(Homological) Knot Invariants

from Mirror Symmetry

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Based on: 2004.14518

2105.06039

There are many beautiful strands in the story of

interactions between

mathematics and physics.

Two of the most fruitful ones

involve

knot theory and mirror symmetry.

In this talk, I will describe a new connection between the two.

We will find a solution to a central problem in knot theory,

the knot categorification problem,

as a new application of

mirror symmetry.

To begin with, it is useful to recall

some well known aspects of knot invariants.

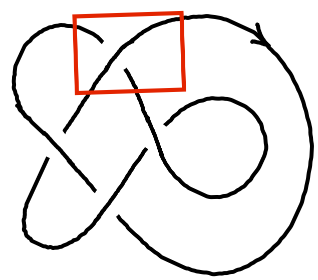
In ’84, Vaughan Jones discovered a polynomial link invariant

$$J_K(q)$$

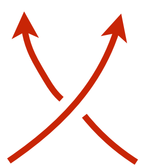
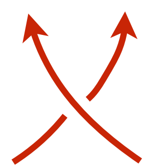
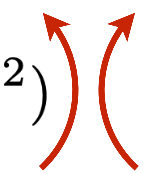
depending on one variable.

The Jones polynomial

is defined by picking a planar projection of the knot

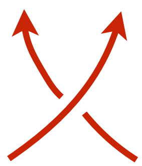
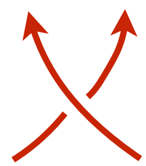
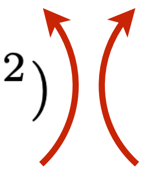


and the "Skein" relation it satisfies as one unties the knot,

$\mathfrak{q}^{n/2}$  $-$ $\mathfrak{q}^{-n/2}$  $= (\mathfrak{q}^{1/2} - \mathfrak{q}^{-1/2})$ 

where one takes $n = 2$.

It turned out other values of n

$\mathfrak{q}^{n/2}$  $-$ $\mathfrak{q}^{-n/2}$  $= (\mathfrak{q}^{1/2} - \mathfrak{q}^{-1/2})$ 

also lead to knot invariants.

Taking $n = 0$ one gets the Alexander polynomial,

which the first know polynomial knot invariant

dating to 1923.

Edward Witten explained in '88 that,

the Jones polynomial comes from

Chern-Simons theory with gauge group based on the Lie algebra

$$^{L}\mathfrak{g} = \mathfrak{su}_2$$

with (effective) Chern-Simons level κ related to $\mathfrak{q}$ by

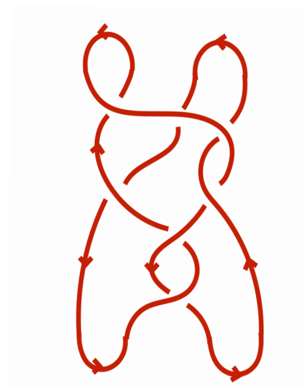
$$\mathfrak{q} = e^{\frac{2\pi i}{\kappa}}$$

The Jones polynomial is the expectation value of

a collection of Wilson loops

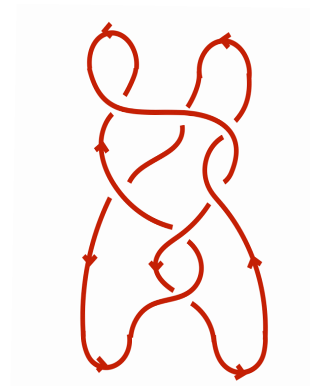
in the fundamental representation of ${}^L\mathfrak{g} = \mathfrak{su}_2$

supported along the link components,



in Chern-Simons theory on $\mathbb{R}^3$.

This placed the Jones polynomial into a more general framework



which one gets by

considering Chern-Simons theory based on

different Lie algebras $^L\mathfrak{g}$ and the representations.

The Alexander polynomial comes from the same setting,

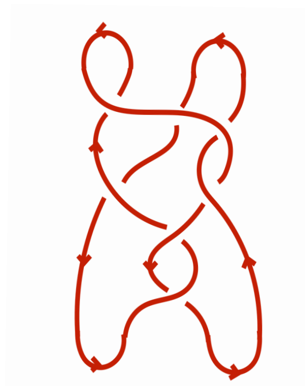
by taking $^L\mathfrak{g}$ to be the

Lie superalgebra:

$$^L\mathfrak{g} = \mathfrak{gl}_{1|1}$$

The resulting invariants of the link

are known as the $U_{\mathfrak{q}}({}^L\mathfrak{g})$ quantum group invariants.



The relation of Witten's link invariants

to quantum groups

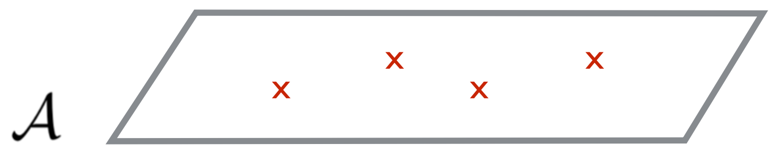
was developed by Reshetikhin and Turaev in '89.

Most works on categorification start with quantum groups.

For us, it will be crucial to recall the precise way

quantum groups came into the story.

Chern-Simons theory

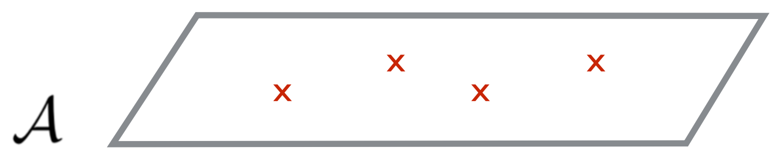


associates to a Riemann surface with punctures

a vector space, its Hilbert space.

The punctures are positions of heavy charged particles,

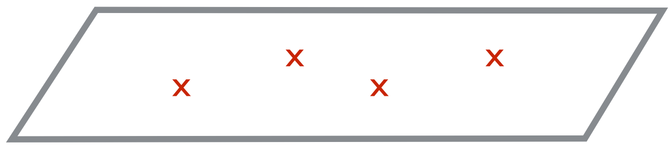
at an instant of time,



so they are "colored" by representations of the Lie algebra.

Witten showed this finite dimensional vector space

is spanned by vectors which have a name:

$\mathcal{A}$ 

they are known as "conformal blocks" of

$$\widehat{{}^L\mathfrak{g}}_{\kappa}$$

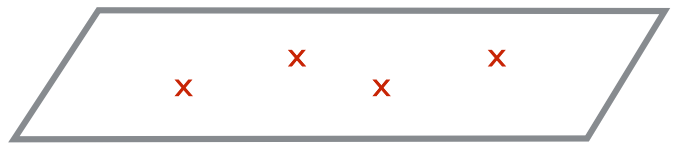
an affine Lie algebra at level κ , associated to

$${}^L\mathfrak{g}$$

Every conformal block of $\widehat{{}^L\mathfrak{g}_\kappa}$ can be produced explicitly,

as solution to a very famous linear differential equation

$\mathcal{A}$



discovered by Knizhnik and Zamolodchikov in ’84:

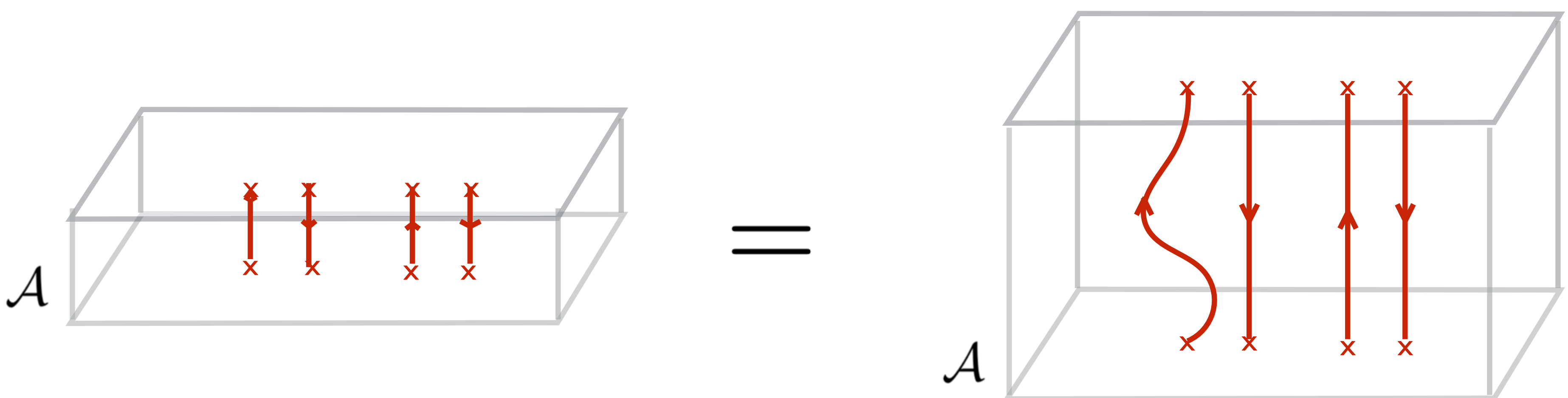
$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell} \mathcal{V} = \sum_{j \neq \ell} r_{\ell i}(a_\ell / a_j)\, \mathcal{V}.$$

The variables in the equation are the positions of punctures on $\mathcal{A}$.

In a topological theory such as Chern-Simons theory

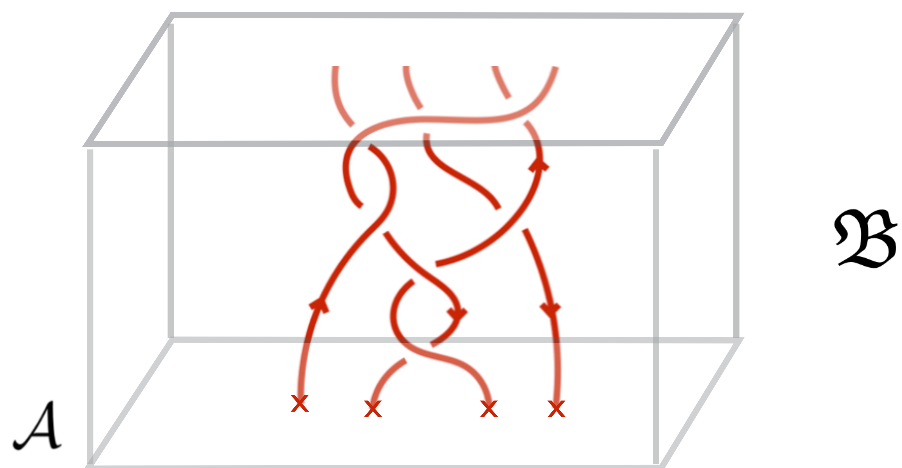
the time evolution

acts trivially on the Hilbert space,



To get something interesting

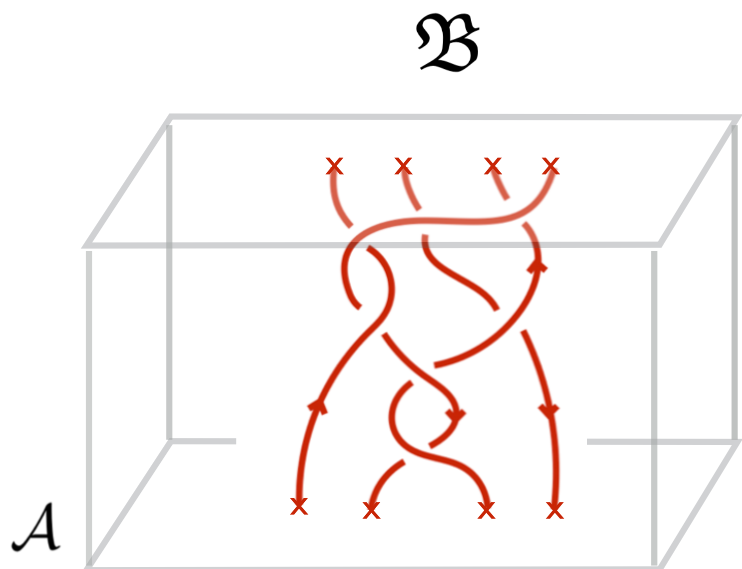
one wants to let the positions of heavy particles vary in time:



and then the path integral computes an invariant of a colored braid in

$$\mathcal{A} \times [0,1]$$

The braid invariant is a matrix



acting on the space of $\widehat{{}^L\mathfrak{g}_\kappa}$ conformal blocks.

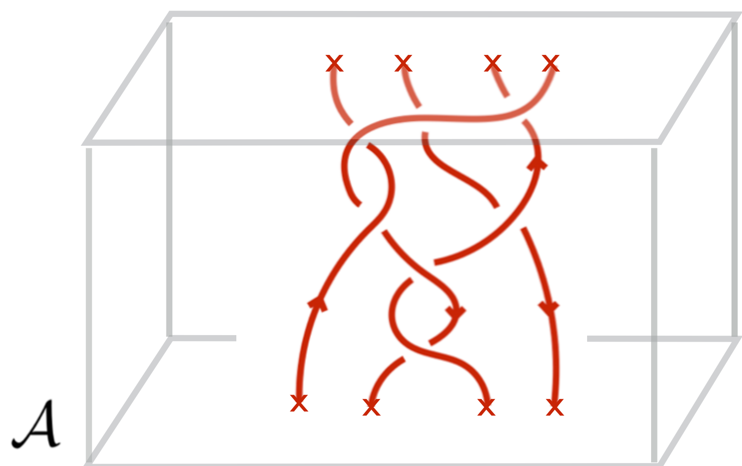
The braiding matrix

$$\mathfrak{B}$$

describes analytic continuation of the space of solutions to the

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell} \mathcal{V} = \sum_{j\neq \ell} r_{\ell i}(a_\ell / a_j)\, \mathcal{V}.$$

Knizhnik-Zamolodchikov equation:



along the path corresponding to the braid.

Such monodromy problems are in general very hard.

In the case of the Knizhnik-Zamolodchikov equation

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell} \mathcal{V} = \sum_{j \neq \ell} r_{\ell i}(a_\ell / a_j)\, \mathcal{V}.$$

the monodromy problem was solved in '89 by

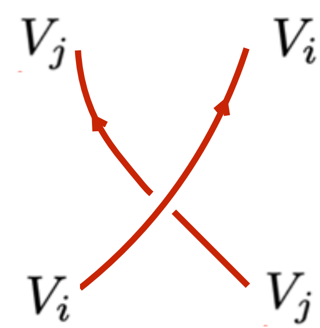
by Drinfeld and by Kazhdan and Lustig,

They showed that monodromy matrix is a product of

"R-matrices" of the quantum group

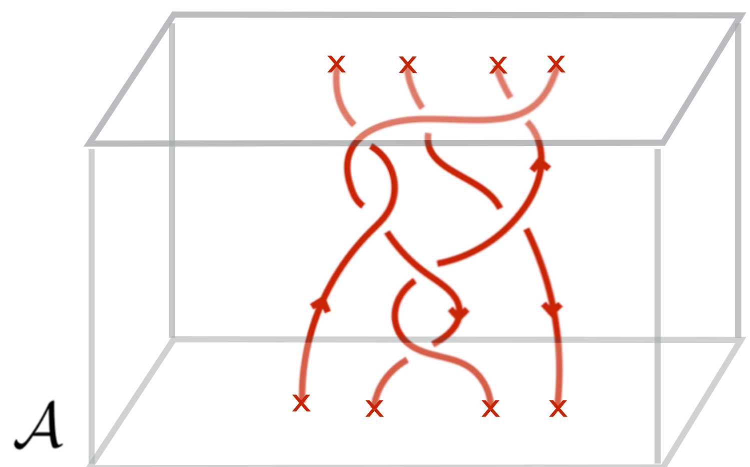
$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

corresponding to ${}^L\mathfrak{g}$



which act by exchanging a neighboring pair of punctures

In this way, Chern-Simons theory



leads to invariants of braid isotopy, based on the

$$U_{\mathfrak{q}}(^L\mathfrak{g})$$

quantum group.

The quantum group symmetry

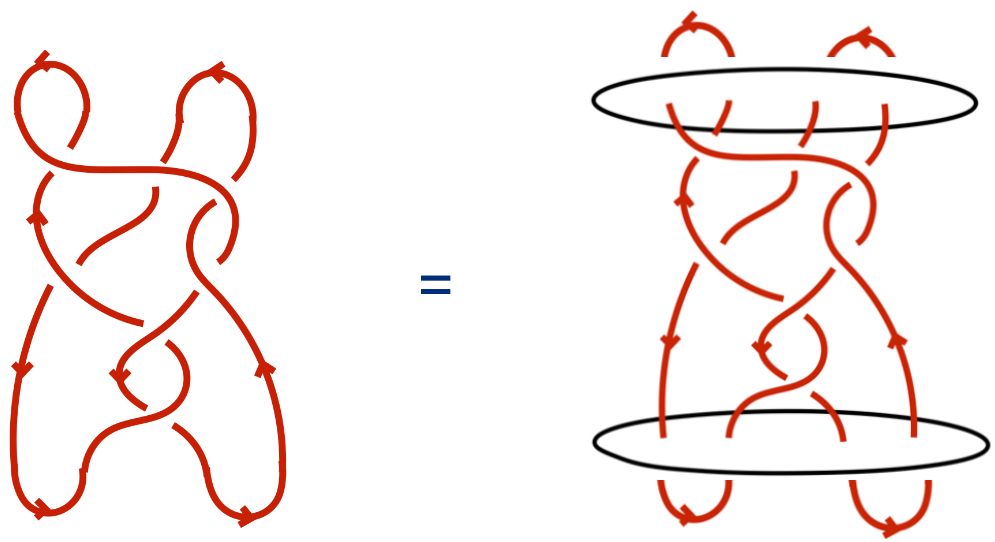
$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

is a quantum symmetry of Chern-Simons theory

which acts on its Hilbert space,

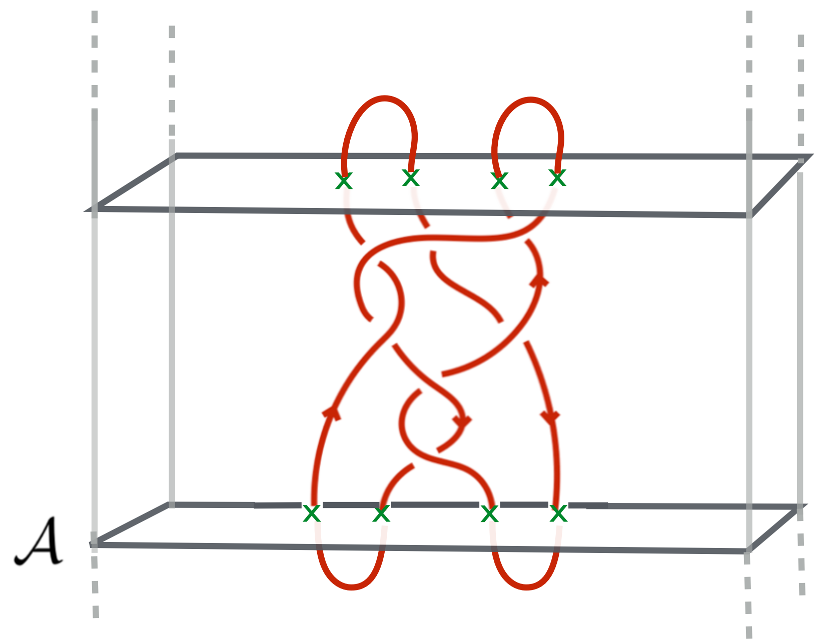
without being a symmetry manifest in the theory classically.

Any link can be represented as a



a closure of some braid.

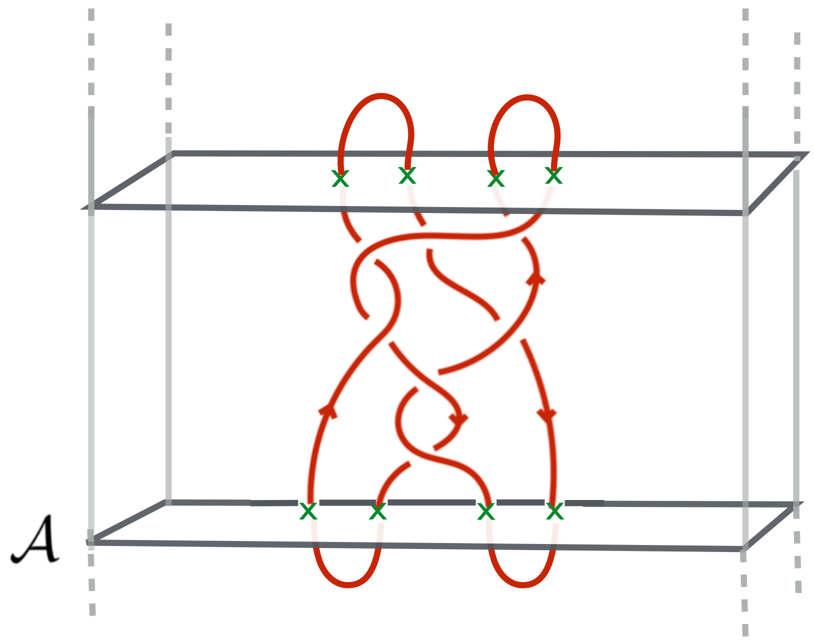
The path integral of Chern-Simons together with the link



computes a very specific braiding matrix element.

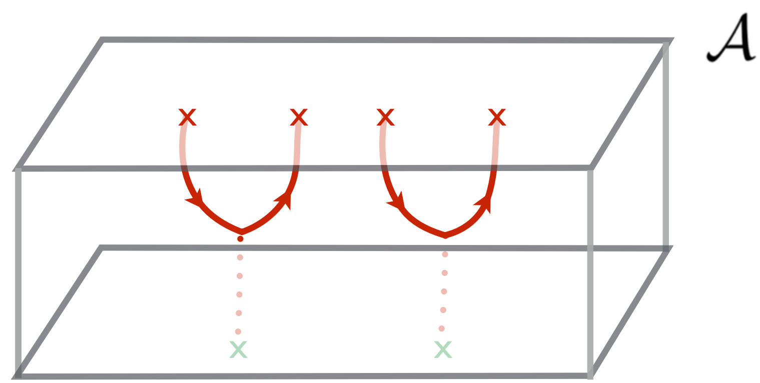
The braiding matrix element

$$(\mathfrak{U}|\,\mathfrak{B}\,\mathfrak{U})$$



is that which is picked out by the states $|\mathfrak{U})$ in the Hilbert space describing a collection of cups or caps.

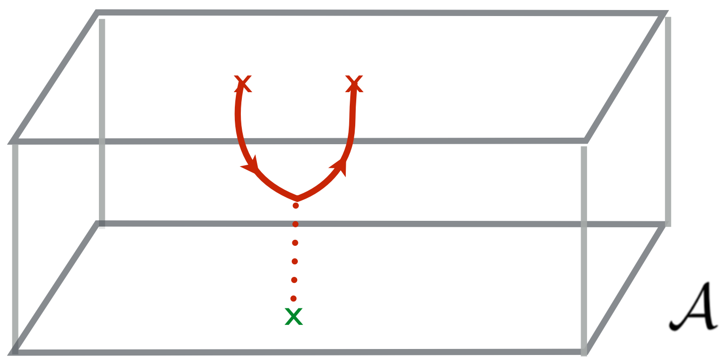
The states $|\mathfrak{U})$ in the Hilbert space



that can close off

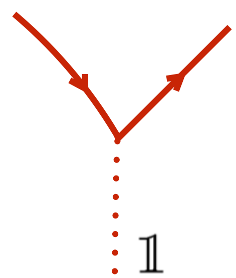
braids into links are very special solutions to the KZ equation.

The state



describes a pair of punctures on the Riemann surface

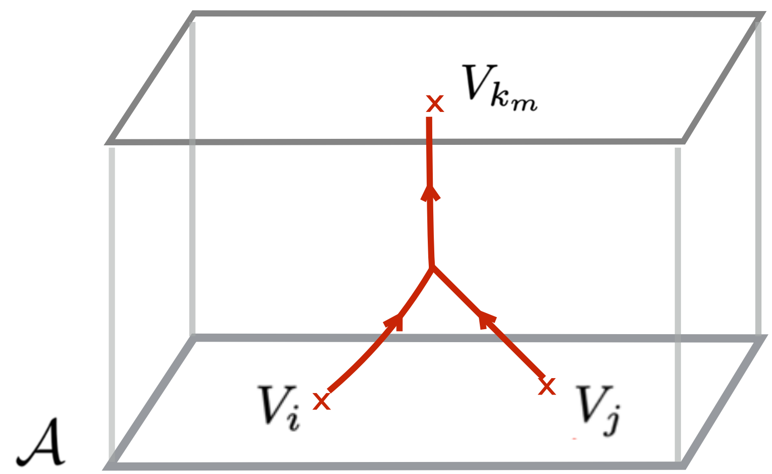
colored by complex conjugate representations



which come together and “fuse” to disappear.

This is a special instance of

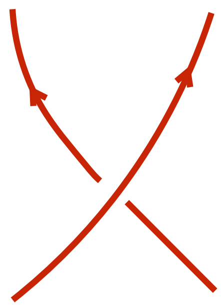
"fusion"

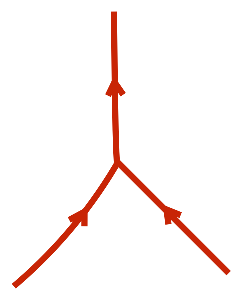


where a pair of charged particles fuse to a single one.

This way,

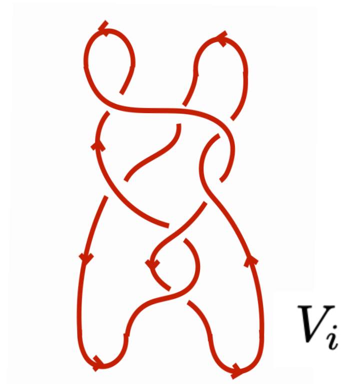
both braiding and fusion in conformal field theory





play an important role in getting knot invariants from Chern-Simons theory.

Chern-Simons knot invariants



are always Laurent polynomials

$$\sum_{j \in \mathbb{Z}} a_j(K)\, \mathfrak{q}^{j/2}$$

with coefficients that turn out to be integers.

$$a_j(K) \in \mathbb{Z}$$

This suggests that Chern-Simons theory,

or at least its link invariants

may be shadows of a deeper, more fundamental theory.

In '98 Khovanov showed one can associate to every link

a "homology theory"

which produces a collection of bi-graded vector spaces,

$$\mathcal{H}_K = \bigoplus_{i,j} \mathcal{H}_K^{i,j}$$

whose graded Euler characteristic is the Jones polynomial

$$J_K(\mathfrak{q}) = \sum_{i,j \in \mathbb{Z}} (-1)^i \mathfrak{q}^{j/2} \dim_{\mathbb{C}} \mathcal{H}_K^{i,j}$$

The integer coefficients of the Jones polynomial

$$a_j(K) = \sum_i (-1)^i \, dim_{\mathbb{C}} \, \mathcal{H}_K^{i,j}$$

are the signed counts of dimensions of knot homology groups.

Khovanov's construction is part of

"categorification program"

pioneered by Crane and I. Frenkel.

Categorification program aims to lift integers

to vector spaces,

vector spaces to "categories",

and maps between vector spaces to "functors" between categories.

A simple toy example of "categorification"

comes from a Riemannian manifold

$$M$$

whose homology groups

$$H_n(M)$$

categorify its Euler characteristic.

$$\chi(M) = \sum_n (-1)^n \dim H_n(M)$$

The homology groups

$$H_n(M) = \frac{\ker \partial_n}{\mathrm{im}\ \partial_{n+1}}$$

are constructed starting with a complex of vector spaces

$$C^\bullet = \ldots C_n \xrightarrow{\partial_n} C_{n+1} \xrightarrow{\partial_{n+1}} \ldots$$

with boundary maps between them that compose to zero.

The complex of vector spaces, obtained by a triangulation of M ,

have far more information about the geometry

than the Euler characteristic.

From physics perspective,

the Euler characteristic is the partition function

$$\chi(M) = \mathrm{Tr}(-1)^F e^{-\beta H}$$

of supersymmetric quantum mechanics with M as a target space.

A collection of vector spaces

$$C^{\bullet} = \dots C_n \xrightarrow{\partial_n} C_{n+1} \xrightarrow{\partial_{n+1}} \dots$$

is provided by

Morse theory approach to supersymmetric quantum mechanics,

as perturbative supersymmetric ground states,

indexed by the fermion number $F = n$.

The action of the Q supercharge

$$Q = \sum_n \partial_n$$

on the complex

$$C^\bullet = \ldots C_n \xrightarrow{\partial_n} C_{n+1} \xrightarrow{\partial_{n+1}} \ldots$$

is generated by instantons.

Q defines a differential as it squares to zero,

$$Q^2 = 0$$

Khovanov showed how to assign to every link

a complex of vector spaces

$$C^j_\bullet(K) = \ldots \to C^j_i(K) \xrightarrow{\partial_i} C^j_{i+1}(K) \to \ldots$$

graded by the fermion number and one additional grading,

such that their

bi-graded homology groups categorify the Jones polynomial

$$H^j_i(K) = \frac{\ker\, \partial_i}{\mathrm{im}\, \partial_{i+1}}$$

and are themselves link invariants.

Khovanov’s remarkable categorification of the Jones polynomial is explicit,

and easily calculable,

although computational complexity

grows exponentially with the number of crossings.

In 2013 Webster showed there is an

abstract algebraic framework

for categorification of quantum link invariants for arbitrary

${}^{L}\mathfrak{g}$

which, unlike Khovanov's construction,

is anything but explicit.

Despite the successes of the program

one is missing a fundamental principle that

explains why is categorification possible:

the construction has no right to exist.

Unlike in

our toy example of categorification of the Euler characteristic

of a Riemannian manifold

$$\chi(M) = \mathrm{Tr}(-1)^F e^{-\beta H}$$

Khovanov's construction and its generalizations

do not come from either geometry,

or physics in any unified way.

The problem Khovanov initiated is to find

a general framework for construction of link homology groups,

that works uniformly for all Lie algebras,

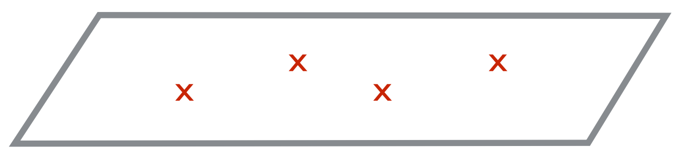
which explains what link homology groups are,

and why they exist.

To categorify quantum knot invariants,

one would like to associate

to the space conformal blocks one obtains at a fixed time slice

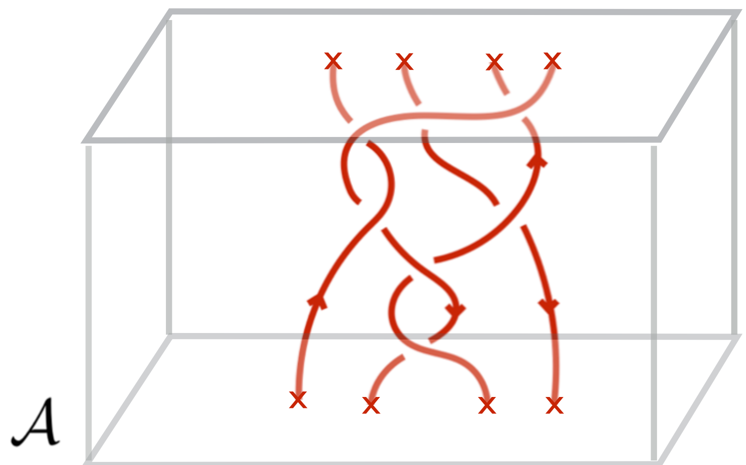
$\mathcal{A}$ 

a bi-graded category.

In addition to the usual fermion number grading

the category should have an additional grading associated to $\mathfrak{q}$

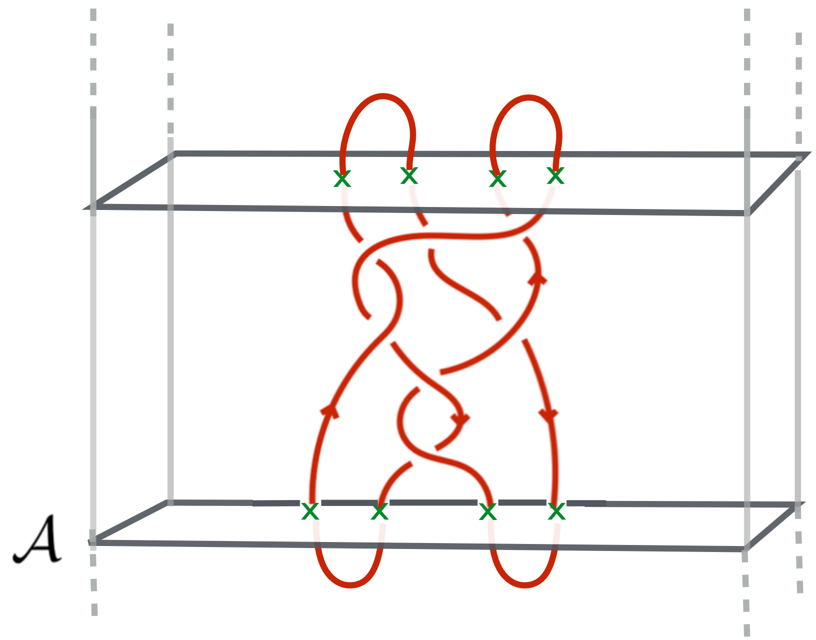
To braids,



one would like to associate functors between the categories

corresponding to the top and the bottom.

To links we would like to associate



a vector space

whose elements are "morphisms"

between the "objects" of the categories the top and the bottom,

up to the action of the braiding functor.

Moreover,

we would like to do that in the way that

recovers the quantum knot invariants upon

de-categorification.

One typically proceeds by coming up with a category,

and then one has to work to prove

that de-categorification gives

the quantum knot invariants one set out to categorify.

As I will explain,

we will find two solutions to the problem,

with virtue that the second step is automatic.

Mirror symmetry is a remarkable duality

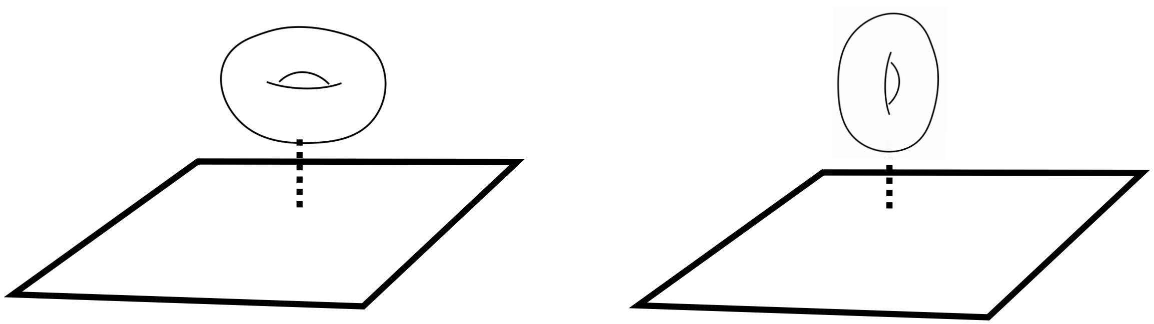
which originates from string theory.

Mirror symmetry,

relates pairs of Calabi-Yau manifolds

$\mathcal{X}$ and $\mathcal{Y}$

which are fibrations by a pair of "dual tori"



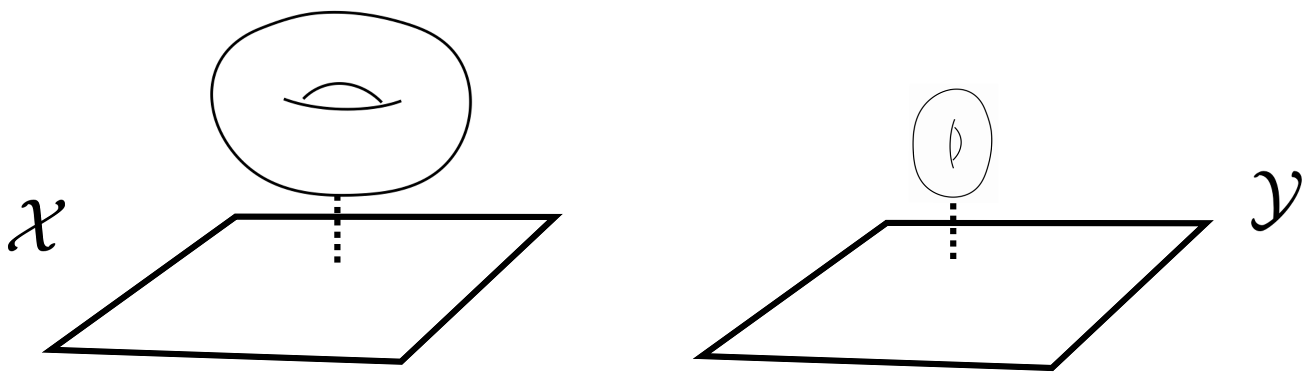
over a common base.

A theory of strings on a pair of dual tori,

has a symmetry that exchanges string winding and momentum modes.

This symmetry is why string theories based

on distinct manifolds



can be equivalent.

Calabi-Yau manifolds like

$\mathcal{X}$ and $\mathcal{Y}$

always come in families.

The family members are parameterized by choices of

complex structure "B-type"	and	symplectic or Kahler structures "A-type"

which modify the metric on the manifold.

Mirror symmetry exchanges

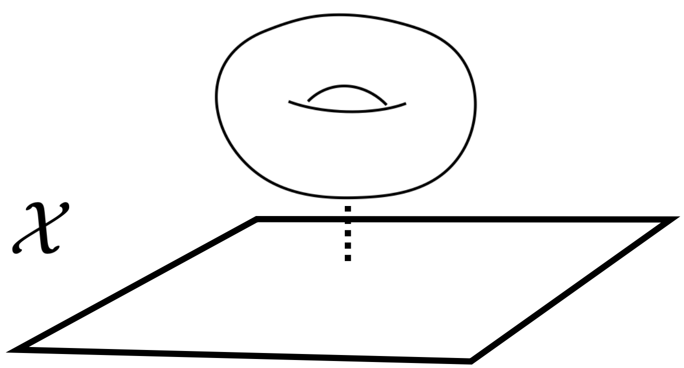
$\mathcal{X}$ and $\mathcal{Y}$

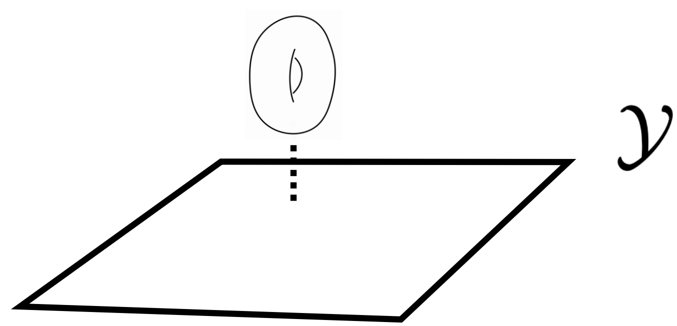
while exchanging variations of

complex structure	and	symplectic or Kahler structures
"B-type"		"A-type"

This way, mirror symmetry exchanges problems in

algebraic geometry and symplectic geometry



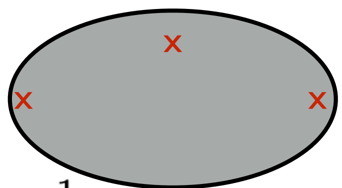


An example of such a correspondence involves

counting "rational curves"

or more precisely, of holomorphic maps

$\phi :$



$D = \mathbb{P}^1$

$$\longrightarrow \qquad \mathcal{X}$$

$$\bar{\partial}\phi = 0$$

to $\mathcal{X}$ which is a Calabi-Yau 3-fold.

This is an infinite series of difficult problems in enumerative geometry,

$$[\phi] \in H_2(\mathcal{X}, \mathbb{Z})$$

one for each degree.

In the mirror,

$$\mathcal{Y}$$

one reproduces all counts at once,

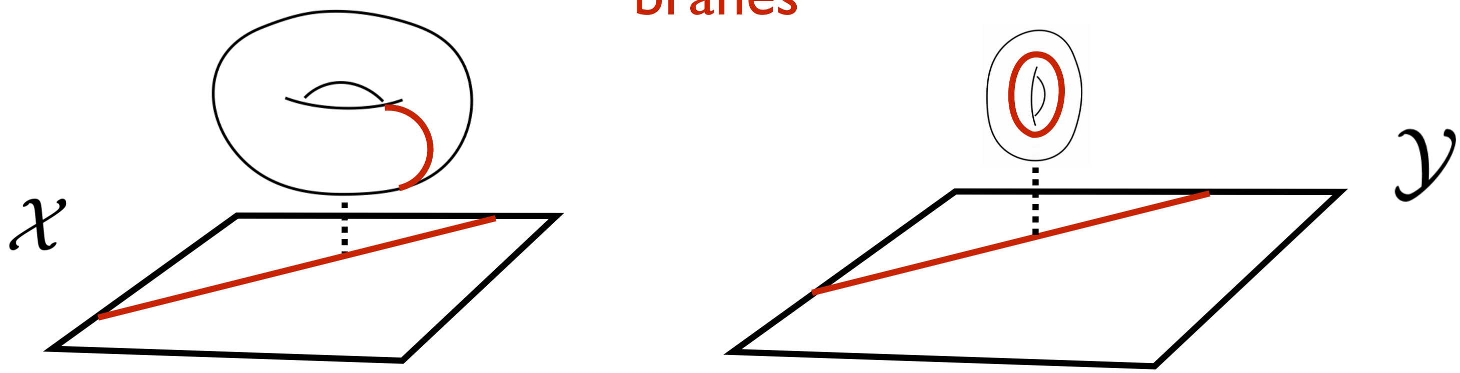
$$\sum_{d \in H_2(\mathcal{X})} (\gamma_i, \gamma_j, \gamma_k)_d \, a^d = \int_{\mathcal{Y}} \Omega \wedge \partial_i \partial_j \partial_k \Omega$$

by computing periods of the top holomorphic form

$$\Omega \in H^{3,0}(\mathcal{Y})$$

Mirror symmetry is enriched by introducing

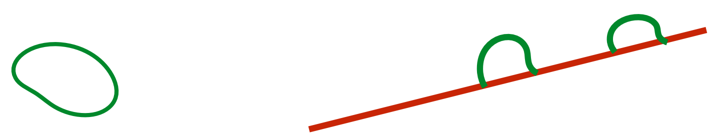
“branes”



supported on submanifolds.

A brane is a boundary condition.

By including them, one allows otherwise closed strings to have boundaries.



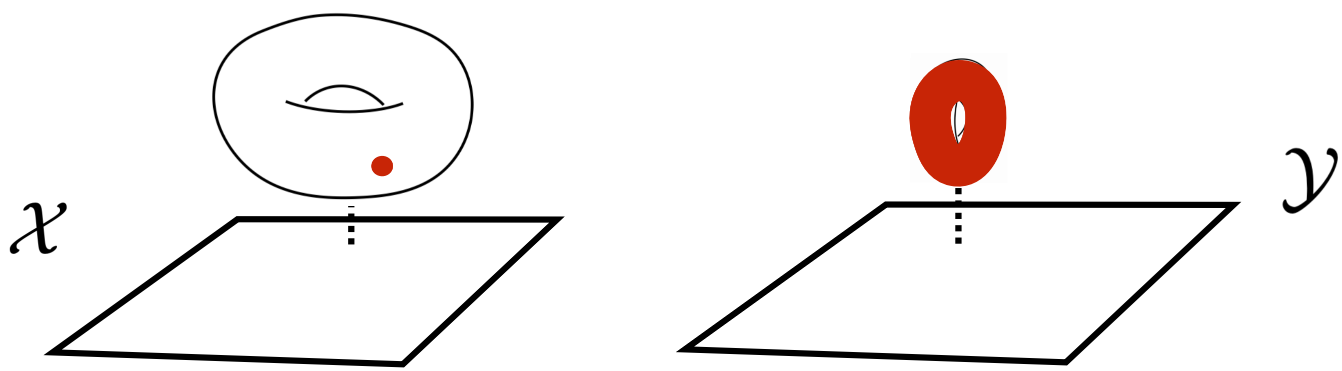
Branes are key objects in string theory,

and asking how mirror symmetry acts on them

turned out to lead to deep insights

into mirror symmetry.

One such insight was due to Strominger, Yau and Zaslow.



They showed that, in order for every pointlike brane on $\mathcal{X}$

to have a mirror brane on $\mathcal{Y}$,

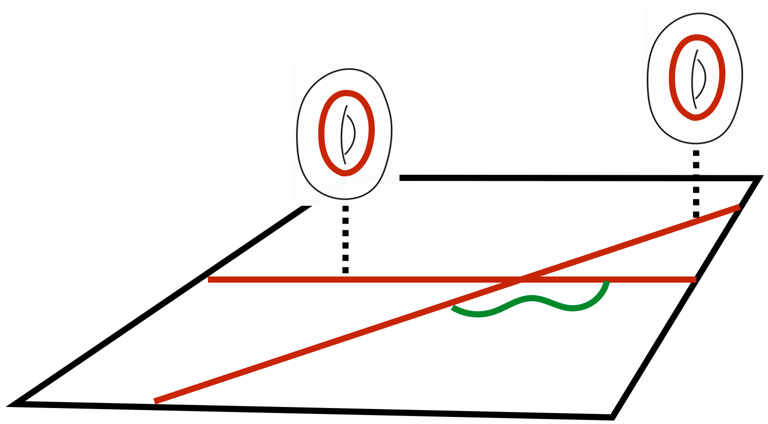
mirror pairs of manifolds must be fibered by dual tori.

A spectacular insight into mirror symmetry was provided by

Kontsevich in his ’94 ICM address.

One can regard branes on a Calabi-Yau manifold

as “objects” of a category,



whose “morphisms”

are open strings stretching between the branes.

Kontsevich conjectured that the way to understand mirror symmetry

is as equivalence

a pair of categories of branes associated to

$\mathcal{X}$ and $\mathcal{Y}$

one of which comes from complex,

and the other from symplectic geometry.

The category of branes coming from complex geometry is

"(derived) category of coherent sheaves"

whose objects are

B-branes

supported on complex submanifolds of

$\mathcal{X}$

The category of branes coming from symplectic geometry is

"derived Fukaya category"

whose objects are

A-branes

supported on real, or "Lagrangian submanifolds" of

$\mathcal{Y}$

Kontsevich's homological mirror symmetry

is a conjecture that

category of B-branes

on

$\mathcal{X}$

(the derived category of coherent sheaves)

category of A-branes

on

$\mathcal{Y}$

(derived Fukaya category)

are equivalent.

Mirror symmetry thus provides a supply of categories of geometric origin.

In an appropriate setting, they solve the knot categorification problem.

In parallel to solving the knot categorification problem

we will discover a new family of mirror pairs,

connected to representation theory,

where homological mirror symmetry

can be made as explicit as in the simplest known examples.

In the end,

I will describe the superstring theory origin

of the two approaches.

The Knizhnik-Zamolodchikov equation

$$\kappa\,\partial_i \mathcal{V}_\alpha - (r_i)_\alpha^\beta\,\mathcal{V}_\beta = 0$$

which plays a central role in knot theory

and in geometric representation theory

has a geometric counterpart.

In the world of mirror symmetry,

there is an equally fundamental differential equation

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \mathcal{V}_\beta = 0.$$

which is sometimes called

“the quantum differential equation.”

The "quantum differential equation"

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \mathcal{V}_\beta = 0.$$

is a linear differential equation for a vector valued function

over the moduli space of either

the symplectic (A-type)

$$\mathcal{X}$$

or complex structures (B-type) on

$$\mathcal{Y}$$

The name

"quantum differential equation"

comes from symplectic geometry where the coefficients in

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \mathcal{V}_\beta = 0.$$

are computed by "quantum multiplication" with a class in

$$C_i \in H^2(\mathcal{X})$$

Quantum product on $H^*(\mathcal{X})$.

$$\langle \gamma_i \star \gamma_j , \gamma_k \rangle = \sum_{d \geq 0, d \in H_2(\mathcal{X})} (\gamma_i, \gamma_j, \gamma_k)_d \; a^d$$

is defined by counting rational curves on

$$\mathcal{X}$$

The first, $d = 0$ term of quantum multiplication

$$\langle \gamma_i \star \gamma_j , \gamma_k \rangle = \sum_{d \geq 0, d \in H_2(\mathcal{X})} (\gamma_i, \gamma_j, \gamma_k)_d \, a^d$$

is the classical product on $H^*(\mathcal{X})$:

$$(\gamma_i, \gamma_j, \gamma_k)_0 = \int_{\mathcal{X}} \gamma_i \wedge \gamma_j \wedge \gamma_k$$

subsequent $d > 0$ terms are quantum corrections.

Both the equation,

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \, \mathcal{V}_\beta = 0.$$

and its monodromy problem,

featured prominently starting with the very first papers on mirror symmetry.

The solutions to the quantum differential equation

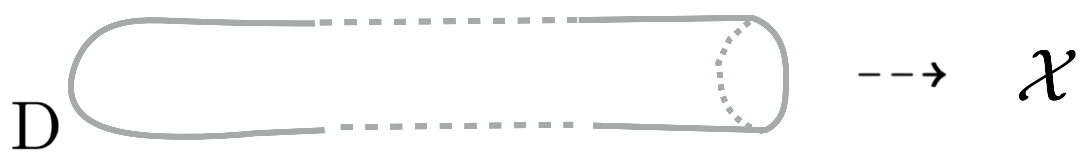
$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \mathcal{V}_\beta = 0.$$

live in a finite dimensional vector space

associated to the manifold

which is spanned by the charges of its branes.

Solutions to the quantum differential equation



are counts of holomorphic maps of all degrees

from a domain curve D which is best thought of an infinite cigar

with an S^1 boundary at infinity.

We get a specific solution of the equation

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \mathcal{V}_\beta = 0.$$

by choosing a B-type brane

as the boundary condition at infinity,

.

The solution depends on the brane only through its charge,

and not the details of its shape.

The Knizhnik-Zamolodchikov equation

$$\kappa\,\partial_i \mathcal{V}_\alpha - (r_i)_\alpha^\beta\,\mathcal{V}_\beta = 0$$

not only has the same flavor

as the quantum differential equation:

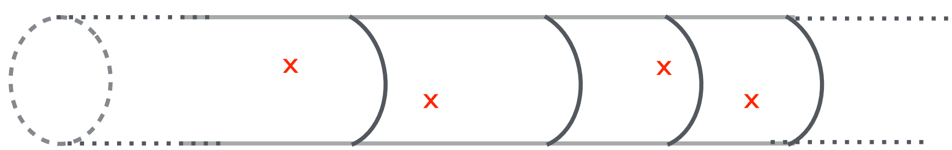
$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta\,\mathcal{V}_\beta = 0.$$

under certain conditions, they coincide.

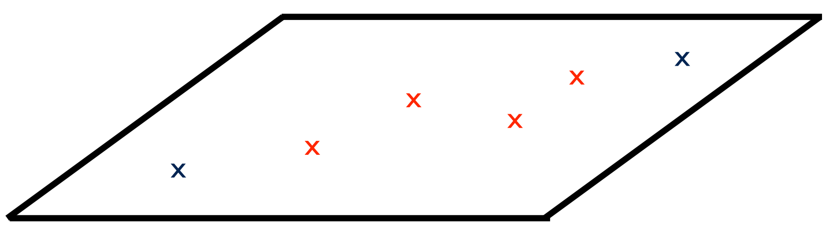
On the knot theory side, we want to take

the Riemann surface to be a punctured infinite cylinder,

$\mathcal{A}$



rather than a complex plane with punctures.



This enriches the theory, allowing it to describe

invariants of knots in

$$\mathbb{R}^2 \times S^1$$

and not only in

$$\mathbb{R}^3$$

On the geometric side, we want to take the target manifold

$$\mathcal{X}$$

to be the moduli space of G -monopoles on

$$\mathbb{R}^3$$

The Chern-Simons gauge group

$$^L G$$

whose Lie algebra was $^L\mathfrak{g}$ is related to

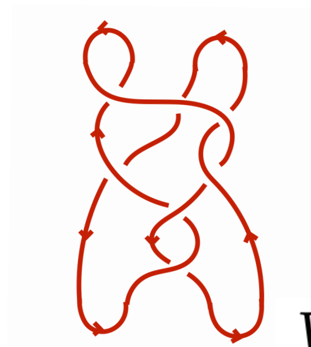
$$G$$

by Langlands, or

electric-magnetic type duality.

In Chern-Simons theory,

view the knots in three dimensional space

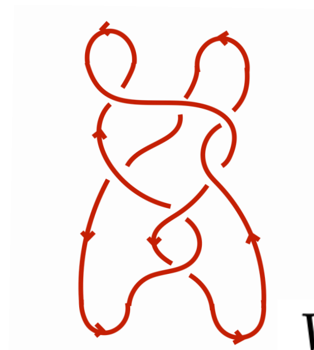


V_i

as paths of heavy particles electrically charged under

^{L}G

In the geometric description, the same heavy particles



V_i

appear as Dirac monopoles of the Langlands dual group

G

This magnetic description is key

to categorification.

The manifold

$$\mathcal{X}$$

has played an important role in mathematics before,

in geometric Langlands correspondence

which is an area of mathematics that studies

incarnations of electric-magnetic duality

in geometry and in representation theory.

There, it is known as a

“transversal slice to affine Grassmanian of G.”

The monopole moduli space

$$\mathcal{X}$$

is parameterized in part by positions of some number of smooth

't Hooft-Polyakov type monopoles on

$$\mathbb{R}^3$$

whereas positions of singular, Dirac-type monopoles are fixed,

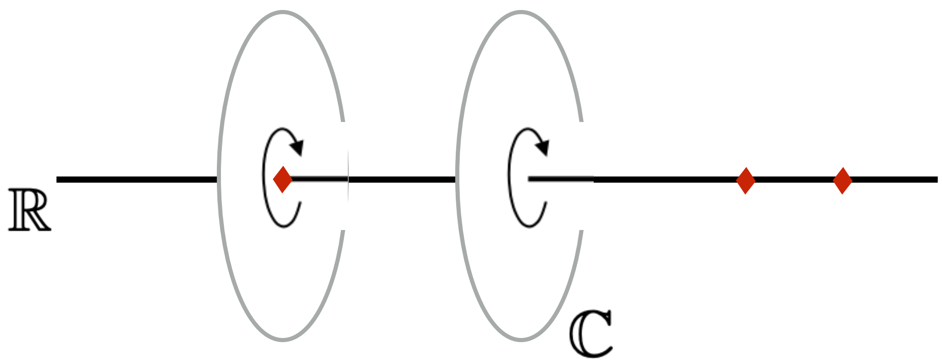
and determine the metric on $\mathcal{X}$

To get the KZ equation to coincide with the quantum differential equation of

$\mathcal{X}$

we want to place all the singular, Dirac-type monopoles at the origin of

a complex plane in $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$

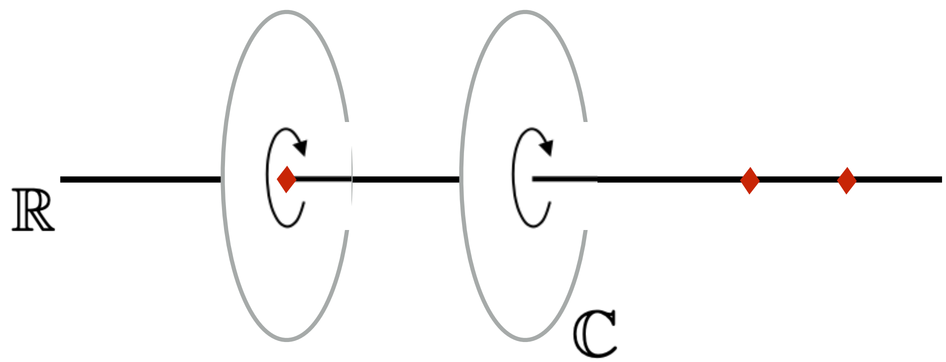


Then, rotations of this plane lead to an isometry of $\mathcal{X}$

The parameter

$\mathfrak{q}$

of knot theory will be related to keeping track of charges of states



under this symmetry.

The fact that quantum group invariants

become interesting only for

$$\mathfrak{q} \neq 1$$

has a geometric counterpart.

Because

$$\mathcal{X}$$

has more symmetries than a typical Calabi-Yau,

(it is "hyper-Kahler" as opposed to just Kahler)

the quantum cohomology theory is interesting,

in that it differs from classical cohomology

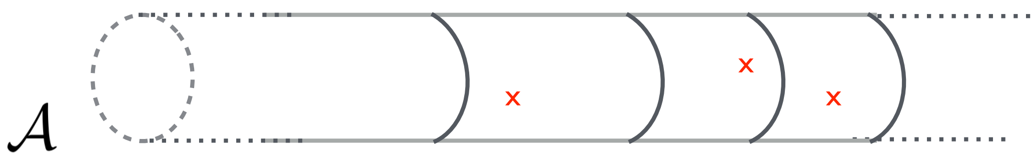
only as long as

$$\mathfrak{q} \neq 1$$

The fact that Knizhnik-Zamolodchikov equation of

$$\widehat{L\mathfrak{g}}$$

for a Riemann surface which is



has a geometric interpretation as the quantum differential equation

$$\mathcal{X}$$

is a recent theorem by Ivan Danilenko.

The positions of punctures on the Riemann surface

$\mathcal{A}$ 

turn out to coincide with the

(complexified) Kahler moduli of

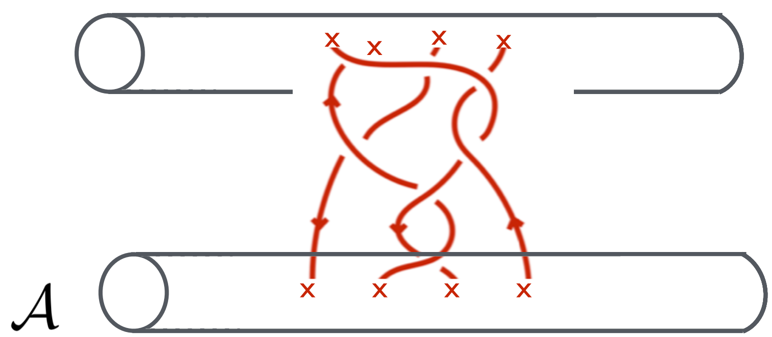
$\mathcal{X}$

It follows that a braid

B

has a geometric interpretation

as a path in (complexified) Kahler moduli



since these moduli of

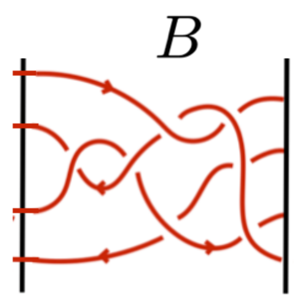
$\mathcal{X}$

are the relative positions of punctures on $\mathcal{A}$

A central expectation in mirror symmetry,

is the fact that

monodromy of the quantum differential equation along the path



is categorified by a functor acting on the category of branes,

$$\mathscr{B} : \mathscr{D}_{\mathcal{X}} \to \mathscr{D}_{\mathcal{X}'}$$

which "transports" the category along the path

and which is an equivalence.

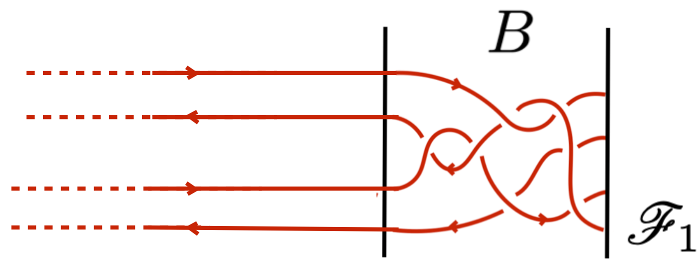
Proving this in the category of B-branes is difficult,

although nobody doubts it is true.

One can understand why one expects that

physically as follows.

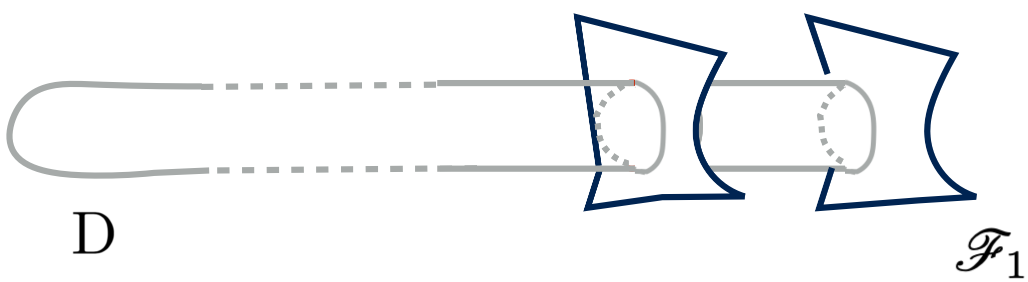
Braid group action is realized physically



in the sigma model on the cigar

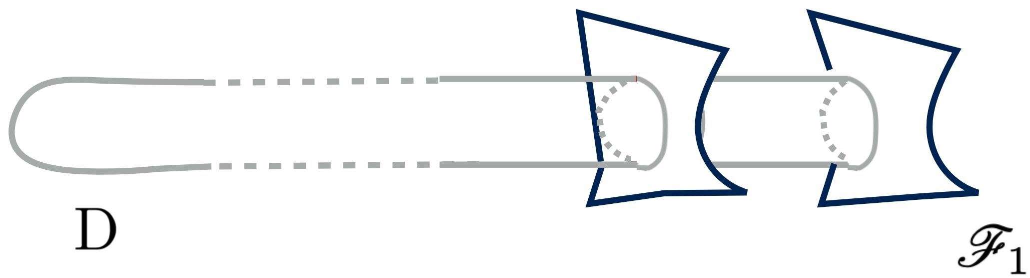
by letting the moduli of the theory vary

according to the braid near the boundary at infinity.

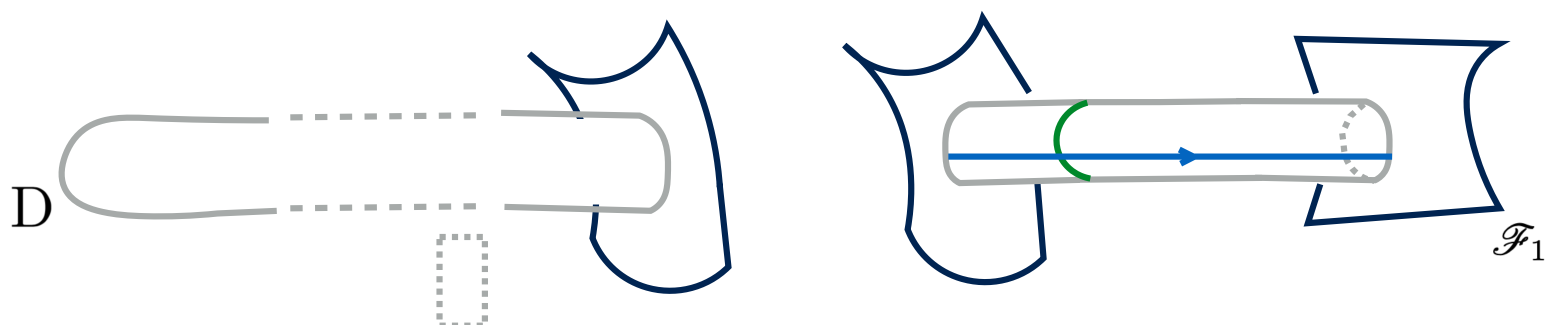


The direction along the cigar coincides with the "time" along the braid.

One can cut the infinite cigar

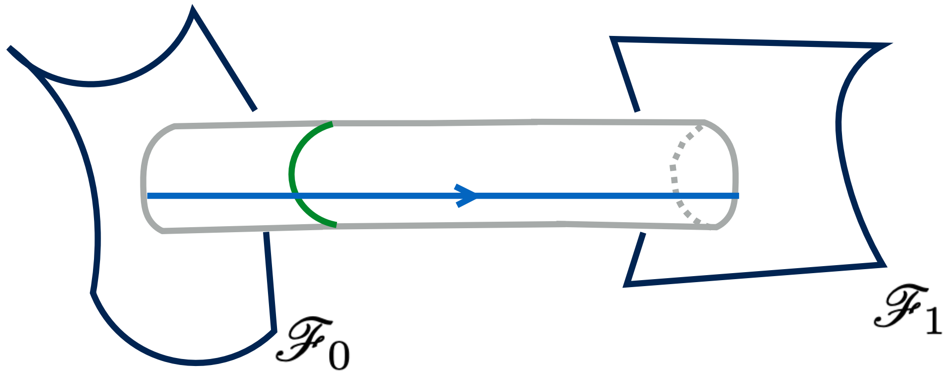


very near the boundary, by inserting a "complete set of branes",



to extract matrix elements.

Thus, sigma model on the annulus



with moduli that vary according to the braid computes

the matrix element of the monodromy

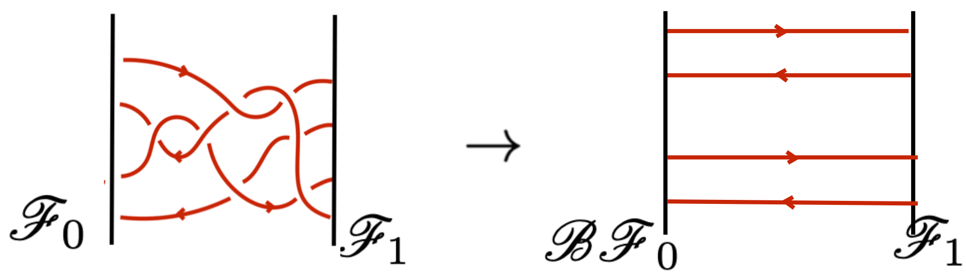
$$\mathfrak{B}$$

between a pair of branes

For this, we view the time to run along the annulus.

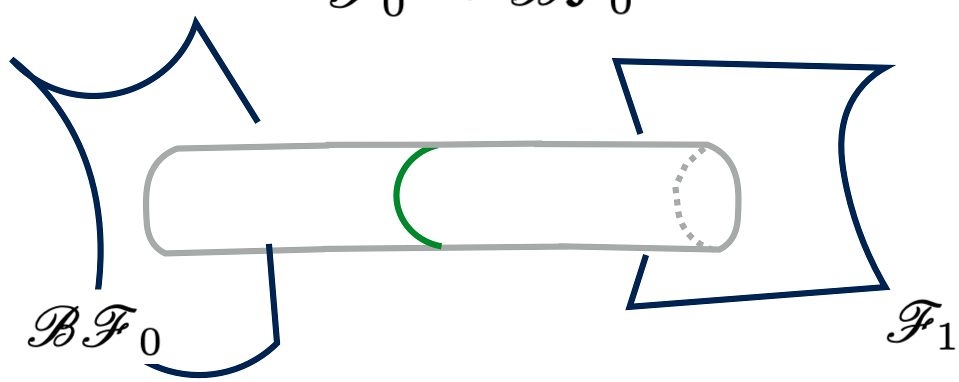
It turns out one can take all the variation of the moduli

to happen near one of two boundaries,



at the expense of changing a boundary condition,

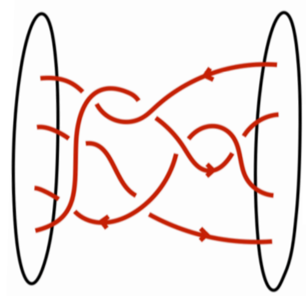
$$\mathscr{F}_0 \rightarrow \mathscr{B}\mathcal{F}_0$$



The change of the boundary condition

$$\mathcal{F} \to \mathscr{B}\mathcal{F}$$

a functor $\mathscr{B}$, associated to the braid



B

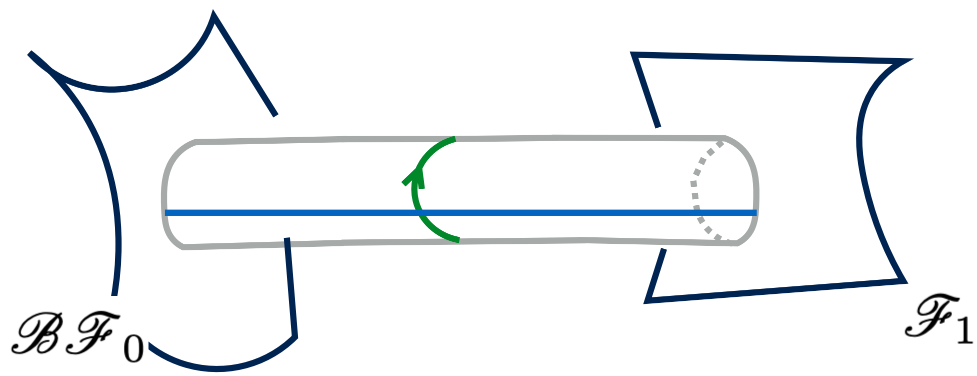
The functor is an equivalence of categories

since the category of B-type branes turns out to be independent of

the moduli being varied.

The theory is euclidian, so we can equally well take the

the time to run around the S^1 ,

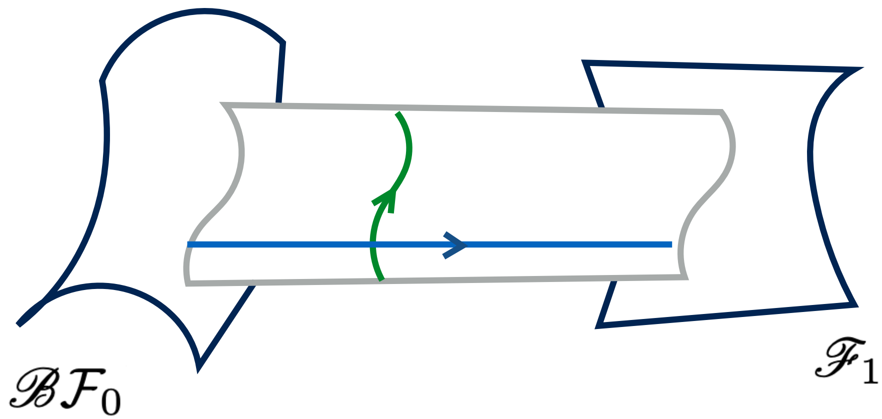


Viewed this way, the path integral computes the supertrace

$$\mathrm{Tr}(-1)^F e^{-\beta H}$$

which is the index of a supercharge Q preserved by the two branes.

The cohomology of the supercharge Q



is the basic ingredient in the category of branes

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

a graded vector space,

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

the space of morphisms between a pair of branes.

While a mathematical proof of this is not available for a general

Calabi-Yau

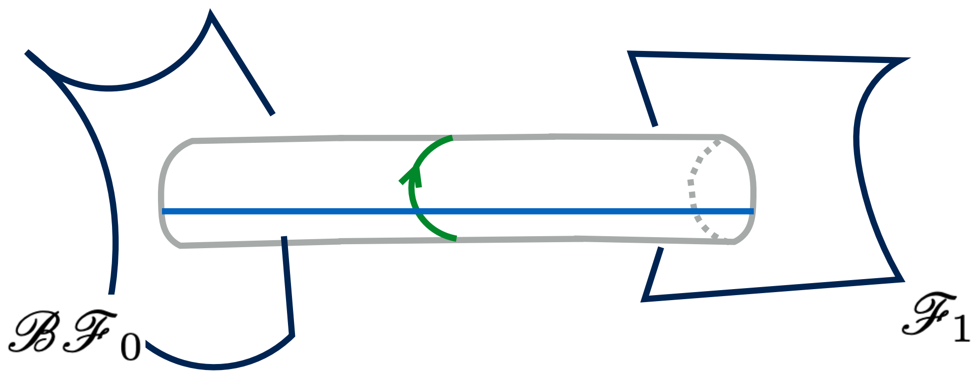
$\mathcal{X}$

in our specific setting,

the proof was given by Bezrukavnikov and Okounkov

(it uses quantization in characteristic p).

Thus, by viewing the same annulus two different ways



we learn that the braid group action on the category of branes

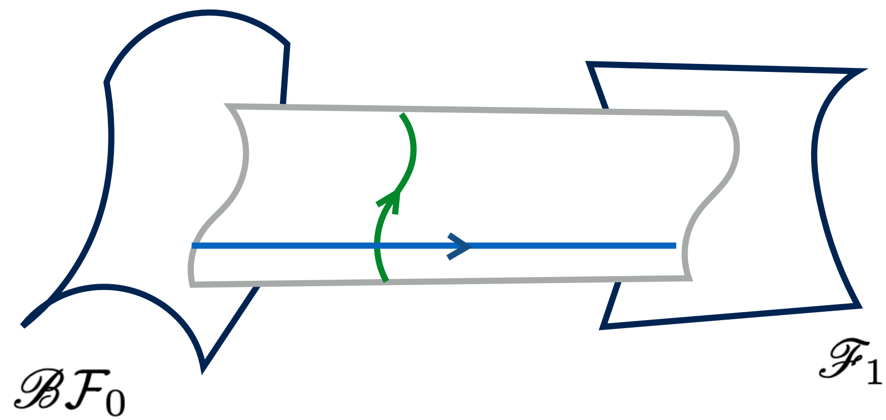
$$\mathscr{B}$$

manifestly categorifies the monodromy matrix

$$\mathfrak{B}$$

of the Knizhnik-Zamolodchikov equation.

By cutting the annulus open,



we find a graded vector space,

$$Hom^{*,*}_{\mathscr{D}\mathcal{X}}(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

the space of morphisms between a pair of branes,

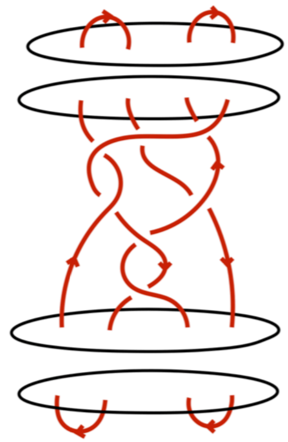
whose Euler characteristic

$$\chi(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

is the braiding matrix element.

The quantum invariants of links should be categorified by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$



since they too can be expressed as matrix elements of the braiding matrix

$$(\mathfrak{U}_1, \mathfrak{B}\,\mathfrak{U}_0)$$

between pairs of conformal blocks.

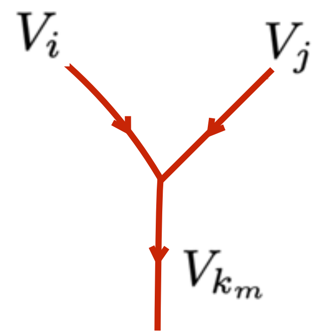
The first step is to find objects of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

whose vertex functions are conformal blocks

in which pairs of vertex operators fuse to trivial representation.

I showed that fusion

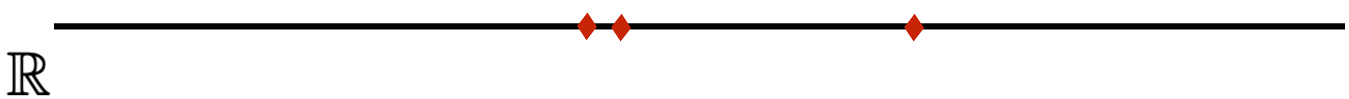


also has a natural geometric interpretation in terms of

$$\mathcal{X}$$

and its category of B-type branes.

As one brings a pair of singular monopoles close together



our manifold

$\mathcal{X}$

develops a singularity where a collection of cycles vanishes,

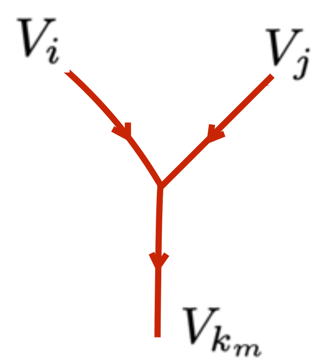
as the distance between the monopoles

controls their size.

The singularities occur due to monopole “bubbling”

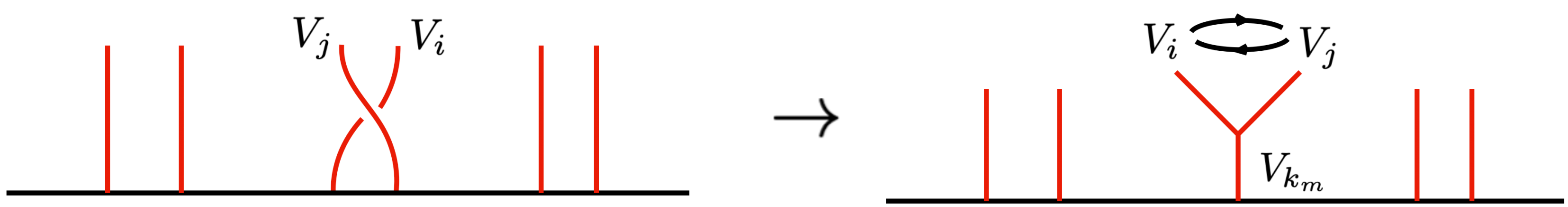
When a pair of singular monopoles come together,

a number of smooth monopoles can bubble off of them and disappear.



to leave behind a single singular monopole of lower charge.

In conformal field theory fusion diagonalizes braiding.



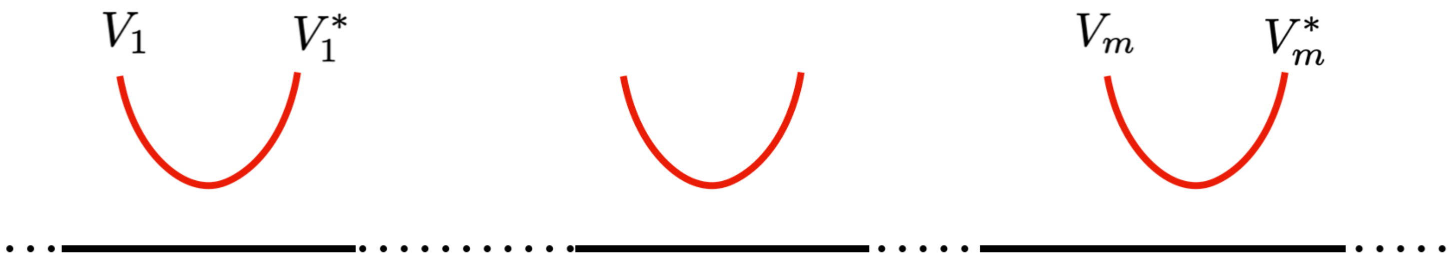
The analogue of this in the category of branes is

turns out to be existence of a “perverse filtration”

envisioned by Chuang and Rouquier

in abstract terms.

One learns from this that cups and caps



come from branes on

$$\mathcal{X}$$

with simple geometric meaning.

They are branes supported on vanishing cycles

$$\mathcal{U} = \mathcal{O}_U$$

known as "minuscule Grassmanians"

which shrink to a point as punctures come together in pairs.

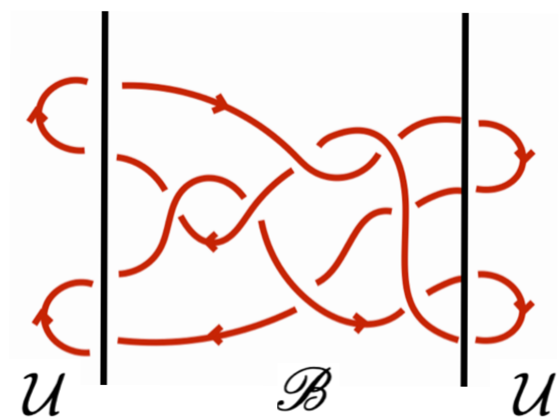
Using very special properties of perverse filtrations

and these vanishing cycle branes

I proved that not only do the homology groups

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{U},\mathcal{U})$$

manifestly categorify the corresponding $U_{\mathfrak{q}}({}^{L}\mathfrak{g})$ link invariants,



they are themselves link invariants.

Recently, Ben Webster proved that homological link invariants

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{U}, \mathcal{U})$$

that come from B-type branes on

$$\mathcal{X}$$

are equivalent to algebraic invariants he defined in '13,

using an algebra

$$\mathscr{A}$$

known as the KLRW algebra.

As stated, neither the approach by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

nor by KRLW algebras is very explicit.

Neither are really amenable to any calculations.

In the rest of the talk I will explain how to

reformulate the problem,

and solve the theory.

The resulting description is completely new.

To solve the theory

we will make use of homological mirror symmetry,

or more precisely,

an equivariant version of it.

Homological mirror symmetry is a statement

that a pair of categories

$\mathscr{D}_{\mathcal{X}}$

category of B-branes

supported on

complex submanifolds of

$\mathcal{X}$

$\mathscr{D}_{\mathcal{Y}}$

category of A-branes

supported on real,

or "Lagrangian" submanifolds" of

$\mathcal{Y}$

are equivalent:

$$\mathscr{D}_{\mathcal{X}} \cong \mathscr{D}_{\mathcal{Y}}$$

In the very best instances,

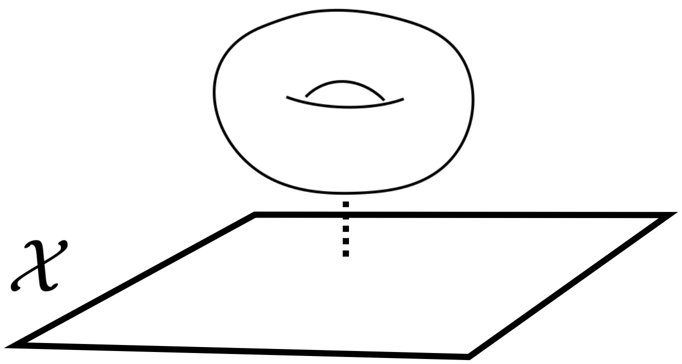
one learns how to make homological mirror symmetry manifest,

and both theories based on

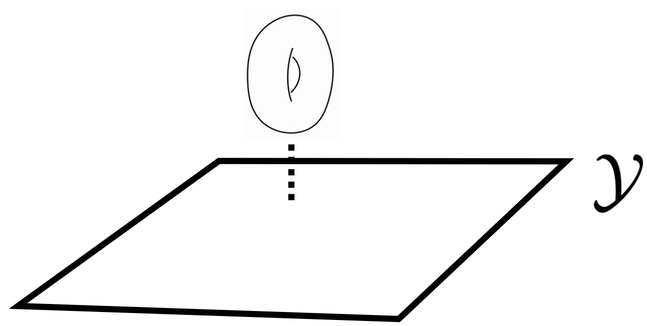
$\mathcal{X}$ and on $\mathcal{Y}$

solvable exactly.

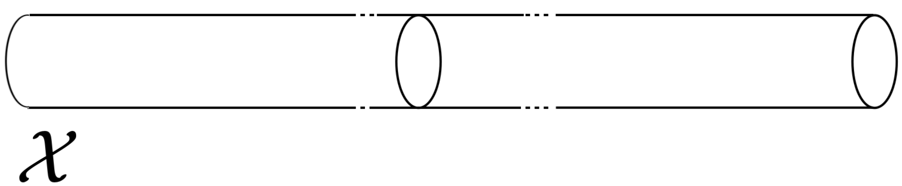
One of the very simplest examples of homological mirror symmetry is when

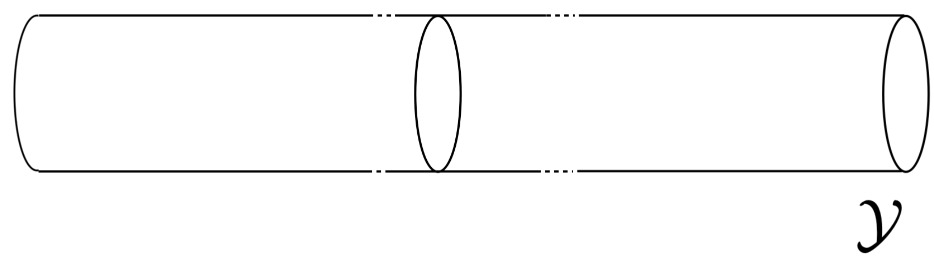


and



are taken to be simply a pair of infinite cylinders,

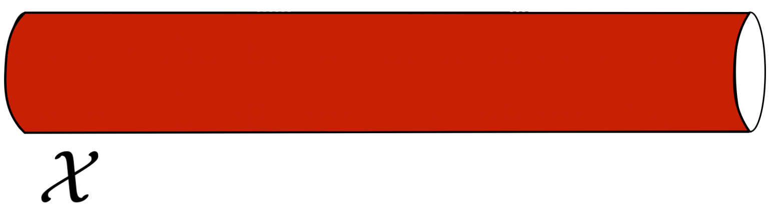


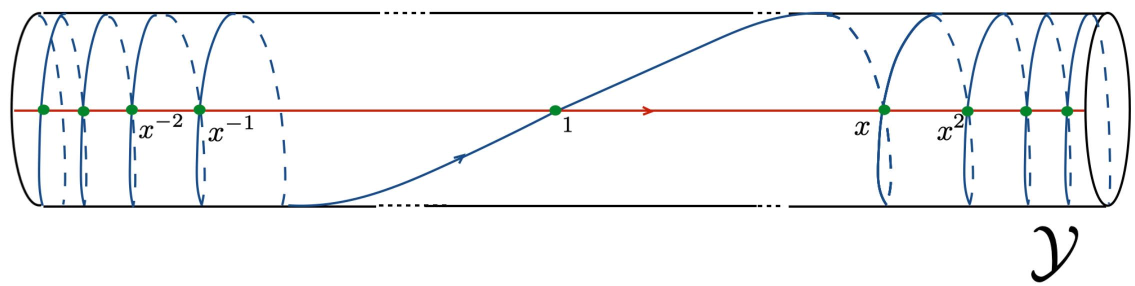


their torus fibers being simply circles.

Categories of branes on the two sides

are each generated by a single brane:





While the branes look different,

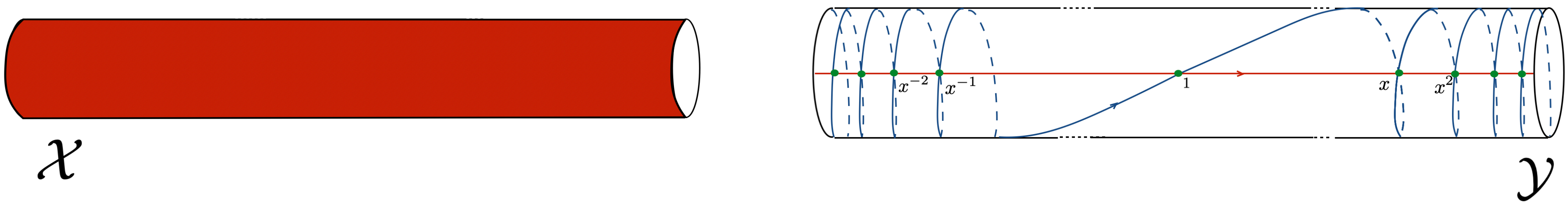
their algebras of open strings ending on this brane

$$\mathscr{A} = \mathbb{C}[x, x^{-1}]$$

are the same.

The algebra is simply the algebra of functions on a complex cylinder

The fact that the algebras of open strings are the same on both sides



turns out to mean that the entire categories of branes are equivalent,

$$\mathscr{D}_{\mathcal{X}} \cong \mathscr{D}_{\mathscr{A}} \cong \mathscr{D}_{\mathcal{Y}}$$

both being equivalent to a (derived) category of modules of the algebra

$$\mathscr{A} = \mathbb{C}[x, x^{-1}]$$

of functions on a complex cylinder.

This simple example is the model for

how one hopes to understand homological mirror symmetry in all cases.

Webster's proof of equivalence of categorification

of $U_{\mathfrak{q}}({}^{L}\mathfrak{g})$ link invariants

via B-type branes on

$$\mathcal{X}$$

and via KLRW algebra

$$\mathscr{A}$$

is really the first of the two equivalences in homological mirror symmetry:

$$\mathscr{D}_{\mathcal{X}} \cong \mathscr{D}_{\mathscr{A}} \cong \mathscr{D}_{\mathcal{Y}}$$

As we are after a simpler and more direct

approach to homological link invariants,

we will not try to describe

$$\mathscr{D}_{\mathcal{Y}}$$

or aim to complete the other half of homological mirror symmetry

$$\mathscr{D}_{\mathcal{X}} \cong \mathscr{D}_{\mathscr{A}} \cong \mathscr{D}_{\mathcal{Y}}$$

Recall that

$$\mathcal{X}$$

is a moduli space of monopoles on

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

where a symmetry that

corresponds to rotations of

$$\mathbb{C}$$

plays a key role — this is how we got $\mathfrak{q}$ into the problem.

This means that all the relevant information about the geometry of

$$\mathcal{X}$$

is much more efficiently contained in a small

"core" subspace,

$$X \subset \mathcal{X}$$

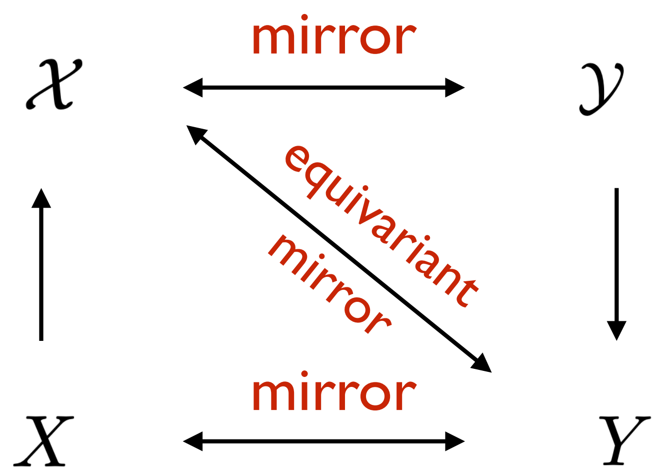
of half the dimension where all monopoles,

singular or not, are at the origin of $\mathbb{C}$ and at points in $\mathbb{R}$

We will define the equivariant mirror of $\mathcal{X}$, call it

$$Y$$

to be the ordinary mirror of its core:



The key fact is that

the bottom row has as much information about the geometry as the top.

The common base torus fibrations of

X and of Y

parameterizes positions of smooth monopoles on a real line:

$\mathbb{R}$ 

in presence of some singular ones.

The smooth monopoles are labeled by simple roots of

${}^L\mathfrak{g}$

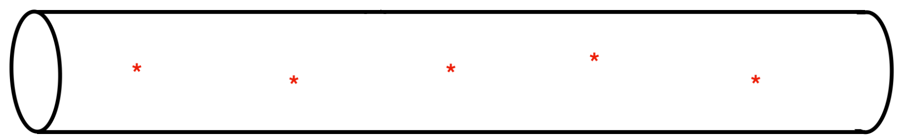
and otherwise identical.

The equivariant mirror

$$Y$$

is (a cousin of) configuration space of points on

$\mathcal{A}$



our Riemann surface with punctures

"colored" by simple roots of ${}^L\mathfrak{g}$ but otherwise indistinguishable,

(with some locus deleted and singularities resolved).

There is a potential on

$$Y$$

which makes the mirror theory into a "Landau-Ginzburg" model,

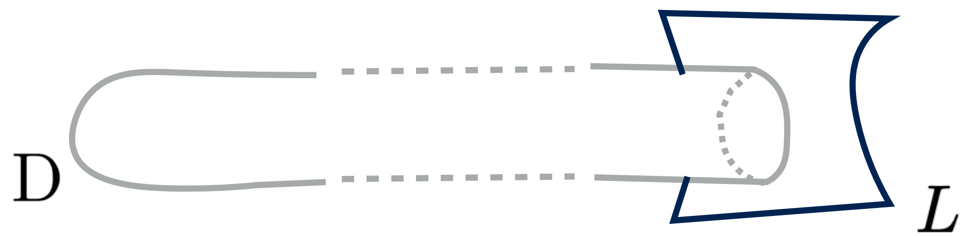
$$W = \lambda_0 W_0 + \sum_{a=1}^{\mathrm{rk}} \lambda_a W^a$$

which is a multi-valued holomorphic function

Corresponding to a solution of the

Knizhnik-Zamolodchikov equation

is an A-brane at the boundary of D at infinity,



The brane is an object of the category of A-branes

$$\mathscr{D}_Y$$

the "derived Fukaya-Seidel category" of Y with potential W .

The mirror description based on

$$Y$$

leads to "integral" formulation of conformal blocks of

$$\widehat{^L\mathfrak{g}}$$

as period integrals.

$$\mathcal{V}_\alpha[L] = \int_L \Phi_\alpha \, \Omega \, e^{-W}$$

This gives a

geometric interpretation to works of Feigin and E.Frenkel in the '80's

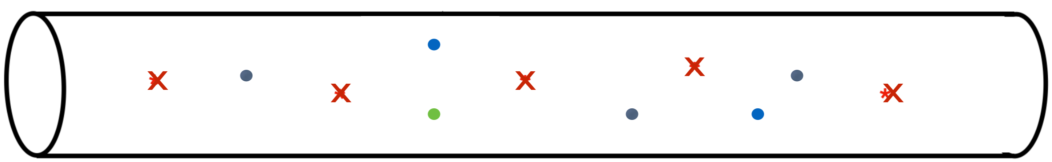
and Schechtman and Varchenko.

One can describe this category very explicitly

thanks to the fact

Y

essentially a configuration space of colored points



$\mathcal{A}$

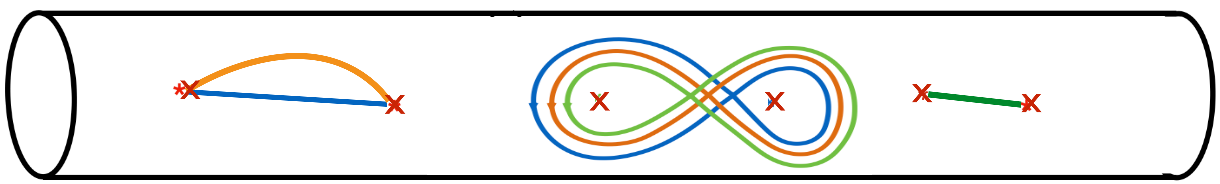
on the punctured Riemann surface.

Objects of the category of boundary conditions,

are A-branes, or Lagrangians on

$$Y$$

are all products of one dimensional curves



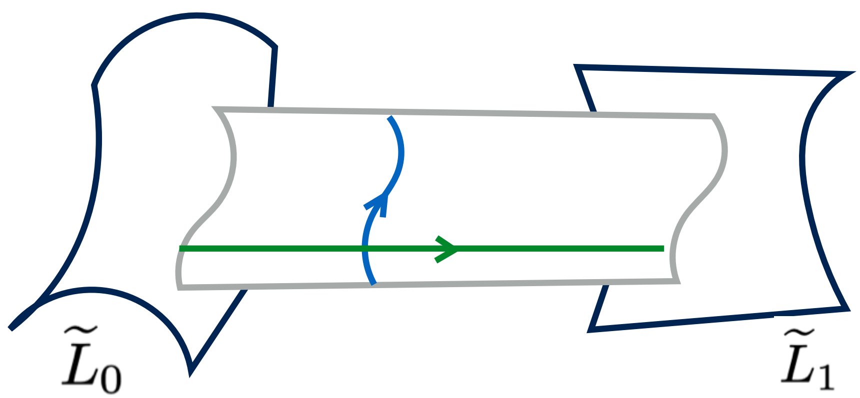
on the Riemann surface, colored by simple roots.

In any category of A-branes

the spaces of morphisms between a pair of branes

$$Hom^{*,*}_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1) = \operatorname{Ker} Q / \operatorname{Im} Q.$$

are defined by Floer theory,



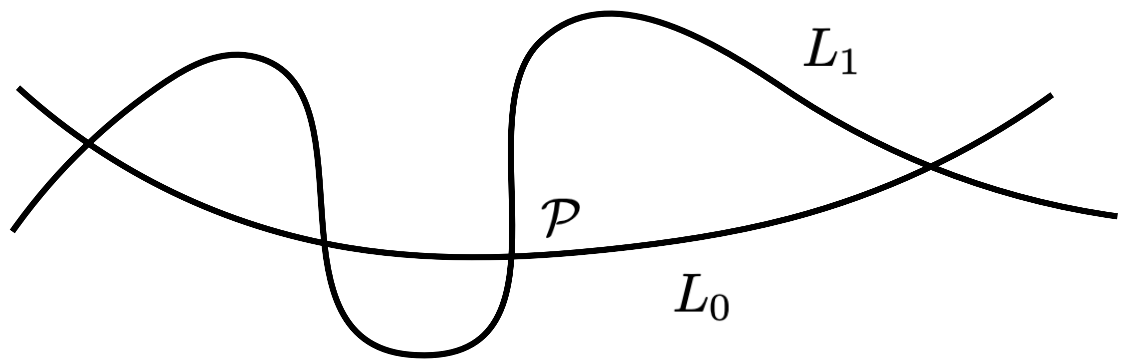
which is modeled after Morse theory approach to supersymmetric quantum mechanics.

The starting point is the Floer complex, which is the vector space

$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P} \in L_0 \cap L_1} \mathbb{C}\mathcal{P}.$$

spanned by the intersection points of the two Lagrangians,

and graded by cohomological and the q-degrees.

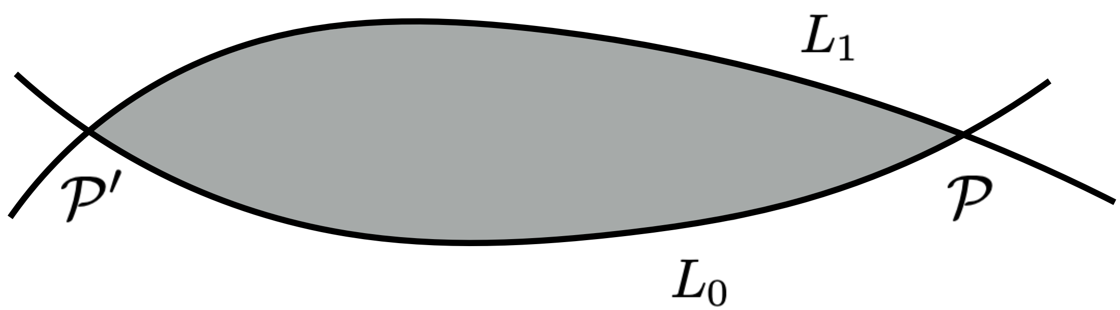


The action of the differential Q on this space

$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P} \in L_0 \cap L_1} \mathbb{C}\mathcal{P}.$$

of perturbative supersymmetric ground states

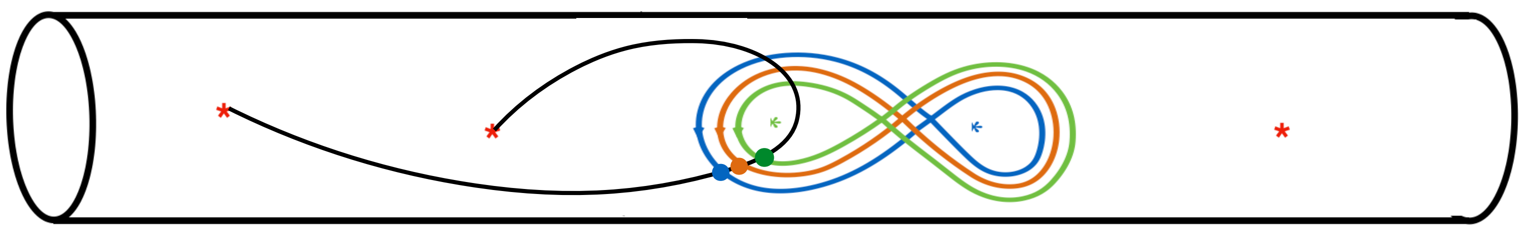
is generated by instantons,



which come from holomorphic maps from the strip to Y

A vast simplification in the present case is that

just as the branes have a description in terms of the Riemann surface



so do their intersection points,

as well as the maps between them.

The theory which results is a generalization of

"Heegard-Floer theory,"

which is associated to

$$^{L}\mathfrak{g} = \mathfrak{gl}_{1|1}$$

to arbitrary

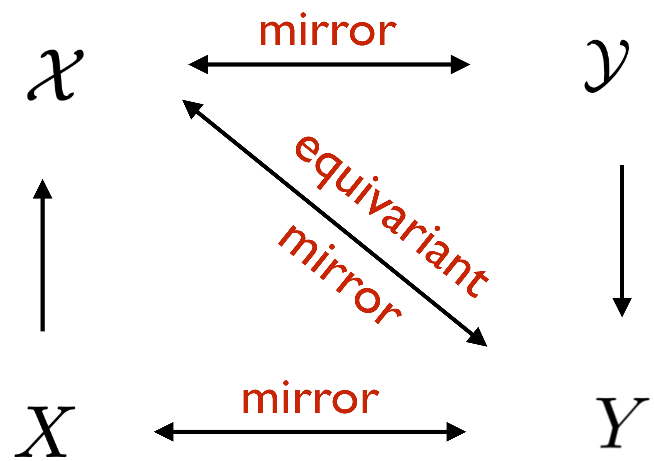
$$^{L}\mathfrak{g}$$

Heegard-Floer theory is phrased in the same, one dimensional, terms,

and categorifies the Alexander polynomial.

Mirror symmetry

helps us understand exactly which questions we need to ask



to recover homological knot invariants from Y ,

for an arbitrary simply laced Lie algebra

$^L\mathfrak{g}$

Equivariant homological mirror symmetry

relating

$\mathcal{X}$ and Y

is not an equivalence of categories,

but a correspondence of branes and associated vector spaces,

which come from a pair of "adjoint functors",

$$\mathscr{D}_{\mathcal{X}} \underset{h_*}{\overset{h^*}{\rightleftarrows}} \mathscr{D}_Y$$

Every B-brane on $\mathcal{X}$ which is relevant to us

"comes from"

an A-brane on Y via a functor,

$$\mathscr{D}_{\mathcal{X}} \xleftarrow{\;h_*\;} \mathscr{D}_Y$$

that maps the brane on Y to its mirror on X ,

and then interprets it as a brane upstairs.

Adjointness implies that

given any pair of branes on $\mathcal{X}$ that come from Y

$$\mathcal{F} = h_* L_F, \qquad \mathcal{G} = h_* L_G$$

the Homs between them, computed upstairs, in

$$\mathscr{D}_{\mathcal{X}}$$

agree with the Hom downstairs, in

$$\mathscr{D}_Y$$

after replacing L_F with $h^* h_* L_F$

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) = Hom^{*,*}_{\mathscr{D}_Y}(h^* h_* L_F, L_G)$$

For any simply laced Lie algebra

$$^{L}\mathfrak{g}$$

branes which serve as "cups" and "caps"   "upstairs" on

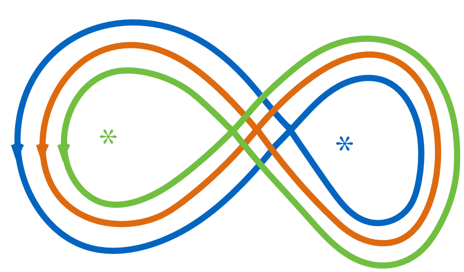
$$\mathcal{X}$$

(associated to "minuscule Grassmanians")

originate from vanishing cycle branes of the downstairs theory

that are generalizations of these interval branes,

and project back down as generalized figure eight branes.

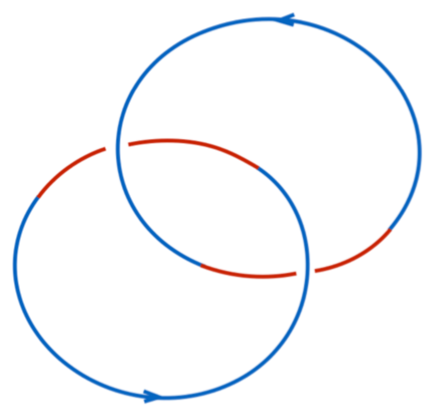


In the description based on

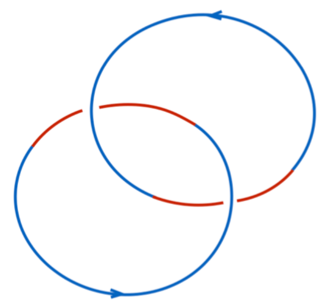
$$Y$$

both the Lagrangians and the action of braiding on them are geometric.

Start with a projection of a link to a the surface $\mathcal{A}$:



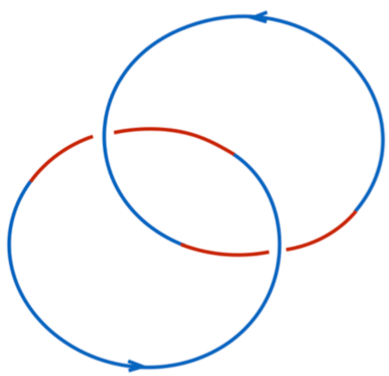
To translate it to a pair of Lagrangians, choose a bicoloring,

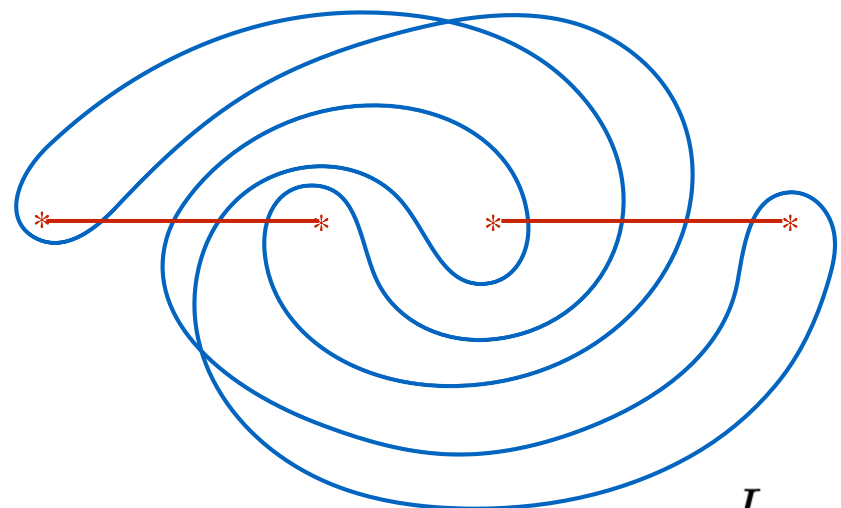


by equal number d of segments of each color,

such that red always underpasses the blue.

The mirror Lagrangians $I_{\mathcal{U}}$ and $\mathscr{B}E_{\mathcal{U}}$ are obtained by replacing all the red segments by the interval-type branes and the blue segments by figure eight-type branes:





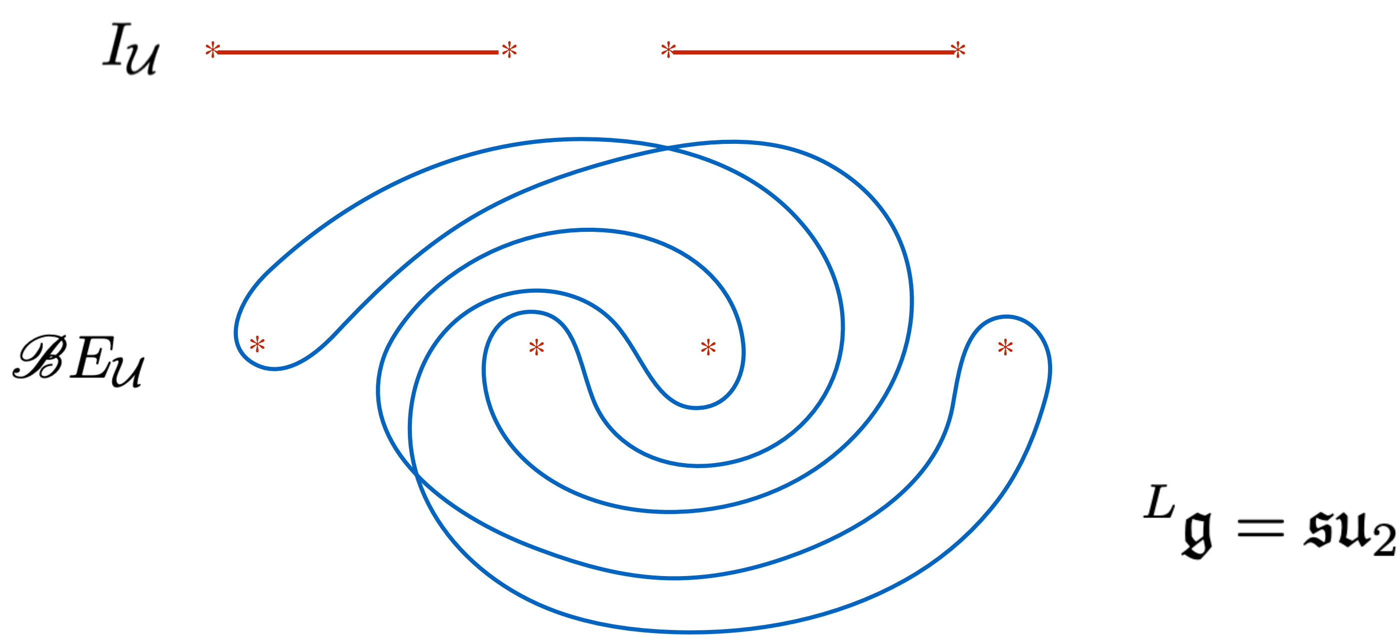
${}^L\mathfrak{g} = \mathfrak{su}_2$

The homological link invariant is

the space of morphisms

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

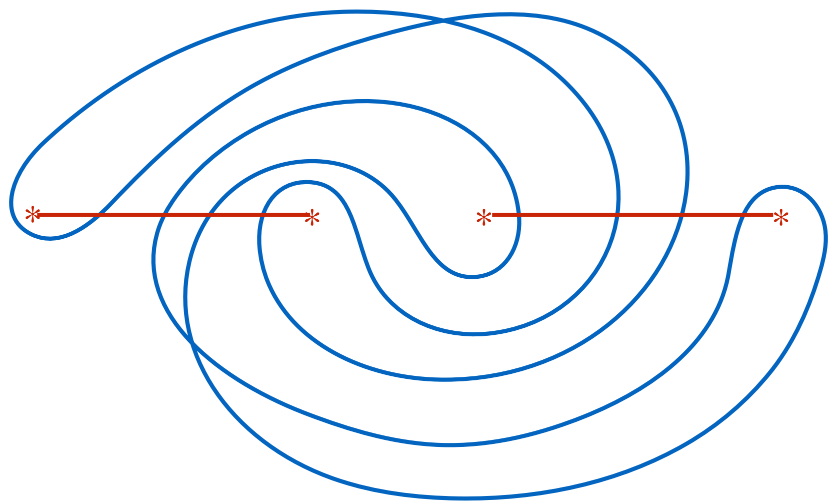
between the pair of branes.



To evaluate the Euler characteristic

one simply counts the intersection points of Lagrangians, keeping track of grading

$$\chi(E,\mathscr{B}I) = \sum_{\mathcal{P}\in E\cap\mathscr{B}I} (-1)^{M(\mathcal{P})}\mathfrak{q}^{J(\mathcal{P})}$$



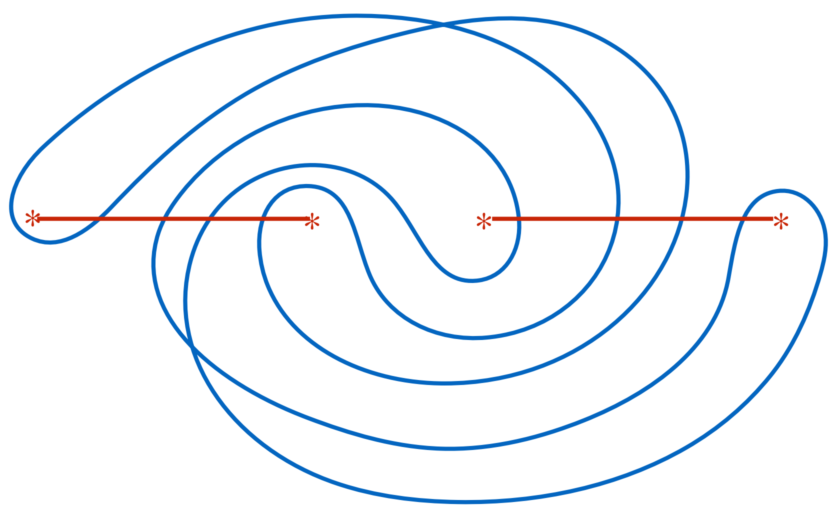
The fact that, for

$$^{L}\mathfrak{g} = \mathfrak{su}_2$$

the graded count of the intersection points in

$$\chi(E, \mathscr{B}I) = \sum_{\mathcal{P} \in E \cap \mathscr{B}I} (-1)^{M(\mathcal{P})} \mathfrak{q}^{J(\mathcal{P})}$$

computes the Jones polynomial is a theorem by



Bigelow from the '90s.

The dimensions of the complex

$$CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{\mathcal{P}\in\mathscr{B}E_{\mathcal{U}}\cap I_{\mathcal{U}}} \mathbb{C}\,\mathcal{P}$$

the vector space with the action of the differential

whose cohomology is the Link homology

grows polynomially with the number of crossings in our case,

which should be compared to exponential growth

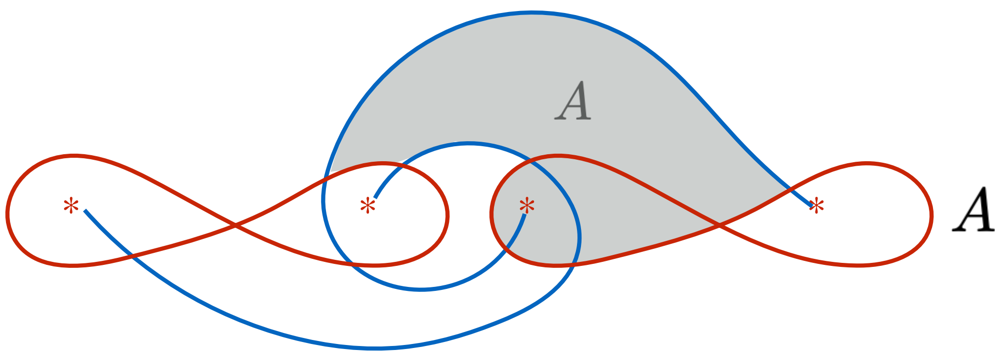
in Khovanov's case.

As in Heegard-Floer theory,

computing the action of the differential

can be translated to a sequence well defined, but hard

problem in complex analysis in one dimension.



Surprisingly, this problem can be solved.

One solves all the disk counting problems

at once,

by making the homological mirror symmetry that relates

“downstairs” mirror pair

$$\mathscr{D}_X \cong \mathscr{D}_A \cong \mathscr{D}_Y$$

manifest.

As in the simplest examples of homological mirror symmetry,

the categories on the two sides

$$\mathscr{D}_X \cong \mathscr{D}_A \cong \mathscr{D}_Y$$

are generated by a finite number of branes.

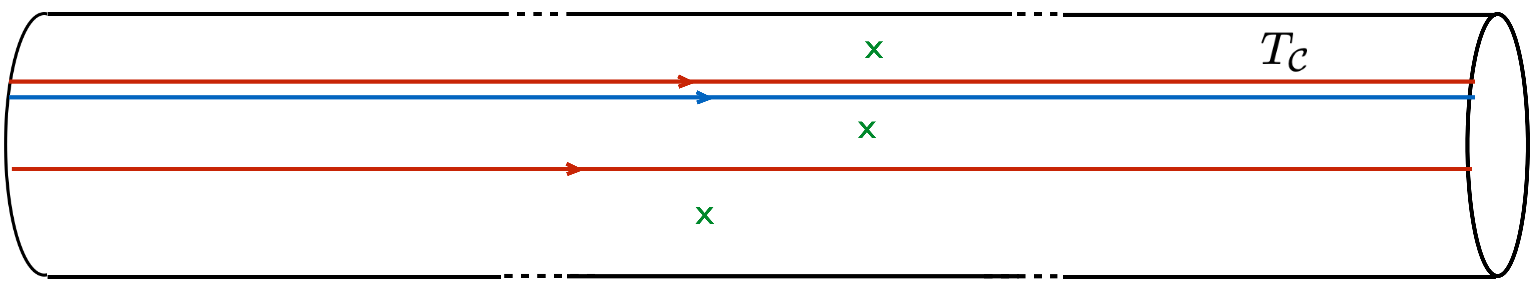
Strikingly, understood in this way,

homological mirror symmetry becomes easier than the topological one.

From perspective of

$$Y$$

the generating set of branes



are products of real line Lagrangians,

$$T_{\mathcal{C}} = T_{i_1} \times T_{i_2} \times \ldots \times T_{i_D}$$

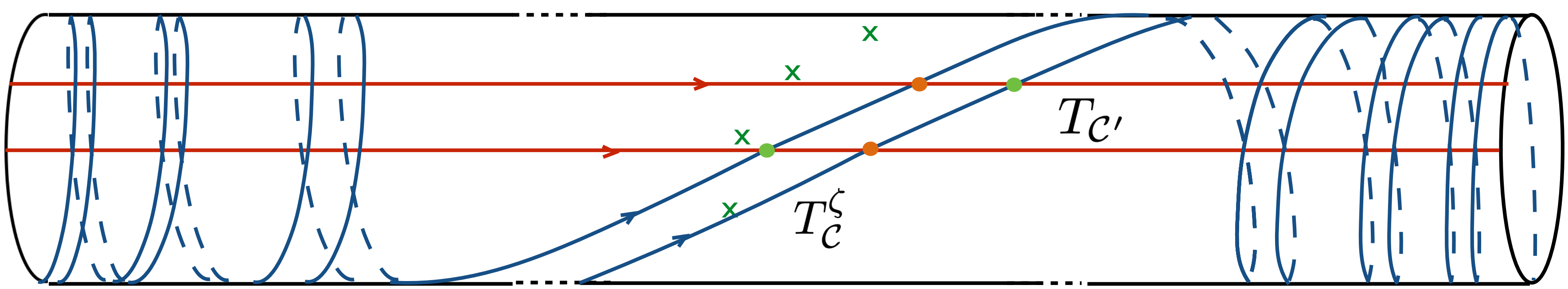
colored by simple roots.

This is a simple generalization of our very simplest example.

The associated “downstairs” algebra of open strings

$$A$$

is computable explicitly



and turns out to be a far simpler cousin of the “upstairs” KRLW algebra.

$$\mathscr{A}$$

In the vast new family of mirror symmetries

we just discovered

homological mirror symmetry also becomes manifest,

with the added benefit that

resulting categories are interesting and rich:

$$\mathscr{D}_X \cong \mathscr{D}_A \cong \mathscr{D}_Y$$

They categorify representations of the $U_{\mathfrak{q}}({}^L\mathfrak{g})$ quantum group.

In the remaining time,

let me try to explain the string theory origin of this construction.

The two dimensional theories we have been

discussing originate directly from string theory.

A helpful observation is another interpretation of

$$\mathcal{X}$$

In addition to being the intersection of slices in the affine Grassmannian

and the moduli space of singular G -monopoles,

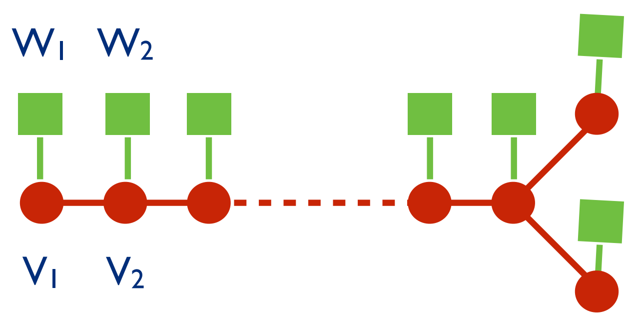
$\mathcal{X}$ is also a Coulomb branch of a three dimensional gauge theory.

The theory is a

three dimensional quiver gauge theory

with quiver $\mathcal{Q}$

$\mathfrak{g} = D_n$

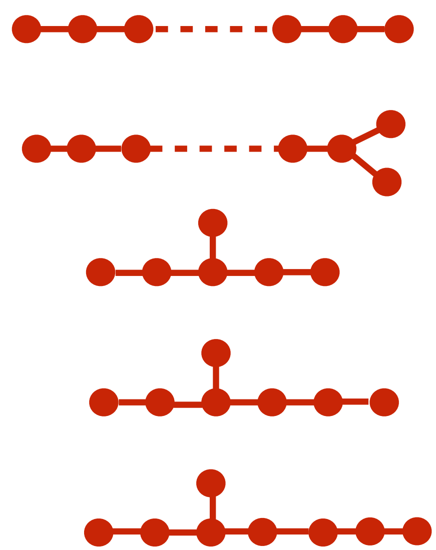


based on the Dynkin diagram of $\mathfrak{g}$

This gauge theory arises on defects, or more precisely, on D-branes

of a certain six dimensional "little" string theory

labeled by a simply laced Lie algebra $\mathfrak{g}$



$\mathfrak{g} = A_n$

$\mathfrak{g} = D_n$

$\mathfrak{g} = E_6$

$\mathfrak{g} = E_7$

$\mathfrak{g} = E_8$

with (2,0) supersymmetry.

The six dimensional string theory is
obtained by taking a limit of IIB string theory on an
ADE surface singularity of type

$$\mathfrak{g}$$

In the limit, one keeps only the degrees of freedom
supported at the singularity and decouples the 10d bulk.

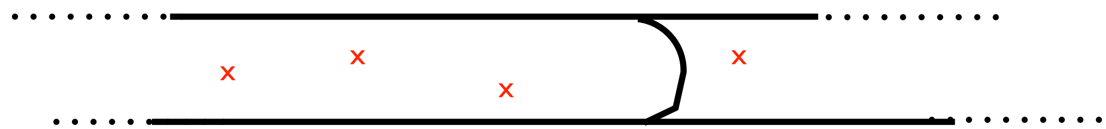
One wants to study the six dimensional (2,0) little string theory on

$$\mathcal{A} \times \mathrm{D} \times \mathbb{C}_{\kappa}$$

where

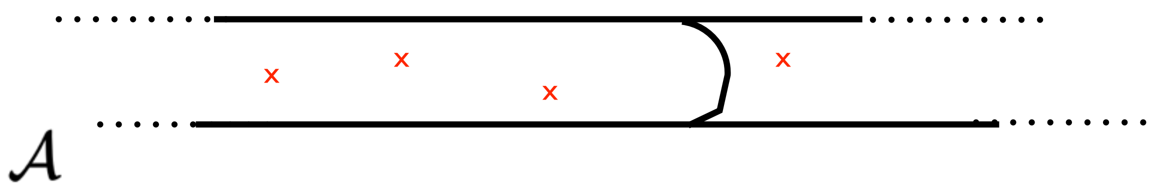
$$\mathcal{A}$$

is the Riemann surface where the conformal blocks live:



and D is the domain curve of the 2d theories we had so far.

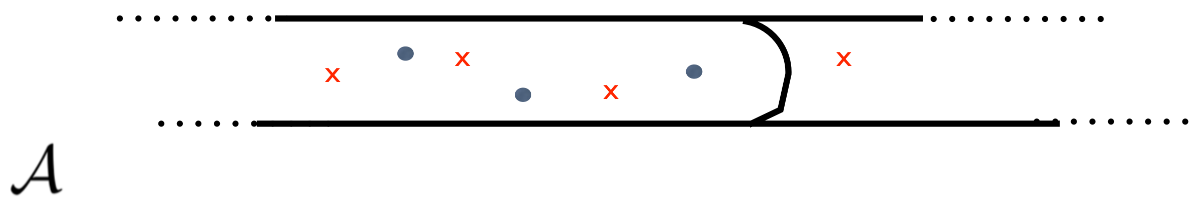
The punctures on the Riemann surface



come from a collection of defects in the little string theory,

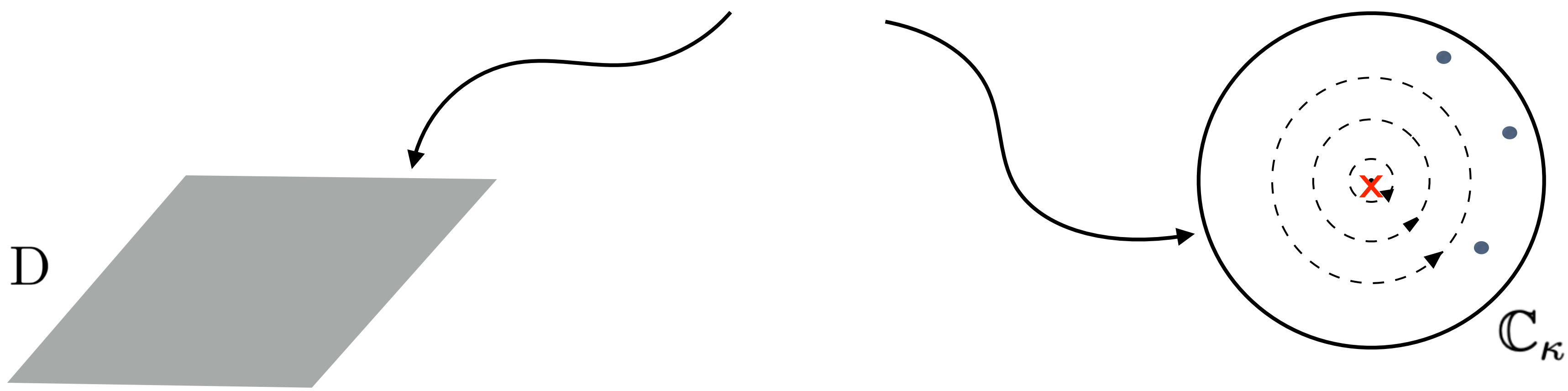
which are inherited from D-branes of the ten dimensional string.

The D-branes needed are



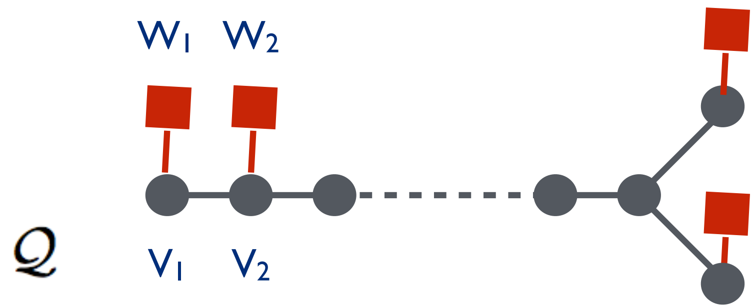
two dimensional defects of the six dimensional theory on

$$\mathcal{A} \times \mathrm{D} \times \mathbb{C}_\kappa$$



supported on D and points on $\mathbb{C}_\kappa$

The theory on the D-branes is the quiver gauge theory

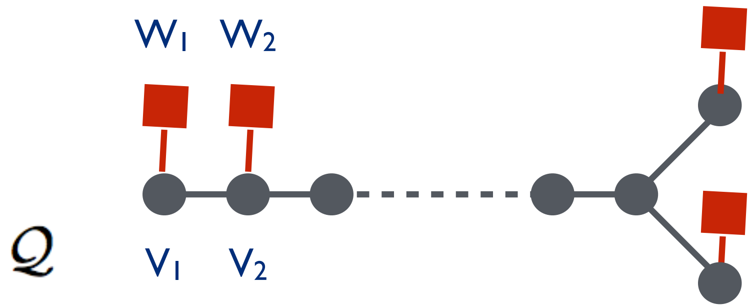


This is a consequence of the familiar description of

D-branes on ADE singularities

due to Douglas and Moore in ’96.

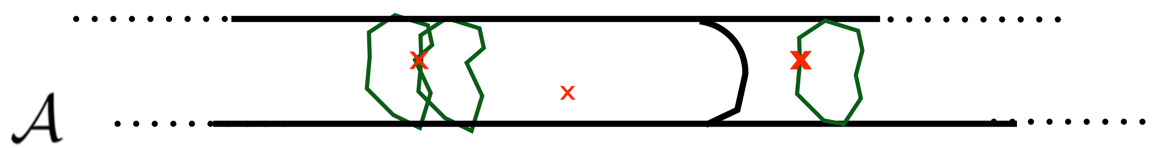
The theory on the D-branes supported on D



is a three dimensional quiver gauge theory on

$$\mathrm{D} \times S^1$$

rather than a two dimensional theory on D



due to string winding modes which one can sum up by T-duality.

One can study the three dimensional theory on

$$\mathrm{D} \times S^1$$

which comes from little string theory,

in much the same way

as we did the two dimensional theory.

The fact that the string scale is finite,

leads to a deformation of the structures

we had found, in particular, it breaks conformal invariance.

Rather than getting conformal blocks

and Knizhnik-Zamolodchikov equation,

from partition functions of the 3d theory on

$$\mathrm{D} \times S^1$$

one obtains their deformation

corresponding to replacing

$$\widehat{{}^L\mathfrak{g}} \longrightarrow U_\hbar(\widehat{{}^L\mathfrak{g}})$$

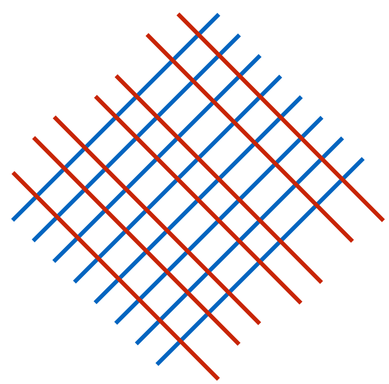
affine Lie algebra | quantum affine algebra

Pursuing our story further,

rather than discovering knot invariants

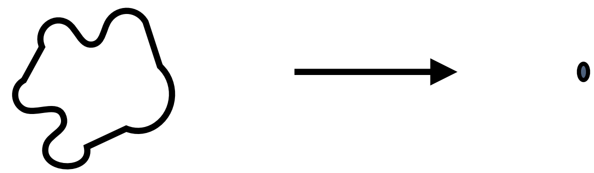
we would discover integrable lattice models,

those of, in some sense, very general kind.



This story is developed in joint works with Andrei Okounkov.

The six dimensional (2,0) string theory has a point particle limit



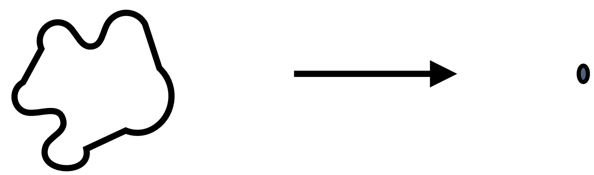
in which it becomes the six dimensional conformal field theory

of type $\mathfrak{g}$

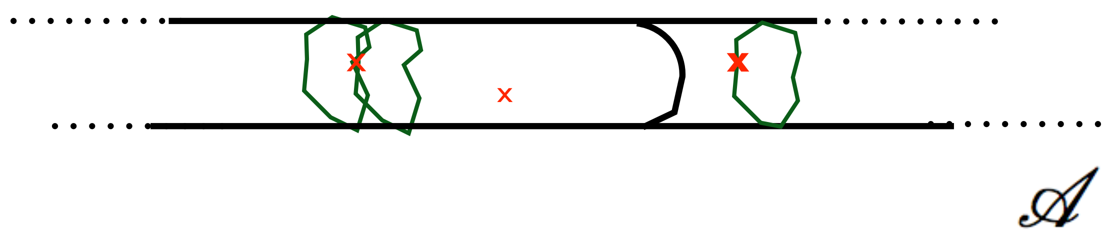
This limit coincides with the conformal limit of the quantum affine algebra

$$U_\hbar(\widehat{{}^L\mathfrak{g}}) \longrightarrow \widehat{{}^L\mathfrak{g}}$$

In the point particle limit,



the winding modes that made the theory
on the defects three dimensional, instead of two,
become infinitely heavy.



As a result, in the conformal limit, the theory on the defects
becomes a two dimensional theory on D

The resulting theory is not a gauge theory,
but it has the two other descriptions,
I described earlier in the talk,
related by two-dimensional mirror symmetry.