbb{Z}_2$ harmonic 1-form Hölder continous. The $\mathbb{Z}_3$ symmetry has been used by Taubes-Wu to construct model examples. Using the extra symmetry, we could find examples of rational homology spheres that exists a $\mathbb{Z}_2$ harmonic 1-forms.

Let $Z \subset M$ be a codimension 2 submanifold and z be the coordinate on the normal bundle with $Z = \{z = 0\}$, then a $\mathbb{Z}_2$ harmonic 1-form could locally have an expansion

$$v \sim \Re(Az^{-\frac{1}{2}}dz + Bz^{\frac{1}{2}}dz) \qquad (1)$$

Suppose there exists an $\mathbb{Z}_3$ action on the normal bundle sending $\sigma : z \to e^{\frac{2\pi i}{3}}z$ such that $\sigma^{\star}v = v$, then

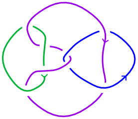
$$z^{-\frac{1}{2}}dz \to \pm e^{\frac{\pi i}{3}}z^{-\frac{1}{2}}dz, \; z^{\frac{1}{2}}dz \to \pm z^{\frac{1}{2}}dz$$

Therefore, a $\mathbb{Z}_3$ invariant singular $\mathbb{Z}_2$ harmonic 1-form is actually a $\mathbb{Z}_2$ harmonic 1-form(Hölder continous along Z).

Now, we will give an example which satisfies the extra symmetry. Let M to be a rational homology 3-sphere, L be an oriented link on M, we write M_k to be the k-fold branched covering of M along L. Suppose the Alexander polynomial $\Delta_L(-1) = 0$, then there exists a $\mathbb{Z}_2$ harmonic 1-forms over M_3.

$$\begin{array}{ccc} M_6 & \xrightarrow{\quad p_3 \quad} & M_3 := M_6/\langle \mathbb{Z}_2 \rangle \\ {\scriptstyle p_2}\downarrow & & \downarrow \\ \alpha \in M_2 := M_6/\langle \mathbb{Z}_3 \rangle & \longrightarrow & M = M_6/\langle \mathbb{Z}_6 \rangle \end{array}$$

One could find examples of links with $b_1(M_2) > 0$ and M_3 be a rational homology 3-sphere. The following link L8n6{0,0} will satisfy the condition. Using connected sum, you could find infinity number of rational homology 3-spheres that admit $\mathbb{Z}_2$ harmonic 1-forms. Moreover, for generic metric, you could make the "B" term of the leading expansion $v \sim \Re(Bz^{\frac{1}{2}}dz)$ nowhere vanishing along Z, which we refer this condition non-degenerate.



Theorem

(H.2022) There exists infinity number of rational homology 3-spheres that admit $\mathbb{Z}_2$ harmonic 1-forms.

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The Calabi-Yau manifold

Now, we will explain an application of $\mathbb{Z}_2$ harmonic 1-form to construct deformation of branched immersed special Lagrangians and using recent work of Abouzaid-Imagi to get a non-existence result for $\mathbb{Z}_2$ harmonic 1-forms.

The Calabi-Yau manifold

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Definition

A Calabi-Yau n-fold is a quadruple (X, J, ω, Ω) such that

(i) (X, J, ω) is a n-dimensional Kähler manifold with a Kähler metric g,

(ii) Ω is a nowhere vanishing holomorphic (n,0)-form which satisfies

$$\Omega \wedge \bar{\Omega} = c_n \omega^n,$$

where c_n is a specific constant depends on n.

The Special Lagrangian Submanifolds

Definition

(Harvey-Lawson)An immersed submanifold $\iota : L \to X$ in a Calabi-Yau (X, J, ω, Ω) is called a special Lagrangian if $\iota^\star\omega = 0$ (Lagrangian condition), $\iota^\star \mathrm{Im}\Omega = 0$ (special condition).

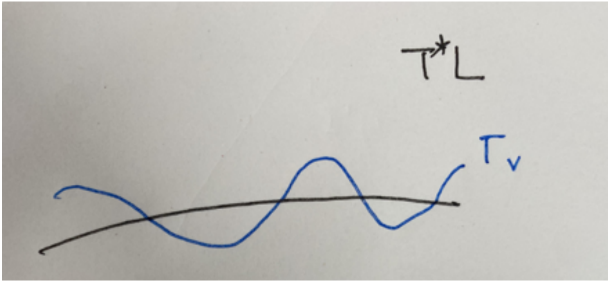
The existence question is a major open problem in general. Some known construction techniques are high symmetrics, gluing constructions and Cartan-Kähler theory. (Joyce, Bryant, Haskins-Kapouleas.)

Example

(Bryant, Doice) Let (L, g) be a real analytic Riemannian manifold with $\chi(L) = 0$, then over a neighborhood of the zero section of $T^\star L$, there exists a Calabi-Yau structure with the zero section a special Lagrangian.

McLean's Deformation Theorem

The special Lagrangians have a beautiful local deformation theory due to R.McLean. Let L be a special Lagrangian manifold in a Calabi-Yau, then by the Weinstein neighborhood theorem, a neighborhood of L could be identified with a neighborhood of the zero section in $T^{\star}L$. Therefore, a C^1 deformations of L is given by the graph of a 1-form v on L.



McLean's Deformation Theorem

Theorem

(R.McLean) The C^1 deformation of a special Lagrangian submanifold L is parametrized by the harmonic 1-forms. Especially, suppose $b_1(L) = 0$, then L is rigid.

Sketch of Proof: Over a neighborhood U of the zero section of $T^\star L$, let $\iota_t : L \to U$ be the graph of tv with t a real parameter, then the linearization of the immersed special Lagrangian condition will be

$$\frac{d}{dt}\iota_t^\star \omega = dv, \ \frac{d}{dt}\iota_t^\star \mathrm{Im}\Omega = d \star v. \tag{2}$$

Then by an implict functional argument, harmonic 1-forms parameterized the nearby C^1 special Lagrangians.

Branched Deformation Question

Question

Let L be a special Lagrangian submanifold in a Calabi-Yau (X, J, ω, Ω), does there exist a family of special Lagrangians $\tilde{L}_t$, which are diffeomorphic to a double branched covering of L, such that $\tilde{L}_t$ convergence to 2L?

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(i) (Joyce)Special Lagrangian enumerative invariants.
(ii) (CH.Liu-ST.Yau 11')Attemps using gluing arguement and obtain some particial results.
(iii) (MT.Wang-CJ.Tasi 18')This problem is negative when L has positive Ricci curvature.

An Example of double branched deformation

Let $\mathbb{C}$ be the complex plane with coordinate z. We identified $T^{\star}\mathbb{C}$ with $\mathbb{C}^2$ and let w be the fiber coordinates.
Let $v_k = \Re(z^{-\frac{1}{2}+k}dz)$, then the defining equation of the graph of tv_k would be

$$\Gamma_t^k := \{(z,w)|w^2 = t^2 z^{2k-1}\}, \tag{3}$$

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which is a family of special Lagrangians.
When $k \geq 1$, we see Γ_t^k convergence to $2\{w=0\}$, and $k=1$, Γ_t^1 is a smooth manifold, which is the graph of the non-degenerate multivalued harmonic 1-form.
When $k=0$, Γ_t^0 is singular.

An Example of double branched deformations

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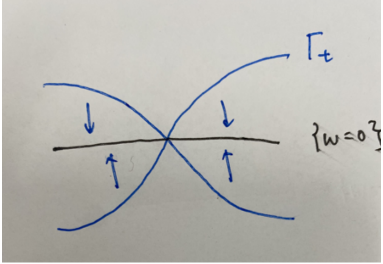

More comments

The first thing to try is to modified McLean's arguement. Suppose $\iota : L \to X$ is a special Lagrangian, we choose $p : \tilde{L} \to L$ be a double branched covering of L, then $\iota \circ p : \tilde{L} \to X$ is a special Lagrangian as

$$(\iota \circ p)^{\star}\omega = (\iota \circ p)^{\star}\Omega = 0,$$

then we apply McLean's theorem to the harmonic 1-forms on $\tilde{L}$, we solve the question.

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Unfortunately, the above arguement is incorrect. As $\iota \circ p$ is no longer an immersion, the induced metric on $\tilde{L}$ is singular.(cone metric with cone angle 4π)

More comments

Moreover, you don't expect you could use some weighted norm to overcome the singular metric issue. For any weight, you might find the linearization operator $d + d^\star$ on $\mathbb{Z}_2$ 1-forms has infinite dimensional cokernel.

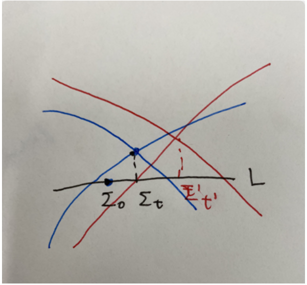
This problem is a free boundary problem, where the branched set itself is a variable that you need to solve. For most of the homology element in the double branched covering, you don't expect them to generate a deformation.

Donaldson's insight

The main idea for this branched deformation is coming from Donaldson during his studied of the deformation problem for $\mathbb{Z}_2$ harmonic 1-forms. As the linearization of a special Lagrangian equation is $d + d^{\star}$, the branched deformation problem for sLags might be considered as a non-linear version of Donaldson's theorem.

As the deformation theory of multivalued harmonic equation is a free boundary problem, you expect that the deformation problem of special Lagrangians is also a free boundary problem, where you could only get a Fredholm theory by perturbing the branched set. Similar idea appears in (Takahashi 15') and (Parker 22').

The previous example in $\mathbb{C}^2$ is in some sense very misleading, as $\mathbb{C}^2$ is hyperKähler. The general picture would be the following:



For different real parameter t, there should be an unique Σ_t and a special Lagrangian L_t which is branching along Σ_t.

Main Difficulties

There are two main difficulties in solving this problem:
(A) The family $\tilde{L}_t$ has unbounded geometry when $t \to 0$. The Riemannian curvatures goes to infinity and the injective radius goes to zero, you need to understand the degenerating behavior.

Main Difficulties

There are two main difficulties in solving this problem:
(A) The family $\tilde{L}_t$ has unbounded geometry when $t \to 0$. The Riemannian curvatures goes to infinity and the injective radius goes to zero, you need to understand the degenerating behavior.
(B) The branched deformation problem needs to keep on moving the branched set to obtain a Fredholm theory.

Branched Deformation Theorem

Theorem

(H. 22') Let L be a special Lagrangian in a Calabi-Yau (X, J, ω, Ω), suppose there exists a non-degenerate $\mathbb{Z}_2$ harmonic 1-form v on L, then there exists a family of special Lagrangian submanifold $\tilde{L}_t$ such that

(i) *$\tilde{L}_t$ convergence to $2L$ as currents and as a $\mathcal{C}^{,\alpha}$ graph, where $0 < \alpha < \frac{1}{2}$.*

(ii) *$\tilde{L}_t$ is close to the graph of tv mod $\mathcal{O}(t^2)$.*

Corollary

Moreover, we find a special Lagrangian with topology rational homology 3-sphere which admits multivalued non-degenerate harmonic 1-form. Therefore, we obtain

Corollary

There exists a special Lagrangian which is C^1 rigid in McLean's sense but have double branched deformations($\mathcal{C}^{,\alpha}$ close special Lagrangians).

Sketch a Proof

We will explain how could we solve the problem.
Step 1. Using tv, we could construct a family of Lagrangians $\tilde{L}'_t$, even the geometry of $\tilde{L}'_t$ is unbounded, we found that the induced Riemannian metric on $\tilde{L}'_t$ will convergence to a cone metric which gives a uniform lower bound on the first eigenvalue of the Laplacian operator. (The unbounded geometry problem.)

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Step 2, Noting that the problem is a free boundary problem, we move the branching sets of $\tilde{L}'_t$ to make a good approximate solutions $\tilde{L}''_t$ with sufficiently small Lagrangian angle. (The free boundary problem.)
Step 3, We perturb $\tilde{L}''_t$ into a real special Lagrangians using Joyce's nearby special Lagrangian method.

The nearby special Lagrangian theorem

Theorem

(Abouzaid-Imagi 2021) Suppose $\pi_1(L)$ is finite, then all $\mathcal{C}^0$ sufficiently close unobstructed(in FOOO sense) immersed sLags must be L.

Combing the branched deformation result with the Abouzaid-Imagi's uniqueness theorem, we obtain extra obstruction for the existence of non-degenerate $\mathbb{Z}_2$ harmonic 1-forms.

Suppose (L, g) be a real analytic manifold with $\chi(L) = 0$ with $\pi_1(L)$ finite, then by Bryant, Doice's Calabi-Yau neighborhood theorem, there exists a Calabi-Yau structure in a neighborhood of the zero section of $T^\star L$ such that the zero section is a special Lagrangian. Suppose over L, there exists a non-degenerate $\mathbb{Z}_2$ harmonic 1-forms, then the $\tilde{L}$ we constructed must be obstructed. However, if $b_2(\tilde{L}) = b_2(L)$, then $\tilde{L}$ has to be unobstructed.

Theorem

(H. 22') If there exists a $\mathbb{Z}_2$ harmonic 1-form on L, then $b_2(\tilde{L}) > b_2(L)$.

Applications to $PSL(2, \mathbb{C})$ Gauge Theory

If you could check that every immersed sLag on $T^\star L$ is unobstructed, then you could conclude the non-existence of $\mathbb{Z}_2$ harmonic 1-form, which is mission impossible. However, the Calabi-Yau neighborhood has an anti-holomorphic involution by sending $v \to -v$ on $T^\star L$ and the sLags we constructed will be preserved under this symmetry. You only need to check the unobstructed condition for the special Lagrangians which is preserved under this extra symmetry. (Soloman, FOOO).

Theorem

Let (L,g) be a real analytic 3-manifold with $\pi_1(L)$ finite, suppose every immersed anti-holomorphic involution invariant special Lagrangian in the Calabi-Yau neighborhood is unobstructed(in FOOO sense), then there doesn't exist any non-degenerate multivalued harmonic 1-form.

Thank You!

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