Chern-Simons theory and non-Abelian topological order

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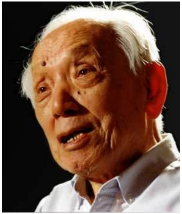
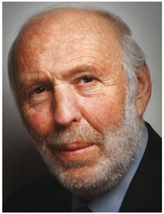
2021/11/17
Chern-Simons and Other Topological Field Theories
Simons foundation and MSRI

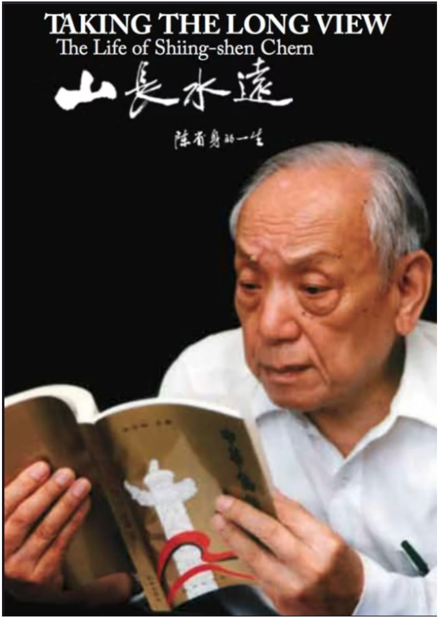
Simons Collaboration on Ultra-Quantum Matter

$$\frac{k}{4\pi}\mathrm{Tr}(A\mathrm{d}A + \frac{2}{3}A^3)$$





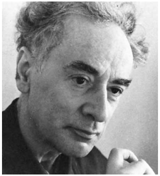
TAKING THE LONG VIEW



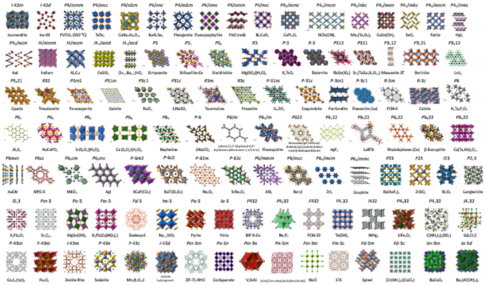
Classify phases of quantum matter ($T = 0$ phases)

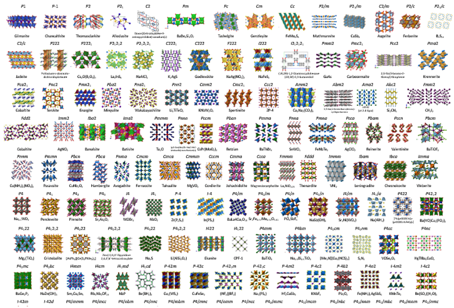
For a long time, we thought that **Landau symmetry breaking theory** *classifies all phases of matter and their transitions*

- **Symmetry breaking phases are classified by a pair of groups** (G_Ψ, G_H) **satisfying** $G_\Psi \subset G_H$
  G_H = symmetry group of the system.
  G_Ψ = symmetry group of the ground states.



- **230 crystals** from group theory





Can symmetry breaking describes all phases of matter?

A **spin-liquid theory** of high T_c superconductors:

- 2d spin liquid $\rightarrow$ spin-charge separation:
  electron $\rightarrow$ *holon* $\otimes$ *spinon*,
  holon: charge-1 spin-0 boson,
  spinon: charge-0 spin-1/2 fermion.

Holon condensation $\rightarrow$ high T_c superconductivity.

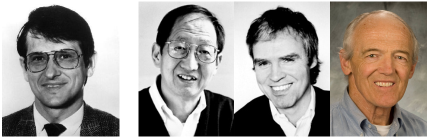
- **But do spin liquid exist? If yes, how to characterize them?**

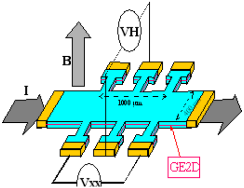
A spin liquid was explicitly constructed, and we found that it is a state that break time reversal and parity symmetry, but not spin rotation symmetry, with order parameter $\boldsymbol{S}_1 \cdot (\boldsymbol{S}_2 \times \boldsymbol{S}_3) \neq 0 \rightarrow$ **Chiral spin liquid** Kalmeyer-Laughlin, PRL **59** 2095 (87); Wen, Wilczek, Zee, PRB **39** 11413 (89)

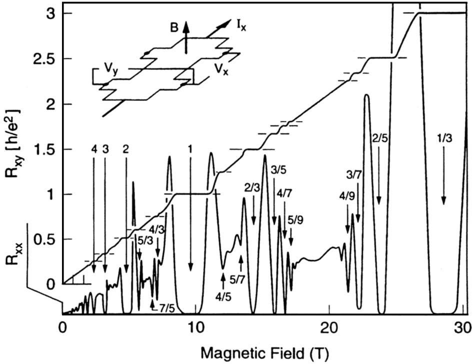
- However, we also discovered several different chiral spin states with identical symmetry breaking pattern.
  How distinguish those chiral spin states with the same symmetry breaking?

Topological orders in quantum Hall effect

- Quantum Hall (QH) states $R_{xy} = V_y/I_x = \frac{m}{n}\frac{2\pi\hbar}{e^2}$
  vonKlitzing Dorda Pepper, PRL **45** 494 (1980)
  Tsui Stormer Gossard, PRL **48** 1559 (1982)







- Fractional quantum Hall (FQH) states have different phases even when there is no symmetry and no symmetry breaking.

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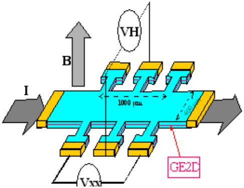
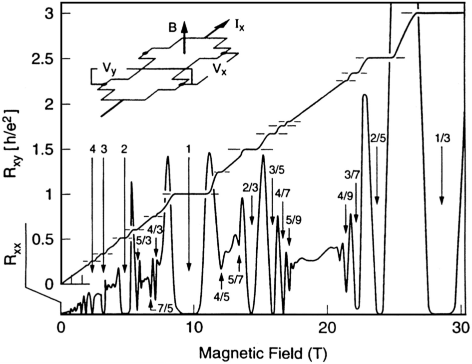




- Fractional quantum Hall (FQH) states have different phases even when there is no symmetry and no symmetry breaking.

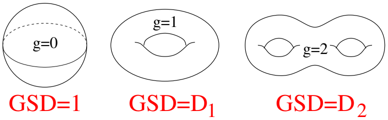
- Chiral spin and FQH liquids must contain a new kind of order, which was named as **topological order**

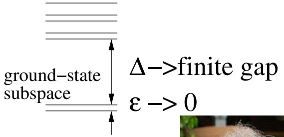
Wen, PRB **40** 7387 (89); IJMP **4** 239 (90)

What is **topological order**? How to characterize it?

- How to extract universal information (topological invariants) from complicated many-body wave function $\Psi(x_1, \cdots, x_{10^{20}})$
  Put the gapped system on space with various topologies, and measure the ground state degeneracy.
  (The dynamics of a quantum many-body system is controlled by a hermitian operator, Hamiltonian H, acting on the many-body wave functions. The spectrum of the Hamiltonian has a gap)
  $\rightarrow$ The notion of **topological order**

Wen PRB **40** 7387 (89)



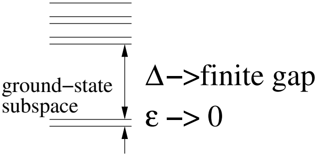


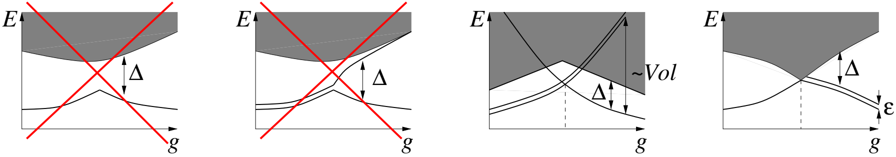
- The name **topological order** was motivated by Witten's **topological quantum field theory** (field theories that do not depend on spacetime metrics), such as **Chern-Simons** theories which happen to be the low energy effective theories for both chiral spin states and QH states.

Witten CMP **121** 351 (1989)

The ground state degeneracy is a topological invariant

- At first, some people objected that the ground state degeneracies are be finite-size effects or symmetry effect, not reflecting the intrinsic order of a phase of matter.
- The ground state degeneracies are robust against any local perturbations that can **break any symmetries**.
  → **topological degeneracy** (another motivation for the name **topological**) Wen Int. J. Mod. Phys. B **04** 239 (90); Wen Niu PRB **41** 9377 (90)
- The ground state degeneracies can only vary by some large changes of Hamiltonian
  → gap-closing phase transition.





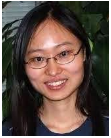
One of the argument of topological degeneracy from low energy effective **Chern-Simons** theories

- It was conjectured that chiral spin states and quantum Hall states are described by **quantum Maxwell-Chern-Simons theories** at low energies, characterized by an integral K-matrix
$$Z = \int D[a_I] \mathrm{e}^{-S(a_I)} = \int D[a_I] \mathrm{e}^{\mathrm{i}\, 2\pi \int_{M^{2+1}} K_{IJ} a_I \mathrm{d} a_J - \int_{M^{2+1}} V_{IJ}^{\text{impurity}}(\boldsymbol{x}) \mathrm{d} a_I \star \mathrm{d} a_J + \cdots}$$
Quantum theory = a theory defined by **path integral** of action e^{-S}.
- The impurity term $V_{IJ}^{\text{impurity}}(\boldsymbol{x})$ does not break the gauge invariance of the Maxwell-Chern-Simons theory. So even in the presence of impurities, the states that correspond to different gauge transformations $a_I \to a_I + \mathrm{d}\phi_I$ have the same energy, but those states are actually the same state and of cause the same state has the same energy.
- The states that correspond to different pseudo-gauge transformations $a_I \to a_I + \alpha_I$, $\mathrm{d}\alpha_I = 0$ also have the same energy, since locally they differ only by "gauge transformations". The local interactions cannot distinguish those states. Those different states give raise to topological degeneracy. Wen Int. J. Mod. Phys. B **04** 239 (90); Wen Niu PRB **41** 9377 (90)

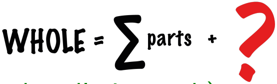
An modern understanding of topological degeneracy

- In 2005, we discovered that topological order has **topological entanglement entropy** Kitaev-Preskill hep-th/0510092 Levin-Wen cond-mat/0510613
  and **long range quantum entanglement** Chen-Gu-Wen arXiv:1004.3835




- For a long-range entangled many-body quantum system, **knowing every overlapping local parts still cannot determine the whole.**



- In other words, there are different "wholes", that their every local parts are identical (Like fiber bundle in math).
- Local interactions/impurities can only see the local parts → those different "wholes" (the whole quantum states) have the same energy.

Topological degeneracy comes from long range entanglement.

The pseudo-gauge transformations → different "wholes" with identical local "parts". **Long-range entanglement → Chern-Simons theory**

Examples of topological order (materials and toy models)

- **Symmetry breaking (SB) phases**:
  -500(bc) Ferromagnet (exp.) ... ...

- **Topologically ordered phases**:
  1904 Superconductor (exp.) Onnes (Z_2 topo. order, but regarded as SB)
  1980 Integral quantum Hall (IQH) states (exp.) von Klitzing (with no topo. exc.)
  1982 Abelian fractional quantum Hall (FQH) states (exp.) Tsui-Stormer-Gossard
  1987 Chiral spin liquids ($d + \mathrm{i}d$ -superconductor for spinons) (theo.)
  Kalmeyer-Laughlin 87; Wen-Wilczek-Zee 89
  $\to$ The notion of topological order (theo.) Wen 90
  1991 Z_2-spin liquids (theo.) Read-Sachdev 91; Wen 91, Kitaev 97
  1991 Non-Abelian FQH states, (theo.) Wen 91; Moore-Read 91 (slave, CFT)
  1992 All 2+1D Abelian topological states (theo.) Wen-Zee 92 (K-matrix)
  2000 $p + \mathrm{i}p$ -superconductor (theo.) Read-Green 00 ($\sqrt{\text{ IQH at } \nu = 1}$)
  2002 Hundreds symmetry enriched topological orders (theo.) Wen 02 (PSG)
  2003 Chiral spin liquid ($p + \mathrm{i}p$ -superconductor for spinons) (theo.) Kitaev 03
  2005 All 2+1D topo. orders with gapped edge (theo.) Levin-Wen 05 (UFC)
  2007 Z_2-spin liquid (exp.?) Helton-Lee-etc 07 Herbertsmithite $ZnCu_3(OH)_6Cl_2$
  2009 $\nu = 5/2$ non-Abelian FQH states (exp.) Willett etal 09; Heiblum 17
  2017 $p + \mathrm{i}p$ Chiral spin liquid (exp.???) Wolter etal 17; Jansa etal 17; Hentrich 17
  α-$RuCl_3$

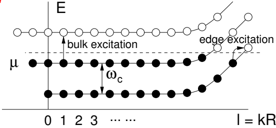
Abelian and non-Abelain FQH states

Abelian topological order $\rightarrow$ **Abelian fractional statistics**

- IQH and Laughlin many-body wavefunction   Laughlin PRL **50** 1395 (1983)

$$\chi_1 = \prod_{1\leq i<j\leq N}(z_i - z_j)\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \quad \Psi^{F,B}_{\nu=1/n} = \prod(z_i - z_j)^n \mathrm{e}^{-\frac{n}{4}\sum |z_i|^2} = (\chi_1)^n$$

where $z_i = x_i + \mathrm{i}\, y_i$ and $\chi_n = n$ filled Landau levels.



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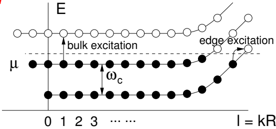


Non-abelian top. order $\to$ **non-Abelian statistics**

- $SU(m)_2$ state via slave-particle and Chern-Simons theory — Wen PRL **66** 802 (1991)

$$\Psi^F_{SU(2)_2} = \chi_1(\chi_2)^2, \ \nu = \frac{1}{2}; \qquad \Psi^F_{SU(3)_2} = (\chi_2)^3, \ \nu = \frac{2}{3};$$

$\to$ $SU(m)_2$ Chern-Simons effective theory $\to$ non-abelian statistics

- Pfaffien state via correlation in CFT — Moore-Read NPB **360** 362 (1991)

$$\Psi^B_{Pfa} = \mathcal{A}[\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots]\prod(z_i - z_j)\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \qquad \nu = 1$$

- *The $SU(2)_2$ and Pfaffien states have the same non-abelian statistics*
- *The $SU(3)_2$ state has the Fibonacci non-abelian statistics*

How to see non-Abelian statistics? Projective construction

Why $\Psi(z_1, ..., z_N) = [\chi_m(z_1, ..., z_N)]^n$ is a non-Abelian QH state?
What kind of non-Abelian statistics?

How to see non-Abelian statistics? Projective construction

Why $\Psi(z_1, ..., z_N) = [\chi_m(z_1, ..., z_N)]^n$ is a non-Abelian QH state?
What kind of non-Abelian statistics?

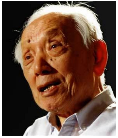
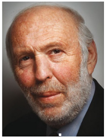
- **Cut an electron into n kinds of partons**

- Each kind of partons z_i^I form a IQH state described by wavefunction of m-filled Landau levels: $\chi_m(z_i^I)$.
- The wavefunction for all partons are $\chi_m(z_i^1)\chi_m(z_i^2)\chi_m(z_i^3)\cdots$
- We then **glue** partons together $z_i^1 = z_i^2 = \cdots = z_i$ to form electrons and obtain electron wave function $\Psi(z_i) = [\chi_m(z_i)]^n$

Doing the gluing via path integral

- Start with path integral for independent partons   Wen PRL **66** 802 (1991)
$$Z = \int D[\psi_I] e^{-\int dt\, d^2\boldsymbol{x}\ i\psi_I^\dagger \partial_t \psi_I - \frac{1}{2M}\psi_I^\dagger (\boldsymbol{\partial} - i\frac{1}{n}\boldsymbol{A}_{B\text{-field}})^2 \psi_I}$$
where $I = 1, \cdots, n$ labels kinds of partons.
- The independent partons motion cause the fluctuations of $SU(n)$ density and current $\rho^a = \Gamma^a_{IJ}\psi_I^\dagger \psi_J, \quad \boldsymbol{j}^a = \Gamma^a_{IJ}\mathrm{Re}(\psi_I^\dagger \frac{\partial}{iM}\psi_J).$
- Gluing partons back to electron kills the $SU(n)$ fluctuations $\rho^a = \boldsymbol{j}^a = 0$.
- This can be achieved via a Lagranian multiplyer – an $SU(n)$ gauge field:

$$Z = \int D[\psi_I] D[(a_\mu)_{IJ}] e^{-\int dt\, d^2\boldsymbol{x}\ i\psi_I^\dagger [\partial_t \delta_{IJ} - i(a_0)_{IJ}]\psi_J}$$
$$e^{-\int dt\, d^2\boldsymbol{x}\ -\frac{1}{2M}\psi_I^\dagger [\boldsymbol{\partial}\delta_{IJ} - i\frac{1}{n}\boldsymbol{A}_{B\text{-field}}\delta_{IJ} - i\,\boldsymbol{a}_{IJ}]^2 \psi_J}$$

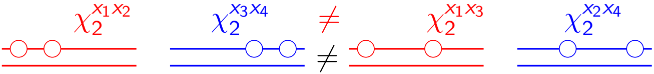



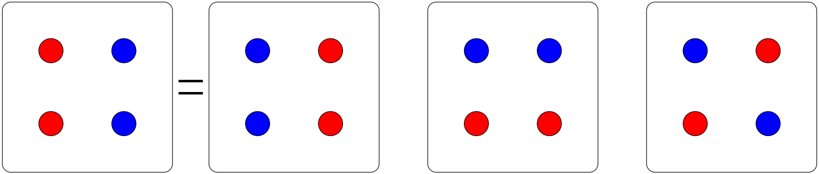
- Integrating out the parton fields $\psi_I \to$ non-Abelian Chern-Simons theory
$$Z = \int D[(a_\mu)_{IJ}] e^{-\int_{M^{2+1}} \frac{m}{4\pi}\mathrm{Tr}(a\, da + \frac{2}{3}a^3)}$$

What is non-Abelian statistics?

Top. degeneracy even when all the quasiparticles are trapped.

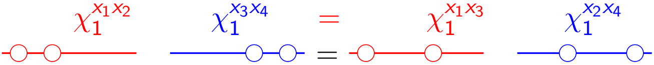
- The ground state $[\chi_2(z_i)]^2 = \chi_2(z_i)\chi_2(z_i)$ is non-degenerate on S^2.
- Degeneracy D_{deg} of 4 trapped quasiparticles at x_1, x_2, x_3, x_4: many different wave functions:





- The above represent a topological degeneracy $D_{\text{deg}} = 2$, since locally the two wave functions $\chi_2^{x_1}\chi_2$ and $\chi_2\chi_2^{x_1}$ are identical.

- However, for Laughlin state $\chi_1\chi_1 \to$ no non-Abelian statistics



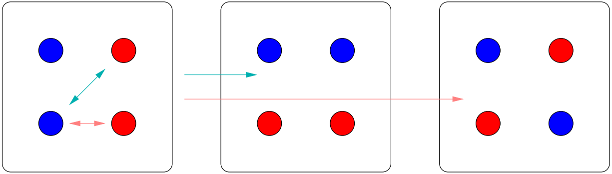
What is non-Abelian statistics?

- Each particle carries a degree of freedom $d = \lim_{N\to\infty}[D_{\text{deg}}(N)]^{N}$, where $D_{\text{deg}}(N)$ is the topological degeneracy for N trapped quasiparticle. d is called the **quantum dimension** of the particle.

- For non-Abelian state $(\chi_2)^2$, the quantum dimension $d = \sqrt{2}$.

- For non-Abelian state $(\chi_2)^3$, the quantum dimension $d = \frac{1+\sqrt{5}}{2}$.

- The presence of the topological degeneracy $\to$ The braiding is described by unitary matrix $U(D_{\text{deg}}) \to$ non-Abelian statistics.



Edge excitations of non-Abelain FQH states $[\chi_m(z_i)]^n$

- Edge state: Independent partons

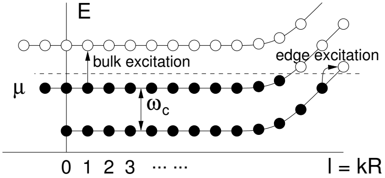
$$\mathcal{L} = \psi^\dagger_{\alpha I}(\partial_t - v\partial_x)\psi_{\alpha I}, \quad \alpha = 1, \cdots, m, \quad I = 1, \cdots, n$$

Excitations are generated by

$U(1) : J = \psi^\dagger_{\alpha I}\psi_{\alpha I}$,

$SU(m) : J^b = \psi^\dagger_{\alpha I} S^b_{\alpha\beta}\psi_{\beta I}$,

$SU(n) : j^a = \psi^\dagger_{\alpha I}\Gamma^a_{IJ}\psi_{\alpha J}$.



Glue partons back to electrons = remove the $SU(n)$ excitations.

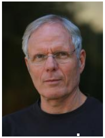
Edge excitations are generated by

$U(1) : J = \psi^\dagger_{\alpha I}\psi_{\alpha I}$,

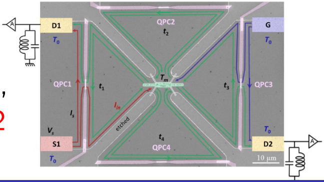
$SU(m) : J^b = \psi^\dagger_{\alpha I} S^b_{\alpha\beta}\psi_{\beta I}$

Edge CFT: $U(1) \times SU_n(m)$ Kac-Moody algebra   Wen PRL 70, 355 (1993)

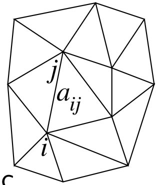
How to realize non-Abelain topological order, *ie* non-Abelain Chern-Simons theories in experiments



- Electrons in **first Landau level (LL)**: $(\chi_1)^3|_{\nu=\frac{1}{3}} \to U^3(1)/SU(3)_1 = U(1)$ Chern-Simons theory = Abelian topological order.
- If first **two LLs** are degenerate, electrons form $\chi_1^2\chi_2|_{\nu=\frac{2}{5}}$ $\to U^4(1)/(SU(2)_1 \times U(1)) = U^2(1)$ Chern-Simons theory = Abelian topological order.
- If first **three LLs** are degenerate, electrons form $\chi_1\chi_2^2|_{\nu=\frac{1}{2}} \to$ $U^5(1)/(SU(2)_2 \times U(1)) = U(1) \times SU(2)_2$ Chern-Simons theory = non-Abelian topological order.
- If first **four LLs** are degenerate, electrons form $\chi_2^3|_{\nu=\frac{2}{3}} \to$ $U^6(1)/SU(3)_2 = U(1) \times SU(2)_3$ Chern-Simons theory = non-Abelian topological order.
- Banerjee, Heiblum, etc arXiv:1710.00492 found the central charge of the edge state: $c = 2.56$ 18mK, $c = 2.64$ 15mK, $c = 2.76$ 12mK, for the $\nu = 5/2$ state in GaAs-AlGaAs hetero structure.



Put $U^\kappa(1)$ Maxwell-Chern-Simons theory on lattice



- The quantum $U^\kappa(1)$ Maxwell-Chern-Simons theory is defined via a path integral (small g large M^{2+1} limit)

$$Z(M^{2+1}) = \int D[a_{I\mu}(x^\mu)] \mathrm{e}^{\mathrm{i}2\pi \int_{M^{2+1}} K_{IJ} a_I \mathrm{d} a_J} \mathrm{e}^{-g^{-1} \int_{M^{2+1}} \mathrm{d}a_I \star \mathrm{d} a_I}$$

where K_{IJ} is a symmetric integral matrix. K = even $\to$ bosonic
K = odd (*ie* some $K_{II} = \text{odd}$) $\to$ fermionic (require M^{2+1} to be spin).

- But the path integral $\int D[(a_{I\mu}(x^\mu)]$ is not well defined. One way to make it well defined is to put the theory on space-time lattice.
- Let $a_I^{\mathbb{R}/\mathbb{Z}}$ be $\mathbb{R}/\mathbb{Z}$-valued (campact $U(1)$) 1-cochain on the space-time splicial complex $\mathcal{M}^3$. On link $\langle ij \rangle$, the value of the 1-cochain $a_I^{\mathbb{R}/\mathbb{Z}}$ is given by $(a_I^{\mathbb{R}/\mathbb{Z}})_{ij}$. Now the path integral can be well defined

$$\int_{a_I^{\mathbb{R}/\mathbb{Z}}} \equiv \prod_{\langle ij \rangle, I} \int_{-1/2}^{+1/2} \mathrm{d}(a_I^{\mathbb{R}/\mathbb{Z}})_{ij}$$

$$Z(\mathcal{M}^{2+1}) = \int_{a_I^{\mathbb{R}/\mathbb{Z}}} \mathrm{e}^{\mathrm{i}2\pi \int_{\mathcal{M}^{2+1}} K_{IJ} a_I^{\mathbb{R}/\mathbb{Z}} \smile \mathrm{d} a_J^{\mathbb{R}/\mathbb{Z}}} \mathrm{e}^{-g^{-1} \int_{\mathcal{M}^{2+1}} \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \smile \star \mathrm{d} a_I^{\mathbb{R}/\mathbb{Z}}}$$

Put $U^\kappa(1)$ Maxwell-Chern-Simons theory on lattice

$$Z(\mathcal{M}^{2+1}) = \int_{a_I^{\mathbb{R}/\mathbb{Z}}} \mathrm{e}^{\mathrm{i}2\pi \int_{\mathcal{M}^{2+1}} K_{IJ} a_I^{\mathbb{R}/\mathbb{Z}} \smile \mathrm{d} a_J^{\mathbb{R}/\mathbb{Z}}} \mathrm{e}^{-g^{-1} \int_{\mathcal{M}^{2+1}} \mathrm{d} a_I^{\mathbb{R}/\mathbb{Z}} \smile\star \mathrm{d} a_I^{\mathbb{R}/\mathbb{Z}}}$$

- But such a well defined path integral does not behave like quantum Chern-Simons theory. It has no $\mathbb{Z}$-gauge invariance $a_I^{\mathbb{R}/\mathbb{Z}} \to a_I^{\mathbb{R}/\mathbb{Z}} + n_I$, where n_I are arbitary $\mathbb{Z}$-valued 1-cochains.
- We find a lattice Maxwell-Chern-Simons quantum model that has the $\mathbb{Z}$-gauge invariance DeMarco Wen, arXiv:1906.08270

$$Z = \int_{a_I^{\mathbb{R}/\mathbb{Z}}} \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}} (\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil) - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil a_J^{\mathbb{R}/\mathbb{Z}}}$$

$$\mathrm{e}^{-\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_J^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil} \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} \mathrm{d}\left(a_I^{\mathbb{R}/\mathbb{Z}} (a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor a_J^{\mathbb{R}/\mathbb{Z}} \rceil)\right)}$$

$$\mathrm{e}^{-\int_{\mathcal{M}^3} \frac{(\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil)\star(\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil)}{g}},$$

dropped $\smile$

if $k_{II} = \frac{1}{2} K_{II}$, $k_{IJ} = K_{IJ}$ are integers. ($\lfloor x \rceil$ is the nearest integer of x.)

- The model also has the $U(1)$-gauge invariance $a_I^{\mathbb{R}/\mathbb{Z}} \to a_I^{\mathbb{R}/\mathbb{Z}} + \mathrm{d}\phi_I$ for closed $\mathcal{M}^3$ and for $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \approx$ int., as expected for Chern-Simons theory.

Exact 1-symmetries in $U^\kappa(1)$ Chern-Simons lattice model

$$Z = \int_{a_I^{\mathbb{R}/\mathbb{Z}}} \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}}(\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}\rceil) - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil a_J^{\mathbb{R}/\mathbb{Z}}}$$

$$\mathrm{e}^{-\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_J^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil} \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} \mathrm{d}\left(a_I^{\mathbb{R}/\mathbb{Z}}(a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor a_J^{\mathbb{R}/\mathbb{Z}}\rceil)\right)}$$

$$\mathrm{e}^{-\int_{\mathcal{M}^3} \frac{(\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil) \star (\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil)}{g}},$$

- With the 1-cup-product term, the lattice model has exact 1-symmetries $a_I^{\mathbb{R}/\mathbb{Z}} \to a_I^{\mathbb{R}/\mathbb{Z}} + \beta_I^{\mathbb{R}/\mathbb{Z}}$, where $\beta_I^{\mathbb{R}/\mathbb{Z}}$ are $\mathbb{R}/\mathbb{Z}$-valued 1-cocycles, which satisfy the following quantization condition

$$\sum_{J=1}^{\kappa} K_{IJ}\beta_J^{\mathbb{R}/\mathbb{Z}} \in \mathbb{Z}, \quad \forall I.$$

- The part of the 1-symmetries that satisfy DeMarco Wen, arXiv:1906.08270

$$\sum_{J=I}^{\kappa} k_{IJ}(\beta_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \beta_J^{\mathbb{R}/\mathbb{Z}}\rceil) = 0, \quad \forall I$$

are anomaly-free. Others are believed to be anomalous.

Dynamics of lattice $U^\kappa(1)$ Maxwell-Chern-Simons model

$$Z = \int_{a_I^{\mathbb{R}/\mathbb{Z}}} \mathrm{e}^{\mathrm{i}2\pi \sum_{I\leq J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}}(\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil) - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil a_J^{\mathbb{R}/\mathbb{Z}}}$$

$$\mathrm{e}^{-\mathrm{i}2\pi \sum_{I\leq J} k_{IJ} \int_{\mathcal{M}^3} a_J^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil} \mathrm{e}^{\mathrm{i}2\pi \sum_{I\leq J} k_{IJ} \int_{\mathcal{M}^3} \mathrm{d}\left(a_I^{\mathbb{R}/\mathbb{Z}}(a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor a_J^{\mathbb{R}/\mathbb{Z}} \rceil)\right)}$$

$$\mathrm{e}^{-\int_{\mathcal{M}^3} \frac{(\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil) \star (\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil)}{g}},$$

The lattice $U^\kappa(1)$ Maxwell-Chern-Simons model is actually a lattice bosonic system.

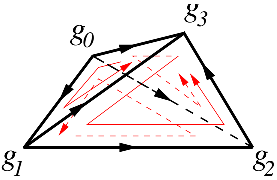
- For **small** g, the bosonic system realizes a topological order, described by the continuum $U^\kappa(1)$ Chern-Simons field theory, characterized by the even integral matrix K.
- For **large** g, due to the anomalous 1-symmetry, the bosonic system may be gapless or have a topological order (spontaneous breaking of the anomalous 1-symmetry).

Framing anomaly in quantum Chern-Simons theory

- For small g, we believe the path integral of Chern-Simons theory to give rise to a **gravitational Chern-Simons term** $\omega_{CS}^{\text{grav}}$ with framing anomaly (*ie* path integral depends on the framing of the spacetime)

$$Z = \int_{a_I^{\mathbb{R}/\mathbb{Z}}} \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_I^{\mathbb{R}/\mathbb{Z}}(\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil) - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil a_J^{\mathbb{R}/\mathbb{Z}}}$$
$$\mathrm{e}^{-\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} a_J^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil} \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^3} \mathrm{d}\left(a_I^{\mathbb{R}/\mathbb{Z}}(a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor a_J^{\mathbb{R}/\mathbb{Z}} \rceil)\right)}$$
$$\mathrm{e}^{-\int_{\mathcal{M}^3} \frac{(\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil) \star (\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil)}{g}} = \mathrm{e}^{-\epsilon V_{M^{3+1}}} \mathrm{e}^{\mathrm{i}2\pi \frac{\mathrm{sgn}(K)}{24} \int_{M^{2+1}} \omega_{CS}^{\text{grav}}}.$$

But how can a spacetime lattice know about the framing and gravitational Chern-Simons term of a 3d manifold M^{2+1}?



- To define cup product, we need to asign a branching structure to the spacetime trianglation.

Branching → Framing?.

From geometry to physics to quantum information (algebra) and back