Holomorphic Floer theory and DT invariants

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Introduction

A conjectural relation between two topics:

- Holomorphic Floer theory, enumerative geometry of holomorphic symplectic/hyperkähler manifolds [see Doan's talk]
- Donaldson-Thomas (DT) invariants, enumerative geometry of Calabi-Yau 3-folds, BPS spectrum of $\mathcal{N} = 2$ 4d field theories.

Reference:

- Bousseau: "Holomorphic Floer theory and Donaldson-Thomas invariants", arxiv:22xx.xxxxx

Plan of the talk

1) DT invariants and BPS states in $\mathcal{N} = 2$ 4d field theories.
2) Analogy with Picard-Lefschetz theory and BPS states in $\mathcal{N} = (2, 2)$ 2d field theories.
3) Holomorphic Floer theory and DT invariants.

Geometry: DT invariants

- DT invariants:

$$\Omega_\gamma(u) \in \mathbb{Z}$$

counts of geometric objects on a Calabi-Yau 3-fold X, with given topology class $\gamma \in \mathbb{Z}^n$ and satisfying a stability condition u.

- Examples:
  - Stable holomorphic vector bundles of Chern character γ for a Kähler parameter u.
  - Special Lagrangian submanifolds of class γ for a complex parameter u.

Physics: BPS states in $\mathcal{N} = 2$ 4d field theories

- $\mathcal{N} = 2$ supersymmetric 4d field theories
  - B: Coulomb branch of vacua of the 4d theory, $B \simeq \mathbb{C}^r$.
  - In a generic vacuum $u \in B \setminus \Delta$, abelian gauge theory $U(1)^r$
  - Supersymmetry: charge γ, central charge $Z_\gamma(u) \in \mathbb{C}$, BPS bound

$$|M| \geq |Z_\gamma(u)|$$

  - Space of BPS states, saturating the BPS bound: $H_\gamma(u)$
  - BPS index

$$\Omega_\gamma(u) = \mathrm{Tr}_{H_\gamma(u)}(-1)^F$$

From geometry to physics

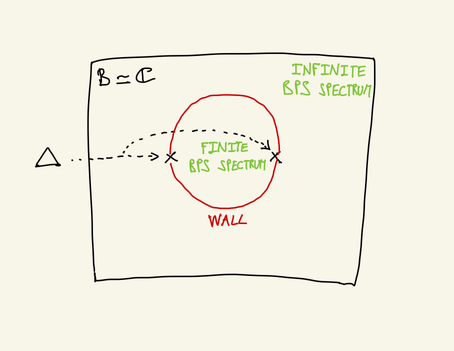
- Geometric constructions from string theory: IIA or IIB string on Calabi-Yau 3-fold X
- DT invariants = BPS indices: stability $u \in B \setminus \Delta$
- From now on: consider $\mathcal{N} = 2$ 4d field theories without gravity.
  - Geometrically: non-compact Calabi-Yau 3-folds.

Wall-crossing

- $\Omega_\gamma(u)$: constant function of u away from codimension one loci in B, called walls, across which $\Omega_\gamma(u)$ jumps discontinuously.
- Jumps controlled by a universal wall-crossing formula [Kontsevich-Soibelman]:

$$\{\Omega_\gamma(u^-)\}_\gamma \to \{\Omega_\gamma(u^+)\}_\gamma \,.$$

- Example: $\mathcal{N} = 2$ $SU(2)$ gauge theory



Seiberg-Witten integrable system

- $\mathcal{M}$: Coulomb branch of the theory on $\mathbb{R}^3 \times S^1$, hyperkähler manifold of complex dimension $2r$, complex integrable system:

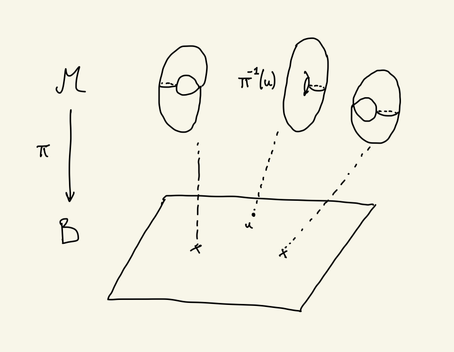
$$\pi \colon \mathcal{M} \longrightarrow B$$

- Low energy: 3d $\mathcal{N} = 4$ sigma model with target $\mathcal{M}$
- Twistor sphere of complex structures I, J, K
  - π I-holomorphic: in complex structure I, generic fibers of π are abelian varieties of dimension r.
  - for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, generic fibers of π are special Lagrangians in complex structure $J_\theta = (\cos\theta) J + (\sin\theta) K$.
- $u \in B \setminus \Delta$, $\gamma \in \pi_2(\mathcal{M}, \pi^{-1}(u)) \to H_1(\pi^{-1}(u), \mathbb{Z}) = \mathbb{Z}^{2r}$,

$$Z_\gamma(u) = \int_\gamma \Omega_I$$

Seiberg-Witten integrable system

- Example: $\mathcal{N} = 2$ $SU(2)$ gauge theory



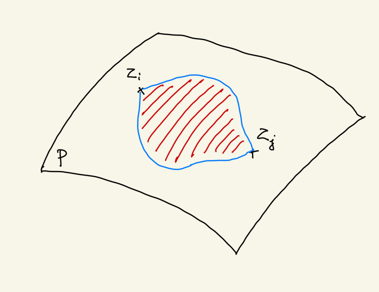
- Class S on C: $\pi : \mathcal{M} \to B$ is (essentially) the Hitchin integrable system for C.
- BPS spectrum $\{\Omega_\gamma(u)\} \to$ hyperkähler geometry of $\mathcal{M}$.
- Wall-crossing formula $=$ smoothness of the hyperkähler geometry [Gaiotto-Moore-Neitzke]

Analogy with $\mathcal{N} = (2,2)$ 2d.

- Simpler wall-crossing story for BPS states in $\mathcal{N} = (2,2)$ 2d field theories [Cecotti-Vafa].
- Example: LG model (P, W)
  - P: Kähler manifold of complex dimension n, $W : P \to \mathbb{C}$ holomorphic Morse function.
  - Finitely many critical points $\{z_i\}$ of W
  - $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, $\mathrm{Re}(e^{-i\theta} W)$ is a real-valued Morse function, critical points $\{z_i\}$, index n.
  - z_i, z_j, $\theta = \theta_{ij} := \mathrm{Arg}(W(z_i) - W(z_j))$, μ_{ij}: count of gradient flow lines between z_i and z_j, 2d BPS indices.
- Wall-crossing formula for μ_{ij} as a function of W [Cecotti-Vafa] (Picard-Lefschetz formula).

Analogy with $\mathcal{N} = (2, 2)$ 2d.

- Space H_{ij} of 2d BPS states?
- Morse homology of the space of paths in P connecting z_i to z_j for the functional $\mathfrak{p} \mapsto \int_{\mathfrak{p}} \left(d^{-1}\omega - \mathrm{Im}(e^{-i\theta} W) dt \right)$. Homology of the complex:
  - generated by gradient flow lines $\mathbb{R} \to P$ between z_i and z_j
  - with differential given by counts of ζ-instantons $\mathbb{R}^2 \to P$ asymptotic to two gradient flow lines.
- $\mu_{ij} = \mathrm{Tr}_{H_{ij}} (-1)^F$



Analogy with $\mathcal{N} = (2,2)$ 2d.

- Category Brane of supersymmetric branes of the $\mathcal{N} = (2,2)$ $2d$ theory: Fukaya-Seidel category of (P, W), can be constructed using similar Morse theory techniques [Haydys, Gaiotto-Moore-Witten].
- Analogy between BPS states in $\mathcal{N} = (2,2)$ 2d and BPS states in $\mathcal{N} = 2$ 4d [Gaiotto-Moore-Neitzke, Kontsevich-Soibelman, Bridgeland-Toledano–Laredo, Bridgeland].

4d	2d
$\{\gamma\}$	$\{z_i\}$
Z_γ	$W(z_i)$
Ω_γ	μ_{ij}
H_γ	H_{ij}
?	Brane

Question

Is it more than an analogy? Can we find a LG model (P, W) recovering the BPS states of a $\mathcal{N} = 2$ 4d field theory?

- LG model (P, W):
  - Critical points $z_i \in P$
  - Gradient flow lines $\mathbb{R} \to P$
  - ζ-instantons $\mathbb{R}^2 \to P$
- General idea: consider LG models (P, W) for infinite dimensional Kähler manifolds P.
- Examples: Holomorphic Floer theory
  - P: space of paths between two holomorphic Lagrangians in a holomorphic symplectic manifold
  - W: holomorphic action functional
  - [B, Doan-Rezchikov, Kontsevich-Soibelman, Khan]

Holomorphic Floer theory

- $(\mathcal{M}, I, \Omega_I)$: holomorphic symplectic manifold.
  - Hyperkähler structure I, J, K, $J_\theta := (\cos\theta)J + (\sin\theta)K$.
  - $L_1, L_2 \subset \mathcal{M}$: I-holomorphic Lagrangian, $\Omega_I|_{L_1} = \Omega_I|_{L_2} = 0$.
- P: space of paths between L_1 and L_2, $W := \int_{\mathfrak{p}} d^{-1}\Omega_I$ (multivalued!)
  - Critical points: intersection points $L_1 \cap L_2$.
  - Gradient flow lines: J_θ holomorphic curves, $u : \mathbb{R}^2 \to \mathcal{M}$.
  - ζ-instantons, $u : \mathbb{R}^3 \to \mathcal{M}$, solutions to Fueter equation

$$\partial_\tau u + I(\partial_s u + J_\theta \partial_t u) = 0\,.$$

- LG model for (P, W):
  - $p, q \in L_1 \cap L_2 \to$ vector space H_{pq} of 2d BPS states of (P, W)
  - $L_1, L_2 \to$ Fukaya-Seidel category $FS(P, W)$
  - $\mathcal{M} \to$ 2-category of I-holomorphic Lagrangians $Ft(\mathcal{M})$ such that

$$\mathrm{Hom}_{Ft(\mathcal{M})}(L_1, L_2) = FS(P, W)\,.$$

Holomorphic Floer theory versus Rozansky-Witten

- $(\mathcal{M}, I, \Omega_I)$: holomorphic symplectic manifold.
  - $Ft(\mathcal{M})$ Fueter 2-category, $\mathbb{Z}$-graded, not Calabi-Yau,

$$\mathrm{Hom}_{Ft(\mathcal{M})}(L_1, L_2) = FS(P, W)\,.$$

  - $RW(\mathcal{M})$ Rozansky-Witten category, $\mathbb{Z}/2\mathbb{Z}$-graded, Calabi-Yau,

$$\mathrm{Hom}_{RW(\mathcal{M})}(L_1, L_2) = MF(P, W)\,.$$

    2-category of boundary conditions of Rozansky-Witten 3d TQFT [Kapustin-Rozansky-Saulina]

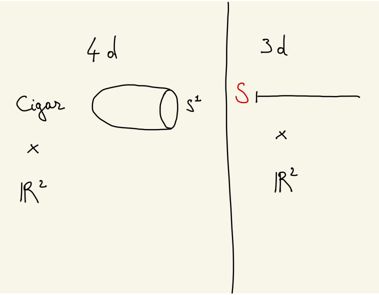
- General principle: known constructions for finite-dimensional $FS(P, W)$ and $MF(P, W)$ should be upgraded to $Ft(\mathcal{M})$ and $RW(\mathcal{M})$.

Holomorphic Atiyah-Floer conjecture

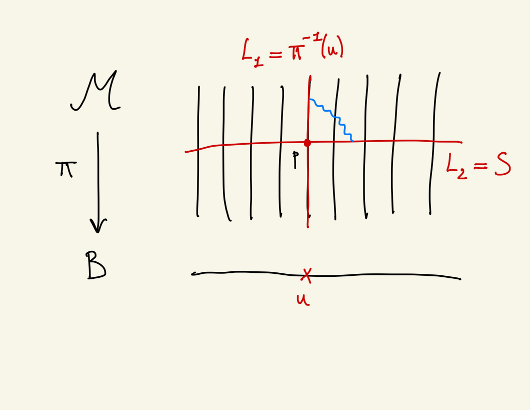
- Another example of infinite-dimensional Fukaya-Seidel: complexified Chern-Simons [Haydys, Witten]
  - Complex-valued Chern-Simons CS on the space A of $G_{\mathbb{C}}$-connections on a 3-manifold X and $G_{\mathbb{C}}$.
  - Expect to construct a category $CS_{G_{\mathbb{C}}}(X) := FS(A, CS)$ attached to X and $G_{\mathbb{C}}$ (from counting solutions to Kapustin-Witten equation on $X \times \mathbb{R}$ and Haydys-Witten equation on $X \times \mathbb{R}^2$).
- Given a Heegard splitting of X along a surface Σ, $X = X_1 \cup_\Sigma X_2$
  - $\mathcal{M}$: $G_{\mathbb{C}}$-character variety of Σ, holomorphic symplectic.
  - L_1, L_2: $G_{\mathbb{C}}$-local systemes extending to X_1 and X_2, holomorphic Lagrangian submanifolds of $\mathcal{M}$
  - Conjecture:

$$CS_{G_{\mathbb{C}}}(X) = \mathrm{Hom}_{Ft(\mathcal{M})}(L_1, L_2)\,.$$

- Back to a $\mathcal{N} = 2$ 4d field theory.
- How to recover the BPS spectrum $\{\Omega_\gamma(u)\}$ from holomorphic Floer theory? Correct holomorphic symplectic manifolds $\mathcal{M}$ and holomorphic Lagrangians L_1, L_2 ?
  - $\mathcal{M}$: Seiberg-Witten integrable system
  - $L_1 = \pi^{-1}(u)$: fiber of $\pi : \mathcal{M} \to B$ over $u \in B$.
  - $L_2 = S$: natural section of π. Physical definition: boundary condition for the 3d sigma model of target $\mathcal{M}$ defined by the cigar geometry [Nekrasov-Witten]. Hitchin system example: Hitchin section.

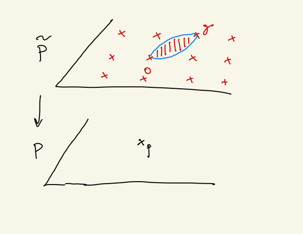


Holomorphic Floer theory and DT invariants



- $L_1 \cap L_2 = \pi^{-1}(u) \cap S = \{p\}$
- But $\pi_1(P) \neq 0$ and W is multivalued.
- $\pi_1(P) = \pi_2(\mathcal{M}, \pi^{-1}(u))$: on $\widetilde{P}$, critical points of W indexed by

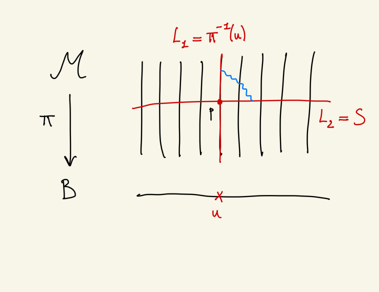
$$\gamma \in \pi_2(\mathcal{M}, \pi^{-1}(u))$$



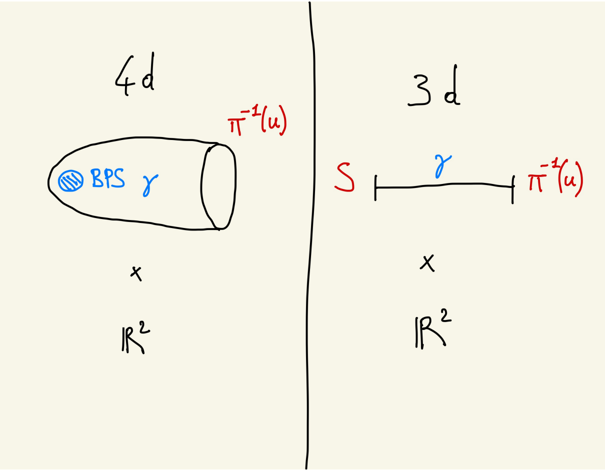
Conjecture (B)

Given a $\mathcal{N} = 2$ 4d field theory, the space of BPS states $H_\gamma(u)$ of class γ in the vacuum u is isomorphic to the vector space $H_{0\gamma}$ associated by holomorphic Floer theory for the Seiberg-Witten integrable system $\mathcal{M}$ to the lifts 0 and γ of the intersection point between the fiber $\pi^{-1}(u)$ and the section S:

$$H_\gamma(u) \simeq H_{0\gamma}$$

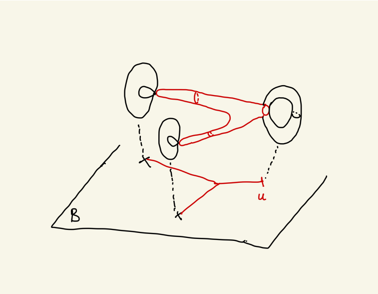


Physics derivation



Numerical limit: BPS indices

- $\Omega_\gamma(u)$= count of J_θ-holomorphic disks in $\mathcal{M}$ with boundary on the fiber $\pi^{-1}(u)$, $\theta = \mathrm{Arg}(Z_\gamma(u))$ [Lu, Kontsevich-Soibelman]
- Projection to B given by attractor flow trees



- Holomorphic disks/Instanton corrections in mirror symmetry:
  - X Calabi-Yau manifold $\to$ counts of holomorphic curves $\to$ mirror Calabi-Yau Y
  - $\mathcal{M}$ $\to$ counts of holomorphic curves = BPS indices $\xrightarrow{GMN}$ $\mathcal{M}$ (self-mirror)

Categories of line operators

- What is $\mathrm{Hom}(S, \pi^{-1}(u))$? A categorical version of the category of IR line operators.
- Conjecture: $\mathrm{Hom}(S, S)$ is a monoidal categorification of the algebra of regular functions on $(\mathcal{M}, J)$ (the character variety for class S). UV line operators.
- $\mathrm{Hom}(S, \pi^{-1}(u))$ is naturally a module over $\mathrm{Hom}(S, S)$. Choice of 0 in $\mathrm{Hom}(S, \pi^{-1}(u))$ gives a functor

$$\mathrm{Hom}(S, S) \to \mathrm{Hom}(S, \pi^{-1}(u)) \,.$$

Conjecture: this functor is a categorification of the UV to IR map [Gaiotto-Moore-Neitzke, Cordova-Neitzke] and is an equivalence of categories.

Thank you for your attention !