From Transgression to Descent

An introduction to Chern-Simons

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Some Cohomology Classes in Principal Fiber Bundles and Their Application to Riemannian Geometry

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Communicated February 9, 1971

ABSTRACT **We define some new global invariants of a fiber bundle with a connection. They are cohomology classes in the principal fiber bundle that are defined when certain characteristic curvature forms vanish. In the case of the principal tangent bundle of a riemannian manifold, they are invariant under a conformal transformation of the metric. They give necessary conditions for conformal immersion of a riemannian manifold in euclidean space.**

Lemma 2.3. $\int_{F(M)_m} Q =$
over any $m \in M$.

Proof. $F(M)_m$ is equival
horizontal, $Q|F(M)_m$ is a n
the volume form on $SO(3$
chosen to normalize this in

Characteristic Forms and Geometric Invariants

Shiing-Shen Chern; James Simons

The Annals of Mathematics, 2nd Ser., Vol. 99, No. 1. (Jan., 1974), pp. 48-69.

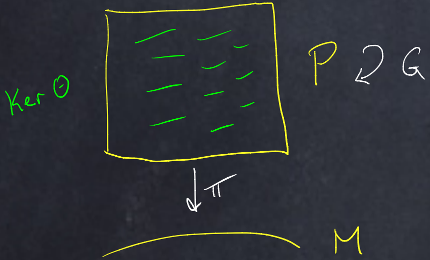
1. Introduction

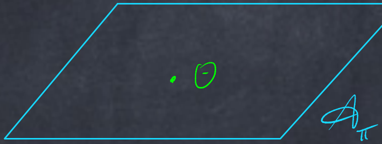
This work, originally announced in [4], grew out of an attempt to derive a purely combinatorial formula for the first Pontrjagin number of a 4-manifold. The hope was that by integrating the characteristic curvature form (with respect to some Riemannian metric) simplex by simplex, and replacing the integral over each interior by another on the boundary, one could evaluate these boundary integrals, add up over the triangulation, and have the geometry wash out, leaving the sought after combinatorial formula. This process got stuck by the emergence of a boundary term which did not yield to a simple combinatorial analysis. The boundary term seemed interesting in its own right and it and its generalization are the subject of this paper.

The Weil homomorphism is a mapping from the ring of invariant polynomials of the Lie algebra of a Lie group, G, into the real characteristic cohomology ring of the base space of a principal G-bundle, cf. [5], [7]. The

The Chern-Weil form

G	Lie group with $\pi_0 G$ finite
$\langle -, \cdots, - \rangle \colon \mathfrak{g}^{\otimes p} \longrightarrow \mathbb{R}$	G-invariant symmetric p-linear form on the Lie algebra $\mathfrak{g}$
$\pi \colon P \longrightarrow M$	principal G-bundle
$\mathcal{A}_\pi$	affine space of connections $\Theta \in \Omega^1_P(\mathfrak{g})$
$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]$	curvature of the connection Θ





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Example: G compact Lie group

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$\langle -, - \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ G-invariant inner product

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Lemma: $\omega = \omega(\Theta) = \langle \Omega \wedge \cdots \wedge \Omega \rangle$ is a *closed* $2p$-form which descends to the base M

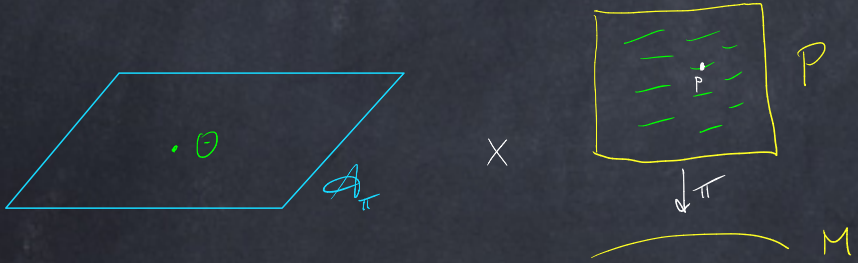
The universal connection on π

Θ_π universal connection on $\mathcal{A}_\pi \times P \longrightarrow \mathcal{A}_\pi \times M$ characterized by

$$\Theta_\pi\big|_{\{\Theta\}\times P} = \Theta, \qquad \Theta_\pi\big|_{\mathcal{A}_\pi \times \{p\}} = 0, \qquad \Theta \in \mathcal{A}_\pi,\ p \in P$$

$\Omega_\pi = \Omega(\Theta_\pi)$ curvature (in $\Omega^2_{\mathcal{A}_\pi \times P}(\mathfrak{g})$)

$\omega_\pi = \omega(\Theta_\pi)$ Chern-Weil form (in $\Omega^{2p}_{\mathcal{A}_\pi \times M}$)



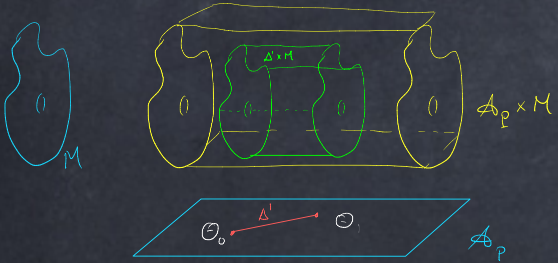
Chern-Simons form for *two* connections

$\Theta_0, \Theta_1 \in \mathcal{A}_\pi$ connections on $P \longrightarrow M$

$\Delta^1 \longrightarrow \mathcal{A}_\pi$ affine map with endpoints Θ_0, Θ_1

Definition: The Chern-Simons $(2p-1)$-form of the connections Θ_0, Θ_1 is

$$\alpha(\Theta_0, \Theta_1) = \int_{\Delta^1} \omega(\Theta_\pi) \qquad \in \Omega_M^{2p-1}$$



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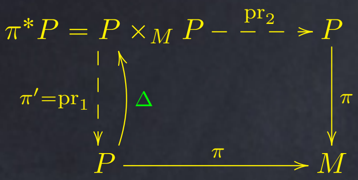
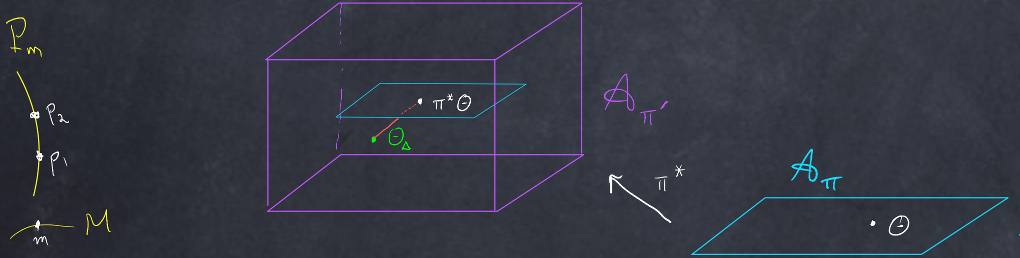
Remark: The de Rham cohomology class of $\omega(\Theta)$ in $H^{2p}_{\mathrm{dR}}(M) \cong H^{2p}(M; \mathbb{R})$ is independent of Θ

Chern-Simons form for *one* connection



- The section Δ determines a *global* parallelism on π'
- A connection Θ on π pulls back to a connection on π'



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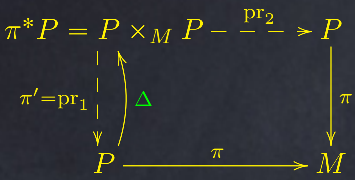

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Chern-Simons form for *one* connection

$$\begin{array}{ccc} \pi^* P = P \times_M P & \overset{\mathrm{pr}_2}{\dashrightarrow} & P \\ {\scriptstyle \pi' = \mathrm{pr}_1} \downarrow \uparrow {\scriptstyle \Delta} & & \downarrow {\scriptstyle \pi} \\ P & \xrightarrow{\pi} & M \end{array}$$

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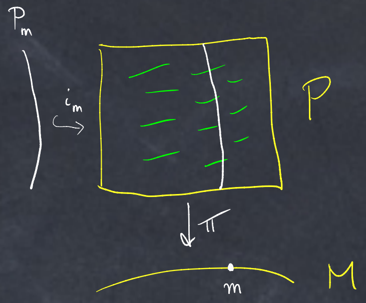
Transgression

Theorem: The Chern-Simons form $\alpha(\Theta)$ satisfies

$$d\alpha(\Theta) = \pi^*\omega(\Theta)$$
$$i_m^*\alpha(\Theta) = c_p \langle \theta_m \wedge [\theta_m \wedge \theta_m]^{p-1}] \rangle$$



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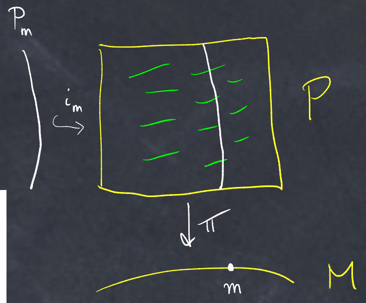

$$l dP(\theta \wedge \varphi_t^{l-1}) = lP(\Omega \wedge \varphi_t^{l-1}) - \frac{1}{2} lP([\theta, \theta] \wedge \varphi_t^{l-1}) + ltP([\theta, \theta] \wedge \varphi_t^{l-1}) = f'$$

by the computation above. This shows (3.4) and the proposition follows from (3.3).

The form $TP(\theta)$ can of course be written without the integral, and, in fact, setting

$$A_i = (-1)^i l!\,(l-1)!/2^i (l+i)!\,(l-1-i)!$$

one computes

$$(3.5) \qquad TP(\theta) = \sum_{i=0}^{l-1} A_i P(\theta \wedge [\theta, \theta]^i \wedge \Omega^{l-i-1}) \,.$$

The operation which associates to $\alpha \in \varepsilon(G)$ the form $TP(\theta)$ is natural; i.e., if $\varphi: \alpha \to \hat{\alpha}$ is a morphism, since $\varphi^*(\hat{\theta}) = \theta$ and thus $\varphi^*(\hat{\Omega}) = \Omega$, clearly

$c_p = A_{l-1}^{(l\to p)}$

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Chern had earlier (1943) introduced another version of transgression with differential forms in his intrinsic proof of the Gauss-Bonnet theorem

Descent

Four differential forms on the total space of $\pi\colon P \longrightarrow M$, each equivariant for $P \times G \longrightarrow P$:

Θ	in $\Omega^1_P(\mathfrak{g})$	(connection form)
Ω	in $\Omega^2_P(\mathfrak{g})$	(curvature form)
$\omega(\Theta)$	in Ω^{2p}_P	(Chern-Weil form)
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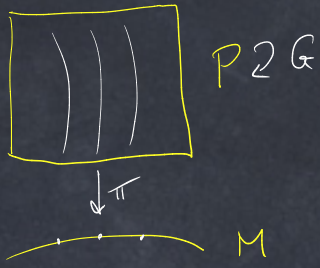

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$\alpha(\Theta)$ also descends to a section of an affine bundle over M, but we can do better…

Descent of the Chern-Simons form

The descent played out in stages over three decades

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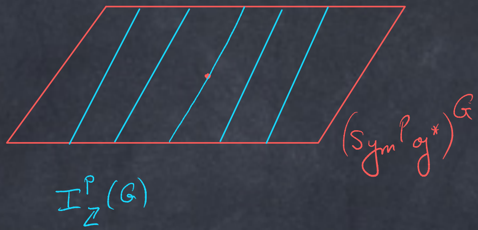
The first steps are in the Chern-Simons paper, where they introduce a subset of *integral* symmetric p-linear forms $\langle -, \cdots, - \rangle \colon \mathfrak{g}^{\otimes p} \longrightarrow \mathbb{R}$ in the vector space $(\operatorname{Sym}^p \mathfrak{g}^*)^G$

More precisely, it is the fiber product

$$\begin{array}{ccc} I^p_{\mathbb{Z}}(G) & \dashrightarrow & (\operatorname{Sym}^p \mathfrak{g}^*)^G \\ \downarrow & & \downarrow \\ H^{2p}(BG;\mathbb{Z})/\text{torsion} & \longrightarrow & H^{2p}(BG;\mathbb{R}) \end{array}$$



Ex: $I^2_{\mathbb{Z}}(SU_2 \times SL_2\mathbb{R})$

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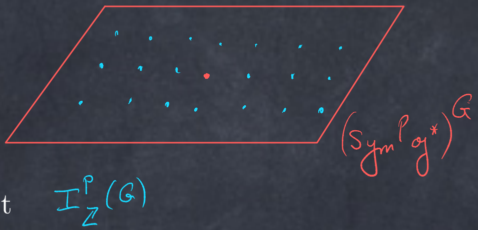

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If $\langle -, \cdots, - \rangle$ lies in $I^p_{\mathbb{Z}}(G)$, then the *mod* $\mathbb{Z}$ *reduction* of the real cochain $\alpha(\Theta)$ descends to M as an $\mathbb{R}/\mathbb{Z}$ cochain, but not canonically and only up to a coboundary

The next steps follow immediately in the work of Cheeger-Simons

DIFFERENTIAL CHARACTERS AND GEOMETRIC INVARIANTS

Jeff Cheeger*

and

James Simons**

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Stony Brook, NY 11794

Abstract

This paper first appeared in a collection of lecture notes which were distributed at the A.M.S. Summer Institute on Differential Geometry, held at Stanford in 1973. Since then it has been (and remains) the authors' intention to make available a more detailed version. But, in the mean time, we continued to receive requests for the original notes. Moreover, the secondary invariants we discussed have recently arisen in some new contexts, e.g. in physics and in the work of Cheeger and Gromov on "collapse" (which was the subject of the first author's lectures at the Special Year). For these reasons we decided to finally publish the notes, albeit in their original form.

The next steps follow immediately in the work of Cheeger-Simons

They introduce abelian groups $\mathrm{CS}^q(M)$ of *differential characters* on a smooth manifold M and abelian groups $\Lambda^p(G)$ of "levels" for a Lie group G (with $\pi_0 G$ finite):

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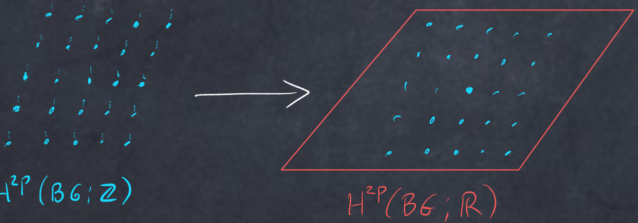

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Given a level, the Chern-Weil form $\omega(\Theta) \in \Omega^{2p}_{\mathrm{closed}}(M)$ of a G-connection Θ on $\pi\colon P \longrightarrow M$ has a *canonical* lift $\omega_{\mathrm{CS}}(\Theta) \in \mathrm{CS}^{2p}(M)$ to a differential character on M

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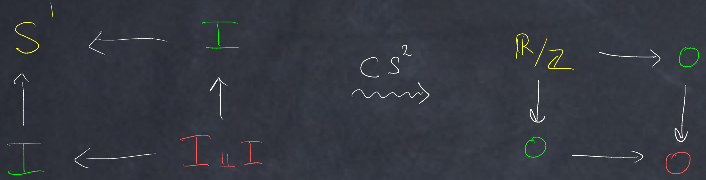
If M is closed oriented of dimension $2p-1$, define the *secondary geometric invariant*

$$\int_M \omega_{\mathrm{CS}}(\Theta) \quad \in \mathbb{R}/\mathbb{Z}$$

Fully *local* descent of the Chern-Simons form

The Cheeger-Simons groups are not local: they do *not* satisfy a sheaf condition

Example:



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J. DIFFERENTIAL GEOMETRY
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QUADRATIC FUNCTIONS IN GEOMETRY, TOPOLOGY, AND *M*-THEORY

M.J. HOPKINS & I.M. SINGER

$$\begin{array}{ccc} \check{C}(q)^*(M) & \longrightarrow & \Omega^{*\geq q}(M) \\ \downarrow & & \downarrow \\ C^*(M;\mathbb{Z}) & \longrightarrow & C^*(M;\mathbb{R}). \end{array}$$

Definition 4.1. A *differential function* $t : S \to (X;\iota)$ is a triple (c, h, ω)

$$c : S \to X, \quad h \in C^{n-1}(S;\mathbb{R}), \quad \omega \in \Omega^n(S)$$

satisfying

$$\delta h = \omega - c^*\iota.$$

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While the connection between (generalized) differential cohomology and invertible field theories has long been clear, it remains to nail a precise theorem

Ramifications of locality

Geometric analysis: Traditionally, one used a global functional which is the integral of a local density and derives local critical point differential equations which are studied using the functional. The fully local Chern-Simons functional offered new possibilities, for example in Floer theory

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Field theory: Traditionally, one wrote an action which is the integal of a local density constructed from the fields and constrained by symmetry. The fully Chern-Simons functional offered new possibilities. They also appear in condensed matter physics: discrete systems are approximated by continuous field theories

Ramifications of locality

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Full locality is central in both contexts

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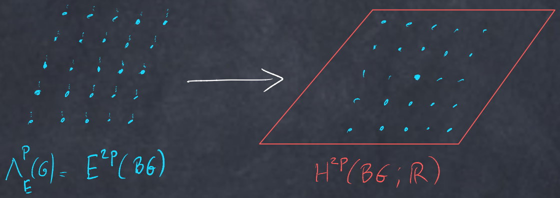

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E cohomology is oriented for *spin* manifolds; E-levels define *spin* Chern-Simons invariants

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Apply to $B\mathrm{SO}_3$ to obtain the short exact sequence of levels

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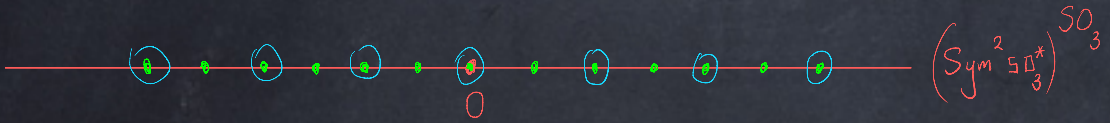

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\end{array}$$

The generator of $\Lambda^2_E(\mathrm{SO}_3)$ is a version of $p_1/2$

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Proof:

$$\begin{array}{ccc} TM & \xrightarrow{\tilde{\Gamma}} & TS^3 \\ \downarrow & & \downarrow \\ M & \xrightarrow[\text{Gauss}]{\Gamma} & S^3 \end{array}$$

$$\Gamma^*(\Theta_{\mathrm{LC}}^{S^3}) = \Theta_{\mathrm{LC}}^{M}$$

$$\mathscr{S}_{\mathrm{SO}_3}(M;\Theta_{\mathrm{LC}}^{M}) = \mathscr{S}_{\mathrm{SO}_3}(S^3;\Theta_{\mathrm{LC}}^{S^3})^{\deg(\Gamma)} = 1$$

Happy Birthday and Thank You