Path Integral Derivations of Vafa-Witten and K-Theoretic Donaldson Invariants

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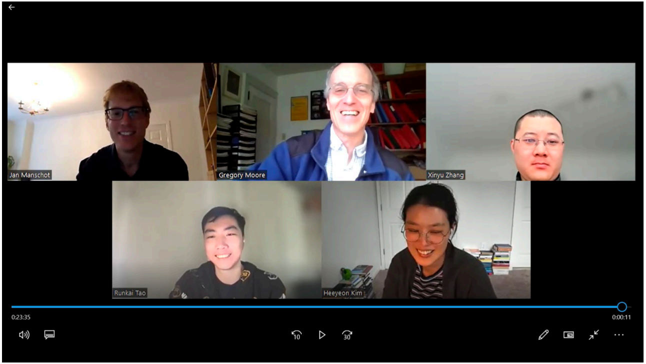
This talk is based on arXiv:2104.06492, joint work with Greg Moore, and *to appear* with Heeyeon Kim, Runkai Tao, and Xinyu Zhang.



Other related papers are Korpas, Manschot (2017), Korpas, JM, Moore, Nidaiev (2019), and JM, Moore, Zhang (2019)

Topologically twisted Yang-Mills theories

Path integrals and correlation functions can be evaluated exactly for topologically twisted $\mathcal{N} = 2$ and $\mathcal{N} = 4$ Yang-Mills theories in many cases. The observables of the theory are a function on its conformal manifold. Such results provide connections to the geometry of four-manifolds and instanton moduli spaces, as well as to analytic number theory.

Using the physical perspective we can evaluate topological invariants of four-manifolds, such as Donaldson-Witten invariants, Vafa-Witten invariants and K-theoretic Donaldson invariants.

$\mathcal{N} = 2^*$ theory

Matter content:

- $\mathcal{N} = 2$ vector multiplet, $SU(2)$ connection A_μ, adjoint complex scalar scalar ϕ
- $\mathcal{N} = 2$ hypermultiplet with scalars $(Q, \tilde{Q}^\dagger)$ in adjoint representation with mass m

Coulomb branch coordinate: $u = \left\langle \mathrm{Tr}\phi^2 \right\rangle_{\mathbb{R}^4}$

Parameters:

- UV coupling constant τ_{uv}, $q_{\mathrm{uv}} = e^{2\pi i \tau_{\mathrm{uv}}}$
- mass m
- scale Λ
- effective coupling $\tau = \tau(u, m, \tau_{\mathrm{uv}})$

Global symmetries:

- $SU(2)_R$
- $U(1)_B$ acting as $Q \to e^{i\varphi} Q$ and $\tilde{Q} \to e^{-i\varphi} \tilde{Q}$

$\mathcal{N} = 2^*$ interpolates between two well-known theories:

- $m \to 0$: $\mathcal{N} = 2^* \to \mathcal{N} = 4$ YM
- $m \to \infty$, $q_{\rm uv}^{1/4} m = \Lambda$ fixed: $\mathcal{N} = 2^* \to \mathcal{N} = 2, N_f = 0$ YM

Seiberg, Witten (1994)

Modular forms

Jacobi theta series:

$\vartheta_2(\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{n^2/2} \qquad q = e^{2\pi i\tau}$

$\vartheta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}$

$\vartheta_4(\tau) = \sum_{n\in\mathbb{Z}}(-1)^n\, q^{n^2/2}$

Half-periods:

$e_1(\tau) = \frac{1}{3}(\vartheta_3(\tau)^4 + \vartheta_4(\tau)^4)$

$e_2(\tau) = -\frac{1}{3}(\vartheta_2(\tau)^4 + \vartheta_3(\tau)^4)$

$e_3(\tau) = \frac{1}{3}(\vartheta_2(\tau)^4 - \vartheta_4(\tau)^4)$

Transformations:

$$\vartheta_2(\tau+1) = e^{2\pi i/8}\,\vartheta_2(\tau)$$
$$\vartheta_3(\tau+1) = \vartheta_4(\tau)$$

$$\vartheta_2(-1/\tau) = \sqrt{-i\tau}\,\vartheta_4(\tau)$$
$$\vartheta_3(-1/\tau) = \sqrt{-i\tau}\,\vartheta_3(\tau)$$

Transform under the congruence subgroup

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a, d = 1 \mod 2, b, c = 0 \mod 2 \right\}$$

Seiberg-Witten solution

SW curve Σ_τ:

$$y^2 = \prod_{j=1}^{3} \left(x - e_j(\tau_{uv})u - \frac{1}{4} e_j(\tau_{uv})^2 m^2 \right)$$

with $e_j(\tau_{uv})$ half-periods of the UV curve Seiberg, Witten (1994)

Discriminant:

$$\Delta = (u - u_1)(u - u_2)(u - u_3)$$

Singularities:

- $u \to \infty$, $\tau \to \tau_{uv}$: limit to $\mathcal{N} = 4$
- $u \to u_1 = \frac{m^2}{4} e_1(\tau_{uv})$, $\tau \to i\infty$: quark becomes massless
- $u \to u_2$, $\tau \to 0$: monopole becomes massless
- $u \to u_3$, $\tau \to 1$: dyon becomes massless

In terms of τ, one can derive

$$u = \frac{m^2}{4}\frac{\vartheta_2(\tau)^4\vartheta_3(\tau_{\rm uv})^4 e_2(\tau_{\rm uv}) - \vartheta_3(\tau)^4\vartheta_2(\tau_{\rm uv})^4 e_1(\tau_{\rm uv})}{\vartheta_2(\tau)^4\vartheta_3(\tau_{\rm uv})^4 - \vartheta_3(\tau)^4\vartheta_2(\tau_{\rm uv})^4}$$

Labastida, Lozano (1998); Huang, Kashani-Poor, Klemm (2011)

Thus u is a bi-modular form, with weight 2 in $\tau_{\rm uv}$ and 0 in τ. u transforms under $\Gamma(2)$, if it acts separately on τ and $\tau_{\rm uv}$; and under $SL(2,\mathbb{Z})$, if it acts simultaneously.

Similarly

$$\Delta = (2m)^6\frac{\eta(\tau_{\rm uv})^{24}\eta(\tau)^{12}}{(\vartheta_4(\tau)^4\vartheta_3(\tau_{\rm uv})^4 - \vartheta_3(\tau)^4\vartheta_4(\tau_{\rm uv})^4)^3}$$

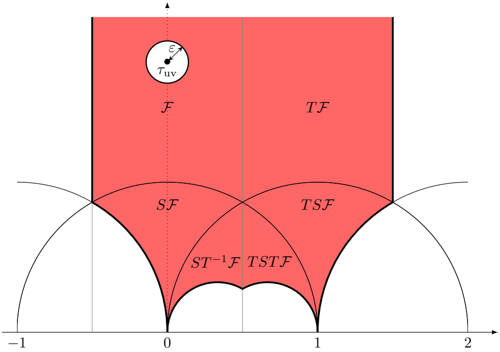
Thus Δ is a bi-modular form, with weight 6 in $\tau_{\rm uv}$ and 0 in τ.

Similar expressions can be obtained for $N_f = 4$

Aspman, Furrer, Manschot (2021)

The Coulomb branch can be mapped to a domain in $\mathbb{H}$ using the change of variables $u(\tau)$. This domain is

$$\mathcal{U}_\varepsilon = (\mathbb{H}/\Gamma(2)) \backslash B(\tau_{\mathsf{uv}}, \varepsilon)$$



Four-manifolds and lattices

$H^2(X,\mathbb{Z})$ together with the intersection form

$$B(\boldsymbol{k}_1,\boldsymbol{k}_2)=\int_X \boldsymbol{k}_1\wedge\boldsymbol{k}_2,\qquad \boldsymbol{k}_{1,2}\in H^2(X,\mathbb{Z})$$

gives rise to an integral, uni-modular lattice L (the image of $H^2(X,\mathbb{Z})$ in $H^2(X,\mathbb{R})$)

The lattice has signature (b_2^+,b_2^-)

For $b_2^+=1$, let J be the normalized generator of the unique self-dual direction in $H^2(X,\mathbb{R})$. It provides the projection of $\boldsymbol{k}\in L$ to $(L\otimes\mathbb{R})^+$,

$$\boldsymbol{k}_+=B(\boldsymbol{k},J)\,J$$

Almost complex four-manifolds

For simplicity, we assume X to be simply connected, $\pi_1(X) = 0$. Real dimension of the $SU(2)$ instanton moduli space is only even for b_2^+ odd.

$\Rightarrow$ Correlation functions of the $SU(2)$, $\mathcal{N} = 2^*$ theory are only non-vanishing for b_2^+ odd. Such four-manifolds admit an almost complex structure $\mathcal{J} : TX \to TX$.

Provides the fundamental (1,1)-form: $\omega(\cdot,\cdot) = g(\mathcal{J}\cdot,\cdot)$ which satisfies

$$d\omega = \theta \wedge \omega$$

with θ the Lee form.

Almost complex four-manifolds

For $\alpha \in \Omega^{0,1}(X)$

$$d\alpha = \partial\alpha + \bar{\partial}\alpha + (N_{\mathcal{J}})^{a}_{bc}\alpha_a\, e^b \wedge e^c$$

with $N_{\mathcal{J}}$ the Nijenhuis tensor and e^a a real orthonormal frame for TX.

If X is Kähler, $\theta = 0$ and $N_{\mathcal{J}} = 0$.

Four-manifolds and Spin^c structures

Let X be an oriented, smooth, compact four-manifold.
Recall

$$\mathrm{Spin}(4) = SU(2) \times SU(2)$$

is a double cover of $SO(4)$, and

$$\mathrm{Spin}^c(4) = \{(u_1, u_2) | \det(u_1) = \det(u_2)\} \subset U(2) \times U(2)$$

A Spin structure on X is a principal Spin(4) bundle, compatible with the principal $SO(4)$ bundle associated to the oriented tangent bundle TX. A Spin^c structure on X is similarly a principal Spin^c(4) bundle.

A Spin structure only exists if $w_2(X) = 0 \in H^2(X, \mathbb{Z}_2)$, however, any oriented four-manifold admits a Spin^c structure.

Four-manifolds and Spin^c structures

Let $W^{\pm}$ be the two chiral spin bundles, corresponding to the two $U(2)$'s. Then the Spin^c line bundle $\mathcal{L}$ is the determinant bundle

$$\mathcal{L} = \det(W^{\pm})$$

and

$$c_1(\mathcal{L}) \in H^2(X)$$

is the characteristic class $c_1(\mathfrak{s})$ of the Spin^c structure. It satisfies $c_1(\mathfrak{s}) = w_2(X) \mod H^2(X, 2\mathbb{Z})$. We introduce $\boldsymbol{k}_m = c_1(\mathfrak{s})/2 \in L$.

There is a canonical Spin^c structure, whose associated line bundle is isomorphic to the canonical bundle with

$$K_X^2 = 2\chi + 3\sigma$$

.

UV $\mathcal{N} = 2^*$ theory on X

The Q-fixed equations are the adjoint Seiberg-Witten equations:

$$F^+_{\mu\nu} + \frac{1}{2}\bar{\sigma}^{\dot{\alpha}\dot{\beta}}_{\mu\nu}[\bar{M}_{(\dot{\alpha}}, M_{\dot{\beta})}] = 0$$

$$\not{D} M = 0$$

Witten (1994); Labastida, Marino (1995); Labastida, Lozano (1998),...

Equations are invariant under $U(1)_B$ symmetry: $M_{\dot{\alpha}} \to e^{i\varphi} M_{\dot{\alpha}}$

$M_{\dot{\alpha}}$ is a spinor $\Rightarrow$ X is spin, or coupling to a Spin^c structure $\mathfrak{s}$ required.

UV $\mathcal{N} = 2^*$ theory on X

For the canonical Spin^c structure:

$$M = \begin{pmatrix} \bar{\beta} \\ \alpha \end{pmatrix}$$

with $\alpha \in \Omega^{0,0}(X, \mathbb{C})$ and $\bar{\beta} \in \Omega^{0,2}(X, \mathbb{C})$

Then the Spin^c Dirac equation reads

$$\not{D} M = \sqrt{2}(\bar{\partial}\alpha + \bar{\partial}^{\dagger}\bar{\beta}) + \frac{1}{4}\theta.M = 0$$

Gauduchon (1997)

Dimension and fixed point locus

The dimension of the moduli space is

$$\mathrm{vdim}(\mathcal{M}^{Q}_{k,\boldsymbol{\mu},\mathfrak{s}}) = \dim(G)\,\frac{c_1(\mathfrak{s})^2 - (2\chi + 3\sigma)}{4} =: 2\dim(G)\,\ell$$

with k the instanton number and $2\boldsymbol{\mu} = \bar{w}_2(P) \in L$ the 't Hooft flux. Thus $\mathfrak{s}$ determined by an ACS are special, since then vdim = 0.

The $U(1)_B$ fixed point locus consists of two components:

- Instanton component: $M_{\dot{\alpha}} = 0$ and $F^+ = 0$
- Abelian component: F diagonal, and $M_{\dot{\alpha}}$ strictly upper or lower triangular

UV correlators

We consider the point observable u and the surface observable

$$u = \frac{1}{16\pi^2}\mathrm{Tr}[\phi^2]$$

$$I(\boldsymbol{x}) = \frac{1}{4\pi^2}\int_{\boldsymbol{x}} \mathrm{Tr}\left[\frac{1}{8}\psi\wedge\psi - \frac{1}{\sqrt{2}}\phi F\right].$$

These observables correspond to differential forms on the moduli space, a 4-form $\omega_{\boldsymbol{u}}$ and a 2-form $\omega_{\boldsymbol{I}}$.

Then a correlator of $\mathcal{N} = 2^*$ becomes an integral of differential forms over the fixed point locus:

$$\begin{aligned}\langle \mathcal{O}_1 \dots \mathcal{O}_p \rangle &= \sum_k q_{\text{uv}}^k \, m^{-\text{Index}(\,\mathbf{D}_A)} \int_{\mathcal{M}^Q_{k,\boldsymbol{\mu},\mathfrak{s}}} \sum_{\ell \geq 0} \frac{c_\ell}{m^\ell} \, \omega_1 \dots \omega_p \\ &= m^{-3\ell + D_\omega} \sum_k q_{\text{uv}}^k \\ &\quad \times \left[\int_{\mathcal{M}^{\text{i}}_{k,\boldsymbol{\mu}}} c_\ell \, \omega_1 \dots \omega_p + \int_{\mathcal{M}^{\text{a}}_{k,\boldsymbol{\mu},\mathfrak{s}}} c_\ell \, \omega_1 \dots \omega_p \right],\end{aligned}$$

where c_ℓ are Chern classes of the matter bundle over the moduli space, i.e. the tangent bundle to the moduli space for $\mathfrak{s}$ associated to an ACS $\Rightarrow$ in the massless limit, the path integral is a generating function of Euler numbers $\chi(\mathcal{M}_{k,\boldsymbol{\mu}})$. In the $m \to \infty$ limit, only $\ell = 0$ contributes

Vafa-Witten twist of $\mathcal{N} = 4$ YM

This topologically twisted theory contains a real scalar field $C \in \Omega^0(X, \mathrm{ad}P)$, a real self-dual 2-form $B^+ \in \Omega^{2+}(X, \mathrm{ad}P)$, field strength F.

The Q-fixed equations are:

$$F^+_{\mu\nu} + \frac{1}{2}[C, B^+_{\mu\nu}] + \frac{1}{4}[B^+_{\mu\rho}, B^+_{\nu\sigma}]g^{\rho\sigma} = 0$$

$$D_\mu C + D^\nu B^+_{\mu\nu} = 0$$

Vafa, Witten (1994)

- No spinors $\Rightarrow$ X can be non-spin
- Difficulty: domain of B^+ is non-compact

Vafa-Witten twist of $\mathcal{N} = 4$ YM

On an almost complex X, we can expand $B = \kappa\,\omega + \beta + \bar{\beta}$
Then with $\alpha = C - i\kappa$, the first of the SW and VW equations are identical

The $(0,1)$ component of the second VW equation gives:

$$\bar{\partial}\alpha + \bar{\partial}^{\dagger}\bar{\beta} - i\kappa\,\pi_{0,1}\circ\theta + \pi_{0,1}\circ(d^{\dagger}\beta) = 0$$

Equivalent to non-abelian Spin^c Dirac equation if X is Kähler!

For such X there is a $U(1)$ symmetry $\Rightarrow$ mathematical definition of Vafa-Witten invariants by Tanaka-Thomas (2017)

While the 2nd of the SW and VW equations are not identical, we have reasons to believe that the invariants of VW theory are identical to those of $\mathcal{N} = 2^*$ coupled to the canonical Spin^c structure.

These reasons include:

1. $\mathcal{N} = 2^*$ equation can be expressed as a deformation of the VW equation:
$$\nabla_{A,\mathcal{J}} C + \nabla_{A,\mathcal{J}} B^+ = 0$$
2. analysis of the low energy effective field theory.

Evaluation using effective field theory

Effective field theory has proven powerful to analyze and evaluate correlation functions. This led for example to the (abelian) Seiberg-Witten equations and invariants. Seiberg-Witten contributions are localized at the singularities u_j, which provide the full correlator for $b_2^+(X) > 1$.

Witten (1994); Moore, Witten (1997),. . .

For manifolds with $b_2^+ \leq 1$, the low energy effective field theory on the Coulomb branch contributes and the full SW solution of the theory is indispensable.

Witten (1995); Moore, Witten (1997); Losev, Nekrasov, Shatashvili (1997),. . .

Schematically

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{u-\text{plane}} + \langle \mathcal{O} \rangle_{\text{SW}}$$

Topological twisting

Assume X is spin, such that the chiral $SU(2)$ spin bundles are well-defined.

Donaldson-Witten twist: Replace $SU(2)_+$ representation by that of the diagonally embedded subgroup in $SU(2)_+ \times SU(2)_R$
$\Rightarrow \phi$ and A_μ remain a vector and scalar, but $(Q, \tilde{Q}^\dagger)$ becomes a space-time spinors $M_{\dot{\alpha}}$, $\bar{M}_{\dot{\alpha}}$

We will restrict to $b_2^+ = 1$: the path integral reduces to an integral over zero modes of the vector multiplet: A_μ, $\phi_0 = a$, η_0, ψ_0, χ_0.

Topological twisting

Spinors $M_{\dot\alpha}$ are problematic for the generalization to non-spin X. We cure this by coupling the hypermultiplet to the Spin^c line bundle $\mathcal{L}$, such that

$$W^+ = S^+ \otimes \mathcal{L}^{1/2}$$

is a well-defined Spin^c bundle

See for Spin^c structures for fundamental matter: Hyun, Park, Park (1995), Labastida, Marino (1997)

For $\mathfrak{s}$ canonically determined by an ACS

$$W^+ \simeq \Lambda^0 \oplus \Lambda^{0,2}, \qquad W^- \simeq \Lambda^{0,1}$$

Lagrangian

Metric dependence of the (general) effective $U(1)^N$ Lagrangian $\mathcal{L}_{DW}$ is $\mathcal{Q}$ exact:

$$\begin{aligned}\mathcal{L} &= \frac{i}{8\pi}\tau_{IJ}F^I \wedge F^J + \{\mathcal{Q}, W\} \\ &= \frac{i}{8\pi}\left(\bar{\tau}_{IJ}F_+^I \wedge F_+^J + \tau_{IJ}F_-^I \wedge F_-^J\right) - \frac{1}{4\pi}y_{IJ}D^I \wedge D^J \\ &\quad + \frac{i\sqrt{2}}{8\pi}\bar{\mathcal{F}}_{IJK}\eta^I\chi^J \wedge (D + F_+)^K.\end{aligned}$$

Here $I, J \in 1, 2 = N$ and $\tau_{IJ} = \partial^2\mathcal{F}/\partial a^I \partial a^J$. We "freeze" the "2" fields, in particular

$$\begin{aligned}&a^{(1)} = a, \qquad F^{(1)} = F, \qquad \tau_{11} = \tau, \\ &a^{(2)} = m, \qquad F^{(2)} = 4\pi \boldsymbol{k}_m, \qquad D^{(2)} = F_+^{(2)}, \qquad \tau_{22} = \xi, \\ &a_D = \frac{\partial\mathcal{F}}{\partial a} \qquad m_D = \frac{\partial\mathcal{F}}{\partial m}\end{aligned}$$

u-plane integrand

The term $\tau_{22} = \xi$ leads to a factor $C^{\boldsymbol{k}_m^2}$

The terms involving $F^{(1)}$ give rise to a sum over fluxes

$$\Psi^J_{\boldsymbol{\mu}}(\tau, \bar{\tau}, \boldsymbol{z}, \bar{\boldsymbol{z}}) = e^{-4\pi y\, \boldsymbol{b}_+^2} \sum_{\boldsymbol{k} \in L + \boldsymbol{\mu}} \partial_{\bar{\tau}} \left(\sqrt{4y}\, B(\boldsymbol{k} + \boldsymbol{b}, J) \right) q^{-\boldsymbol{k}_-^2}\, \bar{q}^{\boldsymbol{k}_+^2} \times\ e^{-4\pi i B(\boldsymbol{k}_-, \boldsymbol{z}) - 4\pi i B(\boldsymbol{k}_+, \bar{\boldsymbol{z}})},$$

with

$$\boldsymbol{b} = \mathrm{Im}(\boldsymbol{z}), \qquad \boldsymbol{\mu} \in L/2, \qquad \boldsymbol{k} = \frac{F^{(1)}}{4\pi}, \qquad \boldsymbol{z} = v\,\boldsymbol{k}_m$$

u-plane integrand

There are in addition topological couplings

$$A^{\chi} B^{\sigma}$$

with

$$A = \alpha \left(\frac{du}{da} \right)^{1/2} \qquad B = \beta \Delta^{1/8}$$

and α, β independent of τ

The integrand

$$da \wedge d\bar{a}\, A^{\chi} B^{\sigma} C^{\boldsymbol{k}_m^2} \frac{d\bar{\tau}}{d\bar{a}} \Psi^J_{\boldsymbol{\mu}}(\tau, \bar{\tau}, v\boldsymbol{k}_m, \bar{v}\boldsymbol{k}_m)$$

is single valued on the u-plane

Labastida, Lozano (1997) considered this integral for $\boldsymbol{k}_m = 0$ (X is spin)

It is natural to change variables to τ and integrate over $\mathcal{U}_\varepsilon$

$$\begin{aligned}&\Phi^J_{\boldsymbol{\mu}}[\mathcal{O}](\tau_{\mathrm{uv}},\bar\tau_{\mathrm{uv}};\boldsymbol{k}_m)\\&=\int_{\mathcal{U}_\varepsilon} d\tau\wedge d\bar\tau\,\nu(\tau,\tau_{\mathrm{uv}})\,\mathcal{O}\,\Psi^J_{\boldsymbol{\mu}}(\tau,\bar\tau,v\boldsymbol{k}_m,\bar v\boldsymbol{k}_m)\end{aligned}$$

We aim to evaluate using Stokes' theorem,

$$\Phi^J_{\boldsymbol{\mu}}(\tau_{\mathrm{uv}},\bar\tau_{\mathrm{uv}};\boldsymbol{k}_m)=\int_{\mathcal{U}_\varepsilon}\Omega=\int_{\partial\mathcal{U}_\varepsilon}\omega$$

with $d\omega=\Omega$

This is possible using mock modular forms.

Korpas, JM, Moore, Nidaiev (2019), JM, Moore (2021)

Some properties can be deduced without explicit evaluation.
$\Phi^J_{\boldsymbol{\mu}}$ transforms as a modular form in τ_{uv} of weight $-\chi/2-4\ell$

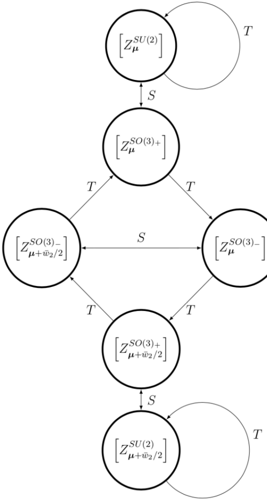
Duality and partition functions for $SU(2)$ and $SO(3)$

$\Phi^J_{\boldsymbol{\mu}}$ transforms as a modular form in τ_{uv} of weight $-\chi/2 - 4\ell$

We combine the $\Phi^J_{\boldsymbol{\mu}}$ to $SU(2)$ and $SO(3)$ partition functions,

$$\begin{aligned} Z^{SU(2)}_{\boldsymbol{\mu}} &= \Phi^J_{\boldsymbol{\mu}} \\ Z^{SO(3)_+}_{\boldsymbol{\mu}} &= \sum_{\boldsymbol{\nu}\in(L/2)/L} e^{4\pi i B(\boldsymbol{\mu},\boldsymbol{\nu})}\Phi^J_{\boldsymbol{\nu}} \\ Z^{SO(3)_-}_{\boldsymbol{\mu}} &= \sum_{\boldsymbol{\nu}\in(L/2)/L} e^{4\pi i B(\boldsymbol{\mu},\boldsymbol{\nu})-2\pi i\boldsymbol{\nu}^2}\Phi^J_{\boldsymbol{\nu}} \end{aligned}$$

Duality diagram



This is identical to the diagram for VW theory for arbitrary X

Related results: Vafa, Witten (1994), Ang, Roumpedakis, Seifnashri (2019), Gukov, Hsin, Pei (2020)

Holomorphic anomaly

$\Phi^J_{\boldsymbol{\mu}}$ is a function of τ_{uv} and $\bar{\tau}_{\mathrm{uv}}$. The $\bar{\tau}_{\mathrm{uv}}$ dependence is Q-exact

$$\frac{\partial}{\partial \bar{\tau}_{\mathrm{uv}}} \Phi^J_{\boldsymbol{\mu}} = \langle [Q, G] \rangle \,,$$

Q-exact observables usually give rise to a total derivative in field space $\Rightarrow$ straightforward to evaluate

We derive from $\Phi^J_{\boldsymbol{\mu}}$ a non-vanishing contribution from reducible connections whose action exceeds the instanton bound.

This reproduces the holomorphic anomaly of VW theory. See for other recent work Dabholkar, Putrov, Witten (2020), Bonelli *et al* (2020)

Evaluation

The main task is to find a function $\widehat{G}^J_{\boldsymbol{\mu}}(\tau,\bar\tau,v,\bar v;\boldsymbol{k}_m)$ such that

$$\frac{\partial}{\partial\bar\tau}\widehat{G}^J_{\boldsymbol{\mu}}(\tau,\bar\tau,v,\bar v;\boldsymbol{k}_m)=\Psi^J_{\boldsymbol{\mu}}(\tau,\bar\tau,v\boldsymbol{k}_m,\bar v\boldsymbol{k}_m)$$

which are regular on $\mathcal{U}_\varepsilon$

$\widehat{G}^J_{\boldsymbol{\mu}}$ is a Jacobi-Maass form with meromorphic part $G^J_{\boldsymbol{\mu}}$
Let again $X=\mathbb{P}^2$ and $\boldsymbol{\mu}=1/2$,

$$G^{\mathbb{P}^2}_{1/2}(\tau,v;1/2)=-\frac{e^{\pi iv}}{\vartheta_4(2\tau)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n\,q^{n^2-\frac14}}{1+e^{2\pi iv}\,q^{2n-1}}$$

$$G^{\mathbb{P}^2}_{1/2}(\tau,v;3/2)=\frac{q^{-\frac14}\,e^{-3\pi iv}}{\vartheta_3(2\tau,v)}\sum_{n\in\mathbb{Z}}\frac{q^{n^2}\,e^{2\pi inv}}{1-e^{-4\pi iv}q^{2n-1}}.$$

Explicit results: $\boldsymbol{k}_m = 3/2$

n	Hol. part of $\underline{\Phi}^{\mathbb{P}^2}_{\frac{1}{2}}[u_{\mathrm{D}}^n/(2\Lambda^2)^n]$
0	$i\,t^3\left(q_{\mathrm{uv}}^{3/4} + 3\,q_{\mathrm{uv}}^{7/4} + 3\,q_{\mathrm{uv}}^{11/4} + 6\,q_{\mathrm{uv}}^{15/4} + \dots\right)$
1	$-i\,t^5\left(\frac{3}{4}\,q_{\mathrm{uv}}^{7/4} + 6\,q_{\mathrm{uv}}^{11/4} + \frac{35}{2}\,q_{\mathrm{uv}}^{15/4} + \dots\right)$
2	$i\,t^7\left(\frac{19}{64}\,q_{\mathrm{uv}}^{7/4} + \frac{31}{8}\,q_{\mathrm{uv}}^{11/4} + \frac{89}{4}\,q_{\mathrm{uv}}^{15/4} + \dots\right)$
3	$-i\,t^9\left(\frac{15}{32}\,q_{\mathrm{uv}}^{11/4} + \frac{971}{128}\,q_{\mathrm{uv}}^{15/4} + \dots\right)$
4	$i\,t^{11}\left(\frac{85}{512}\,q_{\mathrm{uv}}^{\frac{11}{4}} + \frac{15151}{4096}\,q_{\mathrm{uv}}^{\frac{15}{4}} + \dots\right)$

$t = m/\Lambda$

Explicit results: $\boldsymbol{k}_m = 1/2$

n	$\Phi^{\mathbb{P}^2}_{\frac{1}{2}}[u_{\mathrm{D}}^n/(2\Lambda)^n]$
0	$i\,t^3\left(q_{\mathrm{uv}}^{3/4} + 9\,q_{\mathrm{uv}}^{7/4} + 19\,q_{\mathrm{uv}}^{11/4} + 50\,q_{\mathrm{uv}}^{15/4} + \ldots\right)$
1	$-i\,t^5\left(\frac{5}{8}\,q_{\mathrm{uv}}^{7/4} + 3\,q_{\mathrm{uv}}^{11/4} + \frac{43}{2}\,q_{\mathrm{uv}}^{15/4} + \ldots\right)$
2	$i\,t^7\left(\frac{19}{64}\,q_{\mathrm{uv}}^{7/4} + \frac{19}{4}\,q_{\mathrm{uv}}^{11/4} + \frac{581}{16}\,q_{\mathrm{uv}}^{15/4} + \ldots\right)$
3	$-i\,t^9\left(\frac{23}{64}\,q_{\mathrm{uv}}^{11/4} + \frac{2599}{512}\,q_{\mathrm{uv}}^{15/4} + \ldots\right)$
4	$i\,t^{11}\left(\frac{85}{512}\,q_{\mathrm{uv}}^{11/4} + \frac{16025}{4096}\,q_{\mathrm{uv}}^{15/4} + \ldots\right)$

SW contributions

General form of partition function:

$$Z_{\boldsymbol{\mu}}^J = \Phi_{\boldsymbol{\mu}}^J + \sum_{j=1}^{3} Z_{SW,j,\boldsymbol{\mu}}^J$$

The terms on the rhs undergo wall-crossing upon varying J. Wall-crossing from the singularity u_j of $\Phi_{\boldsymbol{\mu}}^J$ is absorbed by the wall-crossing of $Z_{SW,j,\boldsymbol{\mu}}^J$:

$$\left[\Phi_{\boldsymbol{\mu}}^{J^+} - \Phi_{\boldsymbol{\mu}}^{J^-}\right]_j = Z_{SW,j,\boldsymbol{\mu}}^{J^-} - Z_{SW,j,\boldsymbol{\mu}}^{J^+}$$

This makes it possible to derive $Z_{SW,j,\boldsymbol{\mu}}^J$ in terms of SW invariants $\mathrm{SW}(c_{\mathrm{ir}}; J)$ with c_{ir} the IR Spin^c structure. Moreover, it is possible to extend the results to manifolds with $b_2^+ > 1$.

SW contributions

With $c_{\mathrm{ir}} = 2\boldsymbol{x} + c_{\mathrm{uv}}$, the contribution from u_1 is

$$Z_{SW,1,\boldsymbol{\mu}}(\tau_{\mathrm{uv}}) = \left(-2\,\eta(2\tau_{\mathrm{uv}})^{12}\right)^{-\chi_{\mathrm{h}}} \left(4\,t^3\,\eta(\tau_{\mathrm{uv}})^4\,\vartheta_3(2\tau_{\mathrm{uv}})^4\right)^{-\ell} \left(\frac{\eta(\tau_{\mathrm{uv}})^2}{\vartheta_3(2\tau_{\mathrm{uv}})}\right)^{\lambda}$$
$$\times \sum_{\boldsymbol{x}=2\boldsymbol{\mu} \mod 2L} \mathrm{SW}(c_{\mathrm{ir}}) \left(\frac{\vartheta_3(2\tau_{\mathrm{uv}})}{\vartheta_2(2\tau_{\mathrm{uv}})}\right)^{\boldsymbol{x}^2}.$$

This confirms for $\ell = 0$, results from Vafa-Witten (1994), Dijkgraaf, Park, Schroers (1998), Göttsche-Kool (2020).

- Contributions from the other singularities have a similar form, and match expectations of S-duality
- Observables can also be included

5d $SU(2)$ theory on $\mathbb{R}^4 \times S^1$

As another application, we can consider 5-dimensional $\mathcal{N} = 1$ $SU(2)$ gauge theory theory compactified on a circle of radius R. The theory in 4d includes a full KK tower of states.

Work in progress together with Kim, Moore, Tao, Zhang (2022)

Bosonic field content: gauge field A_m, $m = 0, \ldots, 4$, real scalar σ

Global symmetries: $SU(2)_R \times U(1)_I \times U(1)_{KK}$

The current for the $U(1)_I$ instanton symmetry is

$$j = * \frac{1}{8\pi^2} \mathrm{Tr} F \wedge F$$

and the charged particles are instanton particles.

Seiberg (1996); Morrison, Seiberg, Intrilligator (1996),...

Electric BPS particles:

- W-bosons: $m_a = 2a$
- instanton particle: $m_I = \frac{2}{R} \log(\mathcal{R})$
- unit momentum around S^1: $m_K = \frac{2\pi i}{R}$

Dual periods of magnetic objects:

- monopole $a_D = \partial\mathcal{F}/da$
- $m_{DI} = \partial\mathcal{F}/dm_I$
- $m_{DK} = \partial\mathcal{F}/dm_K$

We include a flux $\boldsymbol{n} = [F^I/2\pi] \in H^2(X,\mathbb{Z})$ to the topological global $U(1)_I$ symmetry. In the UV, this is induced by a mixed 5d Chern-Simons action of G and $U(1)_I$

$$S_{\text{mixed CS}} = \frac{1}{8\pi^2}\int_{X\times S^1} F^I \wedge \mathrm{Tr}\left[AdA + \frac{2}{3}A^3\right] + \dots$$

The partition function is a generating function of $\hat{A}$-genera of instanton moduli spaces,

$$Z_{\boldsymbol{\mu}}(\mathcal{R},\boldsymbol{n}) = \sum_{k\geq 0}\int_{\mathcal{M}_k}\hat{A}(\mathcal{M}_k)\, e^{\mu(\boldsymbol{n}_I)}\,\mathcal{R}^{4k},$$

with $\mathcal{R} = R\Lambda$ and $\mu : H^2(X) \to H^2(\mathcal{M}_k)$.

On (almost) complex four-manifolds, the partition function becomes a generating function of holomorphic Euler characteristics

$$Z_{\boldsymbol{\mu}}(\mathcal{R}, \boldsymbol{n}) = \sum_{k\geq 0} \chi(\mathcal{M}_k, \mathcal{L}_{\boldsymbol{n}})\, \mathcal{R}^{4k},$$

with $\mathcal{L}_{\boldsymbol{n}} \to \mathcal{M}_k$ a line bundle determined by F_I.

Order parameter

Vev of Wilson line operator

$$U = \left\langle \mathrm{Tr}_F \mathrm{P} \exp\left(\int_{S_1} (\sigma + iA_5) dx_5 \right) \right\rangle = e^{Ra} + e^{-Ra} + O(\mathcal{R})$$

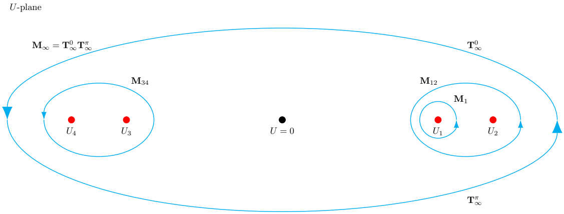
5d $SU(2)$ theory on $\mathbb{R}^4 \times S^1$

SW curve for this theory:

$$\Sigma: \quad Y^2 = P(X)^2 - 4X^2\mathcal{R}^4, \qquad P(X) = X^2 + UX + 1,$$

Nekrasov (1996); Ganor, Morrison, Seiberg (1996); Göttsche, Yoshioka, Nakajima (2006),...

Four singularities: $U = \pm 2(\mathcal{R}^2 \pm 1)$



Using the theory of elliptic curves, one can demonstrate

$$U^2 = -8\mathcal{R}^2 u + 4\mathcal{R}^4 + 4,$$

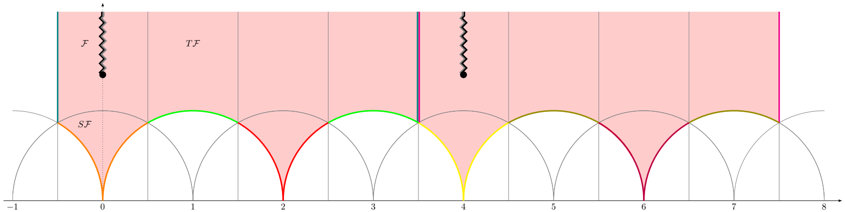
with

$$u(\tau) = \frac{\vartheta_2(\tau)^4 + \vartheta_3(\tau)^4}{2\vartheta_2(\tau)^2\,\vartheta_3(\tau)^2}, \qquad \mathcal{R} = R\Lambda$$

with τ the complex structure of Σ

5d $SU(2)$ theory on $\mathbb{R}^4 \times S^1$

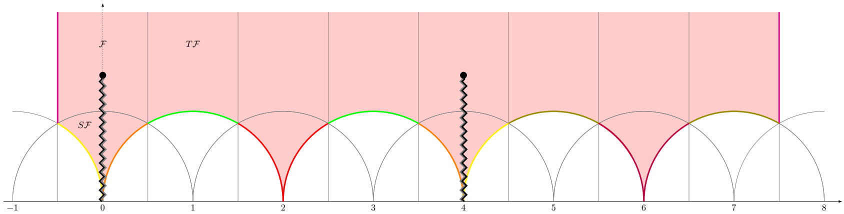
$U(\tau)$ is a function on the double cover of the pure $SU(2)$ domain $(\mathbb{H}/\Gamma^0(4))$. It includes a branch point and cuts:



Such branch points/cuts also appeared for $\mathcal{N} = 2$ SQCD

Aspman, Furrer, Manschot (2021)

We can rearrange the domain to avoid cuts at infinity.



In the limit $\mathcal{R} \to 1$ the branch points disappear and the U is a modular form for (a congruence of) $\Gamma^0(8)$.

Closset, Magureanu (2021)

We can carry out the the U-plane integral for this KK theory coupled to $\boldsymbol{n}$.

$$\Phi_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R}) = K_U \int_{\mathcal{F}_\mathcal{R}} d\tau \wedge d\bar{\tau}\, \nu_\mathcal{R}(\tau)\, C^{\boldsymbol{n}^2}\, \Psi^J_{\boldsymbol{\mu}}(\tau, \bar{\tau}, v\boldsymbol{n}/2, \bar{v}\boldsymbol{n}/2)$$

with

$$\nu_\mathcal{R} = \frac{\vartheta_4(\tau)^{13-b_2}}{\eta(\tau)^9} \frac{1}{\sqrt{-8u\mathcal{R}^2 + 4\mathcal{R}^4 + 1}}$$

$$v = -\frac{1}{2\pi i}\partial_a \partial_{m_l} \mathcal{F}, \qquad C = \frac{\vartheta_4(\tau, v)}{\vartheta_4(\tau)}$$

The integrand can be shown to be invariant under monodromies.

For the evaluation, we first expand in $\mathcal{R}$ and then determine the q^0 term. For example for $X = \mathbb{P}^2$, we obtain

$$\Phi_{1/2,n}(\mathcal{R}) = \begin{cases} 1 - \frac{7}{128}\mathcal{R}^4 - \frac{49}{4096}\mathcal{R}^8 + \dots, & n = 0, \\ 1 + O(\mathcal{R}^{13}), & n = \pm 1, \\ 1 + \frac{33}{128}\mathcal{R}^4 + \frac{543}{4096}\mathcal{R}^8 + \dots, & n = \pm 2, \\ 1 + \mathcal{R}^4 + \mathcal{R}^8 + \mathcal{R}^{12} + \dots, & n = \pm 3, \\ 1 + \frac{345}{128}\mathcal{R}^4 + \frac{21135}{4096}\mathcal{R}^8 + \dots, & n = \pm 4, \\ 1 + 6\mathcal{R}^4 + 21\mathcal{R}^8 + 56\mathcal{R}^{12} + \dots, & n = \pm 5, \\ 1 + \frac{1505}{128}\mathcal{R}^4 + \frac{292255}{4096}\mathcal{R}^8 + \dots, & n = \pm 6, \\ 1 + 21\mathcal{R}^4 + 210\mathcal{R}^8 + 1401\mathcal{R}^{12} + \dots, & n = \pm 7, \\ 1 + \frac{4473}{128}\mathcal{R}^4 + \frac{2253519}{4096}\mathcal{R}^8 + \dots, & n = \pm 8, \\ 1 + 55\mathcal{R}^4 + 1310\mathcal{R}^8 + 19432\mathcal{R}^{12} \dots, & n = \pm 9. \end{cases}$$

For n odd, in agreement with Göttsche, Nakajima, Yoshioka (2006).

Similarly to before, also SW contributions can be determined. In this way we give a physical derivation of the result by Göttsche, Kool, Williams (2019) for the K-theoretic Donaldson invariants of X

$$\frac{2^{2-\chi_{\rm h}(X)+K_X^2}}{(1-\mathcal{R}^2)^{(\boldsymbol{n}-K_X)^2/2+\chi_{\rm h}}}\sum_{c}\mathrm{SW}(c)\,(-1)^{\mu\,(c+K_X)}\left(\frac{1+\mathcal{R}}{1-\mathcal{R}}\right)^{c(K_X-\boldsymbol{n})/2}$$

with K_X the canonical class of X, and $\mathrm{SW}(c)$ the Seiberg-Witten invariant for the basic class c.

Conclusion

- We have explicitly evaluated and analyzed the partition function & correlators of
  1. the $\mathcal{N} = 2^*$ $SU(2)$ theory. The theory interpolates between the Donaldson-Witten and Vafa-Witten topological theories.
  2. 5d $\mathcal{N} = 1$ theory on $X \times S^1$, which gives rise to K-theoretic Donaldson invariants
- To formulate a twisted $\mathcal{N} = 2$ theory on a four-manifold X, extra data, such as $\mathfrak{s}$, is necessary in general

  In progress with J. Aspman, E. Furrer: project on u-plane integral for $\mathcal{N} = 2$ SQCD
- Analysis motivates the study of more general theories

Thank you!