The Chern-Simons functional and Floer homology

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S. S Chern *Complex manifolds without potential theory* 2nd Ed. 1979; with an appendix on the geometry of characteristic classes

The Chern-Simons functional and topological field theories.

- $(2+1)$-dimensional, Jones-Witten:

$$\int e^{ikCS(A)}\mathcal{D}A$$

- $(3+1)$-dimensional, Floer homology $HF_*(M^3)$.

This talk is about the second item.

A. Floer *An instanton invariant for 3-manifolds* Commun. Math.Phys. 1988

A. Floer *Morse Theory for Lagrangian intersections* Jour. Differential Geometry 1988

The background includes work of Casson, Taubes, Gromov, Conley-Zehnder from the 1980's.

G a compact Lie group,

$P \to Y^3$ a G-bundle over a compact oriented 3-manifold.

$\mathcal{A}$ the space of connections on P.

For $A_1, A_2 \in \mathcal{A}$,

$$A_1 - A_2 \in \Omega^1(\mathrm{ad}P).$$

Curvature $F(A) \in \Omega^2(\mathrm{ad}P)$.

This defines a 1-form on the infinite dimensional space $\mathcal{A}$:

$$\Theta_A(a) = \int_Y \mathrm{Tr}(F(A) \wedge a).$$

We claim that Θ is a closed 1-form.

$$F(A+b) = F(A) + d_A b + b \wedge b$$

so
$\Theta_{A+b}(a) - \Theta_A(a) = H(a,b) + O(ab^2)$ where

$$H(a,b) = \int_Y \mathrm{Tr}(d_A b \wedge a).$$

The statement that Θ is closed is equivalent to H being symmetric, which is true because

$$H(a,b) - H(b,a) = \int_Y d(\mathrm{Tr}(a \wedge b)) = 0.$$

The Chern-Simons functional $CS : \mathcal{A} \to \mathbf{R}$ is defined (up to a constant) by $dCS = \Theta$.

It descends to $CS : \mathcal{A}/\mathcal{G} \to S^1$ where $\mathcal{G} = \mathrm{Aut}\, P$ is the group of gauge transformations.

The critical points of CS are the flat connections $F(A) = 0$ (equivalently, representations $\pi_1(M) \to G$).

The Hessian of CS at a critical point is the quadratic form $H(a, a)$.

An alternative definition of the Chern-Simons functional is to choose a 4-manifold X with boundary Y and extend a connection A to $\mathbf{A}$ over X.
Then

$$CS(A) = \int_X \mathrm{Tr}(F(\mathbf{A})^2).$$

There is a long history of the study of relations between critical points of functionals on infinite dimensional space and the homology of the spaces.

e.g. The energy functional $E(\gamma) = \int |\gamma'|^2$ on the loop space $\Omega M = \mathrm{Maps}(S^1, M)$.

In such problems

- the functional is bounded below and there are *compactness properties* of the sub-level sets $\{\gamma : E(\gamma) \leq C\}$.
- The Hessian at a critical point has a finite *index* i.e. dimension of the negative subspace.
- The E-decreasing gradient flow is defined, as a parabolic equation (nonlinear heat equation).

The Chern-Simons functional is completely different.

The analogue in symplectic geometry is the functional on the loop space ΩM of a symplectic manifold (M, ω):

$$A(\gamma) = \int_D \omega,$$

where $\partial D = \gamma$.

If $M = \mathbf{C}^n$ then we can write $\gamma = \sum \gamma_k e^{ik\theta}$ and

$$A(\gamma) = \langle \gamma, i\frac{d\gamma}{d\theta} \rangle = \sum k |\gamma_k|^2.$$

The fixed points of an exact Hamiltonian diffeomorphism $\phi : M \to M$ correspond to the critical points of a deformed functional A_ϕ on ΩM.

The *Arnold conjecture* is that the number of fixed points of ϕ is bounded below by the sum of the Betti numbers of M (assuming transversality).

Conley and Zehnder (1983). Proof of the Arnold conjecture for $M = T^{2n}$.

In this case

$$\Omega T^{2n} = T^{2n} \times H_+ \times H_-,$$

and the functional A_ϕ is asymptotic to

$$\|\gamma_+\|^2 - \|\gamma_-\|^2.$$

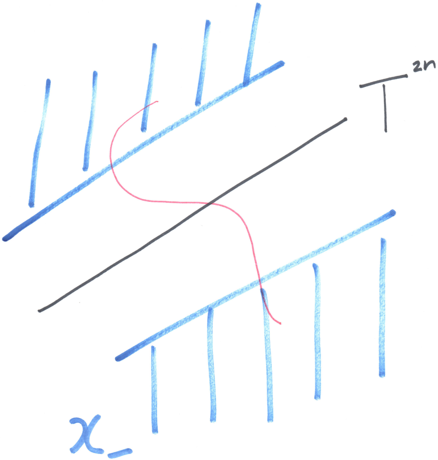
Conley and Zehnder reduce to a finite dimensional problem on a space $\mathcal{X} = T^{2n} \times \mathbf{C}^N_+ \times \mathbf{C}^N_-$ for some large N, and a function a_ϕ on $\mathcal{X}$ asymptotic to

$$(p, z_+, z_-) \to |z_+|^2 - |z_-|^2.$$

Let $\mathcal{X}_- = a_\phi^{-1}(-\infty, -C)$ for some large C.

The relative homology $H_*(\mathcal{X}, \mathcal{X}_-)$ is isomorphic to $H_*(T^{2n})$, shifted in degree by $2N$.

A version of finite-dimensional Morse Theory computes the relative homology with a chain complex generated by the critical points of a_ϕ which correspond to critical points of A_ϕ.


T^{2n}
x_-

Floer's great insight was that, with the right approach, one could "do Morse Theory"with functionals like A_ϕ and CS.

While the gradient flow is not defined, as an initial value problem, the gradient flow lines between critical points are sensible mathematical objects.

In the Chern-Simons case these are 1-parameter families of connections A_t over Y^3 with

$$\frac{\partial A_t}{\partial t} = *_3 F(A_t),$$

where $*_3 : \Omega^2 \to \Omega^1$ is defined using a Riemannian metric.

These solutions A_t can be identified with Yang-Mills instantons on the 4-manifold $Y \times \mathbf{R}$, asymptotic to flat connections at $\pm\infty$. That is, solutions $\mathbf{A}$ of the equation $F(\mathbf{A}) = - *_4 F(\mathbf{A})$ over $Y \times \mathbf{R}$.

In the symplectic case the gradient flow lines in the loop space ΩM are identified with holomorphic maps $S^1 \times \mathbf{R} \to M$.

A formulation of Morse Theory for a function f on a compact finite dimensional manifold Z with nondegenerate critical points.

- Chain complex C_* with generators $\langle p \rangle$ for each critical point p, grading by index $i(p)$.
- $\mathcal{M}_{pq}$ space of gradient flow lines $\dot{x} = -\mathrm{grad} f_x$ from p to q, modulo translation.
- Assuming transversality, $\mathrm{dim} \mathcal{M}_{p,q} = i(p) - i(q) - 1$.
- Boundary map $\partial : C_i \to C_{i-1}$

$$\partial \langle p \rangle = \sum \sharp \mathcal{M}_{p,q} \langle q \rangle .$$

In the infinite-dimensional situation the index of a critical point is not defined but the index difference $i(p) - i(q)$ is defined modulo 8 (in the case $G = SU(2)$).

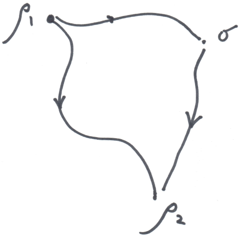
It is the Fredholm index of the linearised instanton equation over $Y \times \mathbf{R}$ or equivalently the spectral flow of the Hessian H.

The upshot was that Floer was able to define $\mathbf{Z}/8$-graded homology groups $HF_*(Y)$ for an oriented homology 3-sphere Y.

Make, if necessary, a generic perturbation of the Chern-Simons functional and then follow the recipe above with a chain complex generated by the irreducible critical points.

We think of this as “middle dimensional” homology of the infinite dimensional space $\mathcal{A}/\mathcal{G}$.

The essential analysis foundation is a compactness statement which implies $\partial^2 = 0$.


ρ1
σ
ρ2


ρ1
σ
ρ2
Y

The Euler characteristic of $HF_*(Y)$ is twice the *Casson invariant* of Y: the invariant introduced by Casson a few years earlier "counting" irreducible flat connections over Y.

Reversing orientation of Y gives dual groups $HF_*(\overline{Y}) = HF^*(Y)$.

The condition that Y is a homology 3-sphere enters because it implies that the trivial flat connection is isolated.

Floer's construction works in certain other situations provided one can avoid reducible flat connections–for example on a non-trivial $SO(3)$ bundle over Y^3.

To define Floer groups for general $P \to Y$ requires the use of equivariant theory or some other device. Such a theory is still not completely worked out, as far as the speaker is aware.

A partial $3+1$-dimensional topological field theory

The solutions of the Yang-Mills instanton equation define invariants of closed 4-manifold X (under suitable technical conditions). For simplicity just consider here numerical invariants $I(X)$ "counting" instantons in 0-dimensional moduli spaces.

For a 4-manifold X_1 with boundary Y we get invariants with values in $HF_*(Y)$. Let $\hat{X}_1 = X \cup_Y (Y \times [0,\infty))$ and set

$$I(X_1) = \sum_\rho n_{X,\rho} \langle \rho \rangle,$$

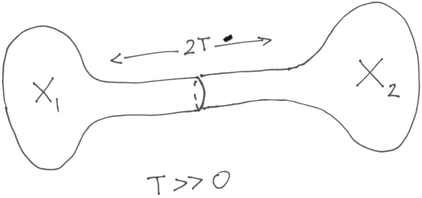
where the sum runs over flat connections ρ and $n_{X,\rho}$ is the number of instantons on $\hat{X}_1$ asymptotic to ρ.

Gluing formula: for a compact manifold X decomposed as $X = X_1 \cup_Y X_2$,

$$I(X) = \langle I(X_1), I(X_2) \rangle, \qquad (*)$$

using the pairing $HF_*(Y) \otimes HF^*(Y) \to \mathbf{Z}$.

The proof goes by "stretching the neck".



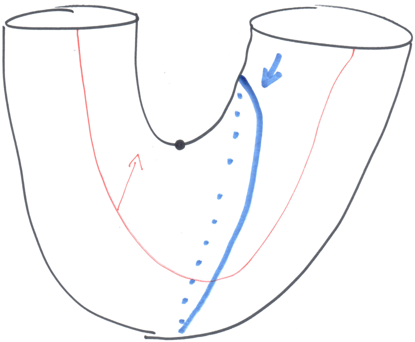
Formal point of view. Let $L(X_i) \subset \mathcal{A}/\mathcal{G}$ be the boundary value of instantons on X_i.

Then $I(X)$ is the intersection number of these "middle dimensional cycles".

Adding a finite cylinder $Y \times [0, T]$ to X_1 has the effect of flowing $L(X_1)$ by the decreasing gradient flow of CS.

Doing the same for X_2 has the effect of flowing $L(X_2)$ by the increasing gradient flow of CS.

Taking $T \to \infty$ moves the intersection $L(X_1) \cap L(X_2)$ close to the critical points of CS, leading to the formula (*).



We do not get a complete topological field theory due to various difficulties with reducible connections.

There are other Floer theories on 3-manifolds based on solutions of the Seiberg-Witten equations (Kronheimer-Mrowka) and more combinatorial Ozsvath-Szabo “Heegard” theory.

An outstanding question is to understand the relation between these Floer theories.

Some of Floer's deepest work lead to his *exact surgery sequence*.

$K \subset Y$ a knot in a homology 3-sphere with $\partial N(K) = S^1_M \times S^1_L$

$+1$-surgery Y_1: cut out $N(K)$ and glue back taking S^1_M to the diagonal in $S^1_L \times S^1_M$.

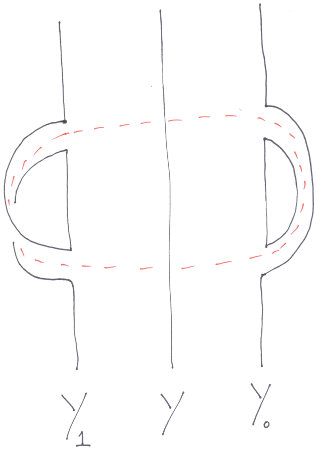
0-surgery Y_0: cut out $N(K)$ and glue back taking S^1_M to S^1_L.

Then there is an exact sequence

$$\dots HF_i(Y_1) \to HF_i(Y) \to HF_i(Y_0) \to HF_{i-1}(Y_1) \dots,$$

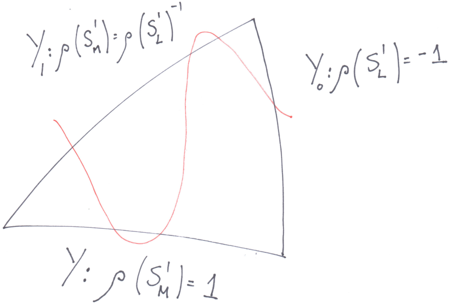
with maps induced by standard cobordisms between the manifolds.

Here Y_0 is a homology $S^1 \times S^2$ and one uses the non-trivial $SO(3)$ bundle.


Y_1
Y
Y_0

Some idea of the proof is given by the picture in the representation variety of the torus $S^1_L \times S^1_M$.

The red curve represents boundary values of flat connections on the complement $Y \setminus N(K)$.



More generally, one would like to extend the theory to dimension 2, with invariants of a 3-manifold-with-boundary taking values in a category associated to the boundary.

This has been achieved in the Seiberg-Witten/Heegard case. ("Bordered Floer Theory" Lipshitz, Ozsvath, D. Thurston)

Another refinement which has been achieved in the Seiberg-Witten case is the definition of *Floer homotopy type* (Manolescu), related to the stable homotopy Seiberg-Witten 4-manifold invariants of Bauer-Furuta.

These constructions use finite-dimensional approximation, somewhat like the Conley-Zehnder work discussed above.

Complexification and higher dimensions

There are two ways one can attempt to complexify the Floer theory on 3-manifolds.

- Replace the compact group G by a complex group G^c—for example $SL(2, \mathbf{C})$.
- Replace the 3-manifold Y^3 by a *Calabi-Yau 3-fold* Z.

We will only discuss the second item here.

There is a non-vanishing holomorphic 3-form Ω on Z.
Define a **C**-valued (local) functional on the space of G connections on $P \to Z$ by, schematically,

$$\mathcal{F}(A) = \int_Z CS(A) \wedge \Omega.$$

That is,

$$\delta\mathcal{F} = \int_Z \mathrm{Tr}\,(\delta A \wedge F(A)) \wedge \Omega.$$

This is a holomorphic function on the infinite-dimensional space of connections.
The critical points are connections with $F^{0,2} = 0$ i.e. $\overline{\partial}_A^2 = 0$, which define holomorphic G^c bundles over Z.

This leads to the “holomorphic Cassom invariant” introduced by R. Thomas, “counting” holomorphic bundles over Z.

More generally, developed in an algebro-geometric framework, to “DT invariants”, counting coherent sheaves on Z.

Just as the ordinary Floer theory is an infinite-dimensional Morse Theory so one would like to develop an infinite-dimensional *Picard-Lefschetz theory* in this situation.

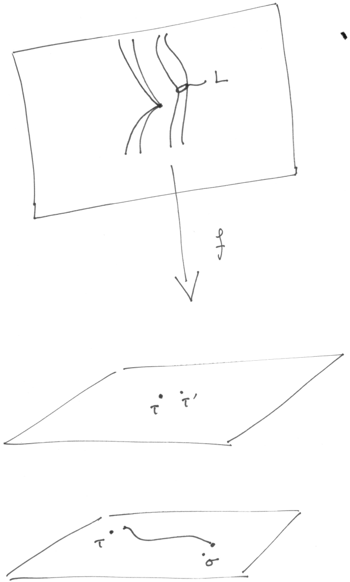
Picard-Lefschetz theory. Let $f : \mathcal{X} \to \mathbf{C}$ be a proper holomorphic function on an $(N+1)$-dimensional Kähler manifold with non-degenerate critical points. The fibres $\mathcal{X}_t$ have complex dimension N. (e.g. $N = 0$, a branched cover of $\mathbf{C}$).

Let $\tau \in \mathbf{C}$ be a critical value of f and τ' be close to τ.

There is a *vanishing cycle* L_τ in the fibre $\mathcal{X}_{\tau'}$ which generates the monodromy around τ.

The original Picard-Lefschetz discussion is in homology, with $[L_\tau] \in H_N(\mathcal{X}_{\tau'})$.

There is also a discussion in *symplectic topology*, where L_τ is a Lagrangian sphere and the monodromy is the generalised Dehn twist about L_τ.


L
f
τ
τ'
τ
σ

Let σ be another critical point and choose a path from τ' to σ'. Then, by parallel transport, the vanishing cycles can be considered as lying in the same fibre and we have:

- The intersection number $L_\sigma . L_\tau$;
- The Lagrangian Floer homology $HF(L_\sigma, L_\tau)$.

Can we define analogues of these in the infinite dimensional situation?

Let ω be the Kähler-Einstein metric on the Calabi-Yau 3-fold (Z, Ω), given by Yau's theorem.

Then $Z \times \mathbf{R}$ has a torsion-free G_2-structure with parallel 3-form $\omega \wedge dt + \mathrm{Re}(\Omega)$ and $Z \times \mathbf{R}^2$ has a torsion-free $\mathrm{Spin}(7)$ structure. More generally there are S^1 families of these structures, replacing Ω by $e^{i\alpha}\Omega$.

There are higher-dimensional Yang-Mills instanton equations over manifolds with torsion-free G_2 and $\mathrm{Spin}(7)$ structures.

Suppose that E, F are (stable) holomorphic bundles over Z. Rotating Θ if necessary we can suppose that $\mathcal{F}(E) - \mathcal{F}(F)$ is real. We can study G_2-instantons on $Z \times \mathbf{R}$ asymptotic to E, F at $\pm\infty$. (More precisely, to the Hermitian-Yang-Mills connections on E, F.)
Formally, a "count" of these should correspond to the intersection number of vanishing cycles.

Suppose that A_I, A_{II} are two such G_2-instantons on $Z \times \mathbf{R}$.

We can study $\mathrm{Spin}(7)$ instantons on $Z \times \mathbf{R}^2 = Z \times \mathbf{R}_t \times \mathbf{R}_s$, asymptotic to A_I, A_{II} as $s \to \pm\infty$ and to E, F as $t \to \pm\infty$.

One can argue that a count of these should correspond to the matrix entries of the Floer differential defining the Lagrangian Floer homology of the vanishing cycles. (In the finite dimensional situation these ideas have been developed by Haydys.)

BUT there are serious, perhaps fatal, difficulties with the compactness properties that would be required to define such counts rigorously.