Categorification: a model example

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Chern-Simons and other Topological Field Theories
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We'll describe a model example that (a) categorifies Kuperberg bracket (quantum $SL(3)$ link invariant), (b) underscores the notion of foams and their role in link homology, (c) categorifies Tait colorings of planar trivalent graphs (Four-Color Theorem).

Based on joint work with Louis-Hadrien Robert, ”Foam evaluation and Kronheimer–Mrowka theories”, *Advances in Math.* 2021.

Also strongly motivated by

- L. H. Robert, E. Wagner, A closed formula for the evaluation of foams, *Quantum Topology* 2020;
  M. Ehrig, D. Tubbenhauer, P. Wedrich, Functoriality of colored link homologies, *Proc. of the LMS* 2018.
- P. Kronheimer, T. Mrowka, Tait colorings, and an instanton homology for webs and foams, *J. of the EMS* 2019.

Chern-Simons functional is fundamental for both 3D and 4D TQFTs:

3D: CS path integral in Witten's construction of WRT 3-manifold invariants (A.Schwarz when $G = U(1)$ abelian.)

4D: CS functional in instanton Floer homology.

4D TQFT is a tensor functor from 3-manifolds and 4D cobordisms between them to an algebraic category, such as abelian groups.

Motivational problem (L. Crane and I. Frenkel): Construct a 4D TQFT that lifts (categorifies) WRT 3-manifold invariants.

An easier setup:

Look for tensor functors from the category of links in $\mathbb{R}^3$ and link cobordisms in $\mathbb{R}^3 \times [0, 1]$ to an algebraic category.

3D: Witten-Reshetikhin-Turaev (WRT) quantum link invariants:

$$P_{\mathfrak{g}}(L) \in \mathbb{Z}[q, q^{-1}].$$

Components of link L are labelled (colored) by irreps of a simple Lie algebra $\mathfrak{g}$.

4D: Link homology (categorifies $P_{\mathfrak{g}}$):

$$\begin{aligned} H_{\mathfrak{g}}(L) &= \oplus_{i,j} H_{\mathfrak{g}}^{i,j}(L), \\ P_{\mathfrak{g}}(L) &= \sum_{i,j} (-1)^i q^j \operatorname{rk} H_{\mathfrak{g}}^{i,j}(L). \end{aligned}$$

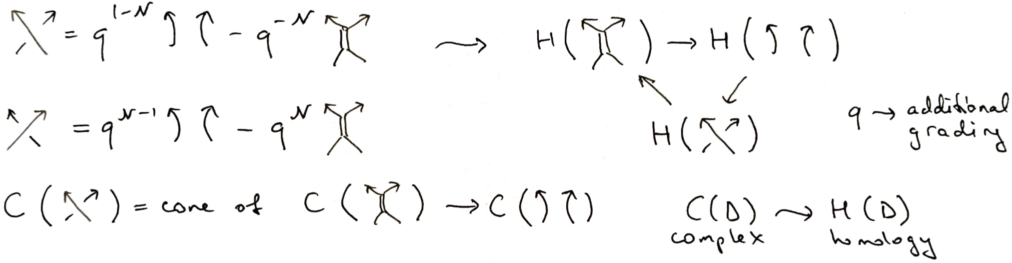
Best-case scenario: $H_{\mathfrak{g}}$ is functorial under link cobordisms. Gives a functor from the category of link cobordisms to the category of (bigraded) abelian groups.

Great many constructions of link homology exist. LH is understood best when $\mathfrak{g} = sl(N)$ and each component is labelled by an exterior power $\Lambda^k(V)$ of the fundamental representation $V \cong \mathbb{C}^N$. $k = 1$ case:

$$P(\bigcirc) = [N] = \frac{q^N - q^{-N}}{q - q^{-1}} = q^{N-1} + q^{N-3} + \dots + q^{1-N}$$

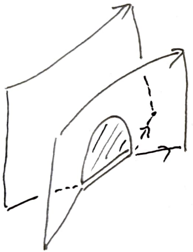
$N = 0$ Alexander polynomial (Ozsvath-Rasmussen-Szabo knot Floer homology)
$N = 2$ Jones polynomial (Khovanov homology)
$N = 3$ Kuperberg bracket (M.K., Mackaay-Vaz, Morrison-Nieh, Clark)
HOMFLYPT polynomial if $a = q^N, b = q - q^{-1}$. (Triply graded Khovanov-Rozansky homology).

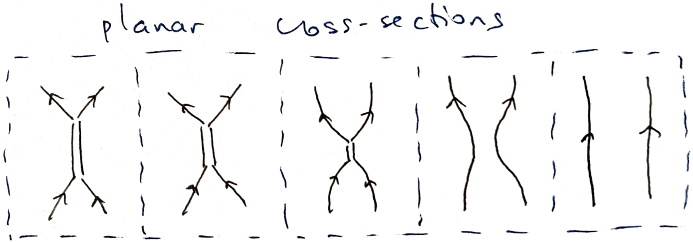
3-term skein relations on quantum invariants lift to short exact squences of homology groups. $sl(N)$, N-dimensional representation:



Via iterated cones, construction of link homology reduces to that for homology of suitable planar graphs.

WRT (Murakami-Ohtsuki-Yamada) invariant of a planar graph Γ has positivity property, $P(\Gamma) \in \mathbb{Z}_+[q, q^{-1}]$, and homology lives in one homological degree. Maps between homology of graphs are induced by "foams".



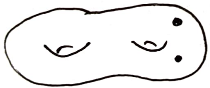


Want category of foams in $\mathbb{R}^3$: cobordisms between planar trivalent graphs, and a functor from it to graded "vector spaces".

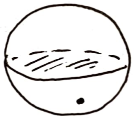
A foam F (closed unoriented $SL(3)$ foam) is a two-dimensional combinatorial CW-complex with generic singularities embedded in $\mathbb{R}^3$. Consists of

Neighbourhood of a vertex is homeomorphic to a cone over a complete graph on 4 vertices. Foam may carry dots (observables) on facets.

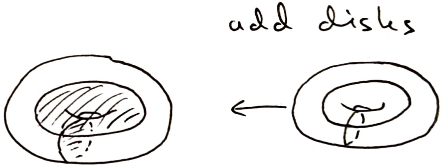
$s(F)$ - graph of singular points (seams and vertices). 4-valent. May contain circles.
Examples of foams:



$s(F) = \emptyset$



$s(F) = \mathbb{S}^1$

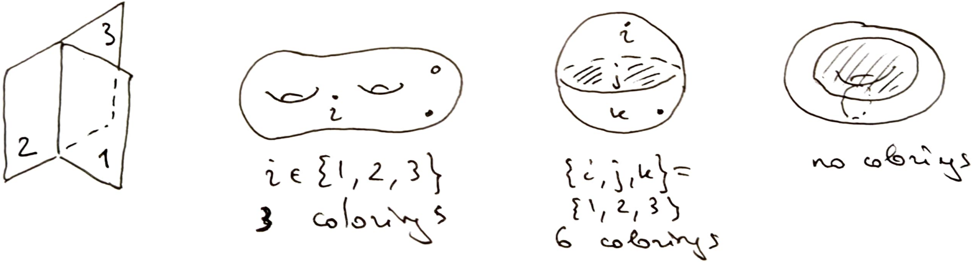


$s(F)$ – bouquet of 2 circles.

$f(F) = \text{connected components of } F \setminus s(F)$ is the set of facets of F.
An *admissible* or *Tait* coloring of F is a map

$$c \ : \ f(F) \longrightarrow \{1,2,3\} \quad \text{set of colors}$$

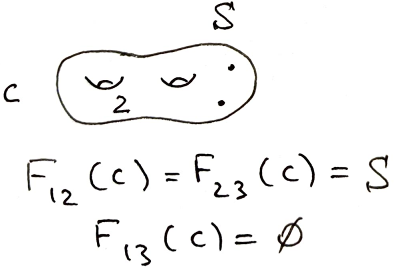
such that along each seam colors of the three adjacent facets are distinct.
$\mathrm{adm}(F)$ – set of admissible colorings.

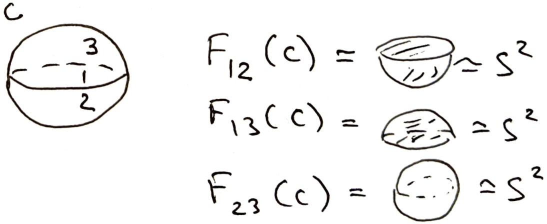


S_3 acts on $\mathrm{adm}(F)$ by permuting the colors.

For $c \in \mathrm{adm}(F)$ and $i \neq j,\ i, j \in \{1, 2, 3\}$ denote by $F_{ij}(c)$ the union of facets colored i or j. This is an ij-bicolored surface of F for the coloring c.

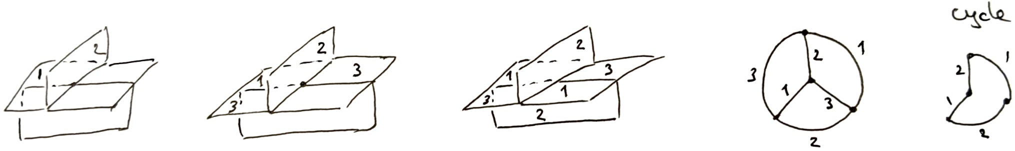
Examples:





Theorem $F_{ij}(c)$ is a closed orientable surface in $\mathbb{R}^3$.

Proof: Examine $F_{ij}(c)$ along seams and vertices of F:



A closed surface $S \subset \mathbb{R}^3$ is orientable. $\square$

Corollary: $F_{ij}(c)$ has even Euler characteristic,

$$\chi_{ij}(c) \;:=\; \chi(F_{ij}(c)) \;\in\; 2\mathbb{Z}.$$

Algebraic side setup: work over any field $\mathbf{k}$ of characteristic 2.

Base rings:

$R' = \mathbf{k}[x_1, x_2, x_3]$. Variables $x_1, x_2, x_3 \longleftrightarrow$ colors $\{1, 2, 3\}$.

$R \subset R'$, $R = \mathbf{k}[x_1, x_2, x_3]^{S_3} = \mathbf{k}[E_1, E_2, E_3]$ symmetric functions.

E_i – elementary symmetric functions, $E_1 = x_1 + x_2 + x_3$, $E_2 = x_1 x_2 + x_1 x_3 + x_2 x_3$, $E_3 = x_1 x_2 x_3$.

Construct evaluation function $\langle F \rangle \in R$ as sum over evaluations $\langle F, c \rangle$, $c \in \mathrm{adm}(F)$.

$$\langle F \rangle = \sum_c \langle F, c \rangle.$$

Functions $\langle F, c \rangle$ are rational functions in x_1, x_2, x_3, but $\langle F \rangle$ is a symmetric polynomial.

$$
\begin{aligned}
\langle F, c\rangle &= \frac{X_1^{d_1(c)} X_2^{d_2(c)} X_3^{d_3(c)}}{(x_1+x_2)^{\chi_{12}(c)/2}(x_1+x_3)^{\chi_{13}(c)/2}(x_2+x_3)^{\chi_{23}(c)/2}} \\
&= \frac{\prod_{k=1}^{3} X_k^{d_k(c)}}{\prod_{1\le i<j\le 3}(x_i - x_j)^{\chi_{ij}(c)/2}}, \\
\langle F\rangle &= \sum_{c\in \mathrm{adm}(F)} \langle F, c\rangle .
\end{aligned}
$$

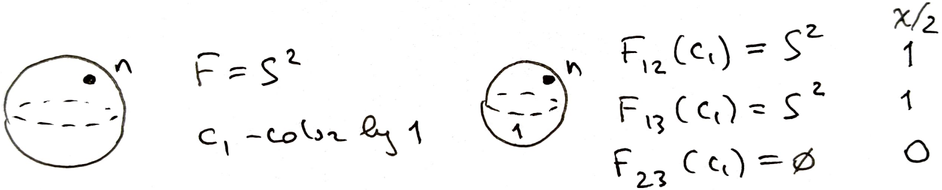
$d_k(c)$ total number of dots on all facets colored k in coloring c.

We can divide Euler characteristic χ by 2 since it's even.

In a field $\mathbf{k}$ of char 2 plus equals minus, $+ = -$. With minus signs, reminiscent of Weyl character formula.

Theorem: $\langle F\rangle \in R$ (symmetric function; denominators vanish).

Example: evaluation of dotted sphere. 3 colorings.



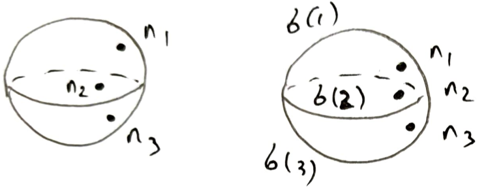
$$\langle F, c_1\rangle = \frac{x_1^n}{(x_1 - x_2)(x_1 - x_3)},$$

$$\langle F\rangle = \sum_{i=1}^{3} \frac{x_i^n}{(x_i - x_j)(x_i - x_k)} = h_{n-2}(x_1, x_2, x_3) = \sum_{a_1+a_2+a_3=n-2} x_1^{a_1} x_2^{a_2} x_3^{a_3}.$$

Complete symmetric function h_{n-2}. If $n = 0, 1$, sphere evaluates to 0.

Two-dotted sphere evaluates to 1. Three-dotted to $x_1 + x_2 + x_3$.

Example: evaluation of theta-foam. 6 colorings.



$\sigma \in S_3$ defines a coloring

$F_{ij}(\sigma) = S^2$

$\chi(S^2)/2 = 1$

$$\langle F \rangle = \sum_{\sigma \in S_3} \frac{x_{\sigma(1)}^{n_1} x_{\sigma(2)}^{n_2} x_{\sigma(3)}^{n_3}}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} = s_\lambda(x_1, x_2, x_3).$$

Schur function s_λ for partition $\lambda = (n_1 - 2, n_2 - 1, n_3)$ (assume $n_1 \geq n_2 \geq n_3$).

Individual terms are rational functions, the sum is a symmetric polynomial (element of R).

Work mod 2, but there's a version over $\mathbb{Z}$ (for oriented foams).

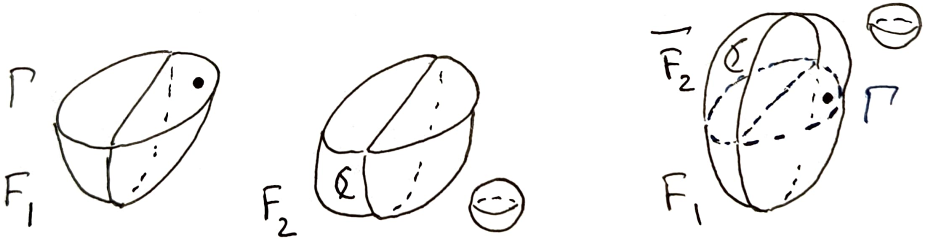
Universal construction. Given an R-valued evaluation of n-dimensional objects, can construct state spaces (R-modules) for their "generic cross-sections", $(n-1)$-dimensional objects.

A generic cross-section of a foam F by a plane is a planar trivalent graph Γ. Given Γ, consider a free R-module $\mathrm{Fr}(\Gamma)$ with a basis of foams F in $\mathbb{R}^3_-$ with $\partial F = \Gamma$. Call basis element the symbol of F and denote $[F]$.

$\partial F = \Gamma$

$F \subset \mathbb{R}^3_- = \mathbb{R}^2 \times (-\infty, 0]$

Given F_1, F_2 with $\partial F_1 = \Gamma = \partial F_2$, glue F_1, F_2 along Γ and evaluate closed foam $\overline{F_2} F_1$.



$$([F_1],[F_2]) \;:=\; \langle \overline{F_2} F_1 \rangle \;\in\; R.$$

Extend to R-linear combinations of symbols of such F. Get a symmetric bilinear form on the free R-module $\mathrm{Fr}(\Gamma)$.
Define the state space $\langle \Gamma \rangle$ of a planar trivalent graph Γ by

$$\langle \Gamma \rangle \;:=\; \mathrm{Fr}(\Gamma)/\ker((,)).$$

A linear combination $\sum_i a_i[F_i] = 0 \in \langle \Gamma \rangle$ iff for any F with boundary Γ,

$$\sum_i a_i \langle \overline{F} F_i \rangle \ = \ 0 \in R.$$

Generic choices of $\langle F \rangle$ for closed objects F will result in state spaces of infinite rank (as R-modules).

Very general construction, arguably underexplored.
Lax TFTs: $\langle \Gamma_1 \rangle \otimes \langle \Gamma_2 \rangle \longrightarrow \langle \Gamma_1 \sqcup \Gamma_2 \rangle$ inclusion, not isomorphism.

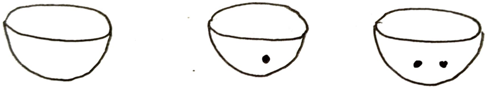
Theorem $\langle \Gamma \rangle$ is a finitely-generated R-module.

R and $\langle \Gamma \rangle$ are additionally $\mathbb{Z}$-graded, with $\deg(x_i) = 2$.

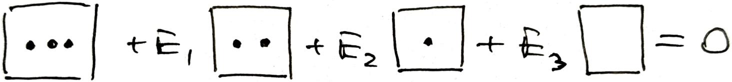
Examples of state spaces:

1) $\Gamma = \emptyset_1$ empty graph, $\langle \emptyset_1 \rangle$ is a free rank one R-module with a basis $[\emptyset_2]$ (symbol of empty foam).

2a) If Γ is a circle, $\langle \Gamma \rangle$ is a free R-module of rank 3 with basis



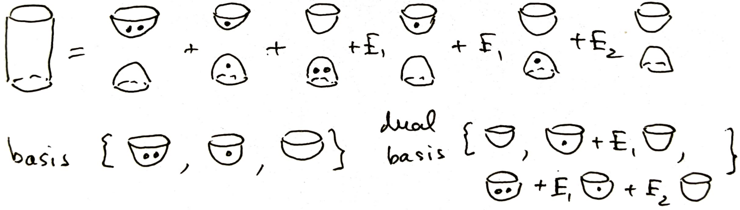
How to get rid of 3 dots on a facet:



$$x^3 + E_1 x^2 + E_2 x + E_3 = 0$$

$$(x - x_1)(x - x_2)(x - x_3) = 0$$

A better reason (neck-cutting):



Decomposition of identity in a Frobenius algebra A for dual bases $\{u_i\}, \{v_i\}$ (trace $\varepsilon(u_i v_j) = \delta_{ij}$):

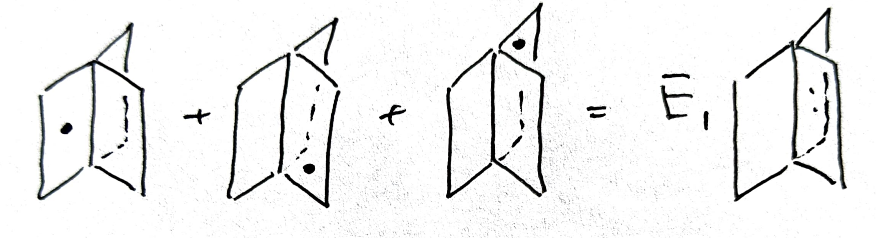
$$x = \sum_i u_i \otimes \varepsilon(v_i x)$$

2b) if Γ' is given by removing an innermost circle from Γ,

$$\langle\Gamma\rangle \;\cong\; \langle\Gamma'\rangle\{-2\} \oplus \langle\Gamma'\rangle \oplus \langle\Gamma'\rangle\{2\}.$$

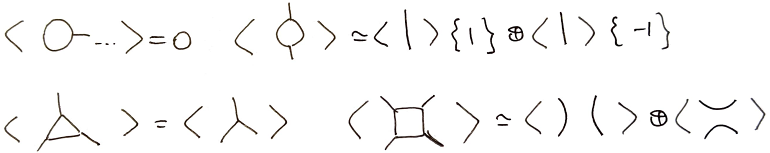
Here $\{n\}$ is the grading shift by n.

Another example of skein relation



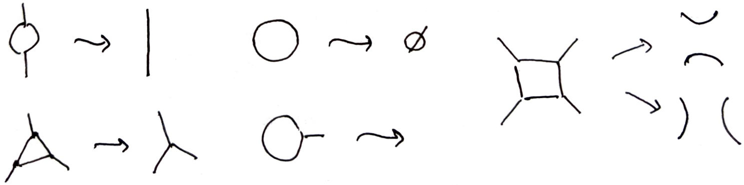
For each coloring c, the three dots contribute x_1, x_2, x_3 respectively to the 3 foam evaluations, and $E_1 = x_1 + x_2 + x_3$.

Other direct sum decompositions:



If Γ has no Tait colorings, $\langle\Gamma\rangle = 0$ (bilinear pairing is identically zero). A graph with a bridge has no Tait colorings.

Call Γ *reducible* if one can get to the union of empty graphs by inductively simplifying regions of Γ with at most four edges:



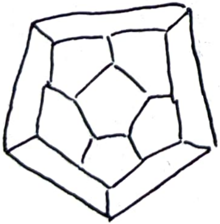
Theorem: For a reducible Γ, the state space $\langle\Gamma\rangle$ is a free R-module of rank equal to $|\mathrm{Tait}(\Gamma)|$.

$\mathrm{Tait}(\Gamma)$ is the set of Tait colorings of edges of Γ (three colors, at each vertex edge colors are distinct).

$\mathbb{Z}$-grading of $\langle\Gamma\rangle$ allows to quantize $|\mathrm{Tait}(\Gamma)|$ to an element of $\mathbb{Z}_+[q, q^{-1}]$, via graded rank of $\langle\Gamma\rangle$. This quantum Tait counting is not understood. For bipartite Γ, it can be interpreted via intertwiners for quantum group $sl(3)$.

Naive Conjecture: The state space $\langle \Gamma \rangle$ is a free R-module of rank $|\mathrm{Tait}(\Gamma)|$.

Holds for reducible graphs. Not known even for the dodecahedron graph (first non-reducible graph). See D. Boozer ”Computer bounds for Kronheimer-Mrowka foam evaluation”.



Functoriality: A foam with $\partial F = \Gamma_0 \sqcup \Gamma_1$ induces a map

$$[F] \ : \ \langle \Gamma_0 \rangle \longrightarrow \langle \Gamma_1 \rangle$$

Get a homology theory for planar trivalent graphs (objects) and foams between them (morphisms or cobordisms).

Works the same way for any universal construction.

Evaluation $\langle F \rangle \in R$ can be composed with a commutative ring homomorphism $\psi : R \longrightarrow S$ to get evaluation $\langle F \rangle_\psi \in S$. Universal construction produces S-modules $\langle \Gamma \rangle_\psi$ (graded modules if ψ is degree-preserving).

Let $S = \mathbf{k}[t]$ and $\psi : R \longrightarrow \mathbf{k}[t]$,

$$\psi(E_1) = \lambda_1 t, \ \psi(E_2) = \lambda_2 t^2, \ \psi(E_3) = \lambda_3 t^3, \ \ 0 \neq \lambda_1 \lambda_2 + \lambda_3 \in \mathbf{k}.$$

Theorem $\langle \Gamma \rangle_\psi$ is a free S-module of rank $|\mathrm{Tait}(\Gamma)|$ for any Γ.

Get a quantization of $|\mathrm{Tait}(\Gamma)|$ for any such ψ.

P. Kronheimer and T. Mrowka, ”Tait colorings, and an instanton homology for webs and foams,” arxiv 2015, *Journal EMS* 2019.

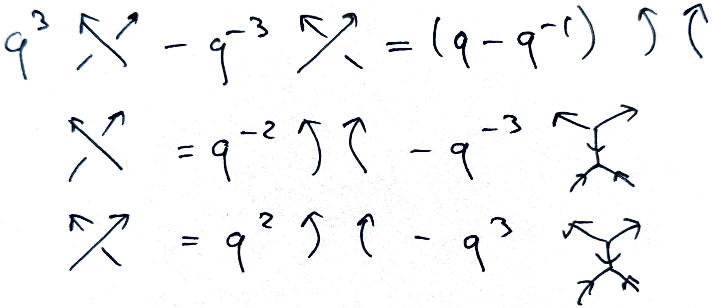
They define and study $SO(3)$ instanton homology theory over $\mathbb{F}_2$ for certain 3-orbifolds (via Chern-Simons functional for orbifolds). Trivalent vertex comes from orbifold $\mathbb{R}^3/V_4$, for the Klein four group V_4 of even coordinate reflections $(x, y, z) \to (\pm x, \pm y, \pm z)$ in $SO(3)$.

In particular, they get homology theory for trivalent graphs in $\mathbb{R}^3$. For planar trivalent graphs their homology $J^\sharp(\Gamma)$ has $\langle\Gamma\rangle_\psi$ as a subquotient, for

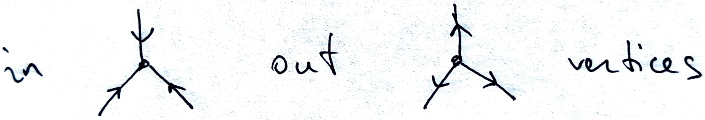
$$\psi : R \longrightarrow \mathbb{F}_2,\ \psi(E_i) = 0, i = 1, 2, 3.$$

Foams in $\mathbb{R}^3 \times [0, 1]$ induce maps of KM homology groups. Kronheimer and Mrowka have a nonvanishing result for $J^\sharp(\Gamma)$ strongly reminiscent of the Four-Color Theorem.

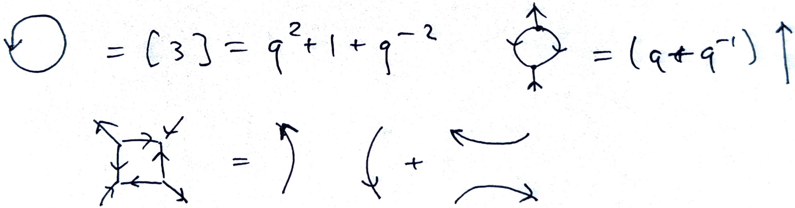
$sl(3)$ link homology ($N = 3$) lifts quantum $sl(3)$ link invariant, aka Kuperberg bracket.



Bipartite planar trivalent graphs. Out vertex represents a generating vector in $U_q(sl(3))$-invariants of $V^{\otimes 3}$.



Bipartite, no pentagon regions. Always reducible.



Graph $\Gamma \longrightarrow P(\Gamma) \in \mathbb{Z}_+[q, q^{-1}] \longrightarrow H(\Gamma) = \langle \Gamma \rangle$ homology (state space) of Γ.

Link $L \longrightarrow P(L) \in \mathbb{Z}[q, q^{-1}] \longrightarrow H(L)$ homology of L. Bigraded, with Kuperberg bracket $P(L)$ as Euler characteristic.

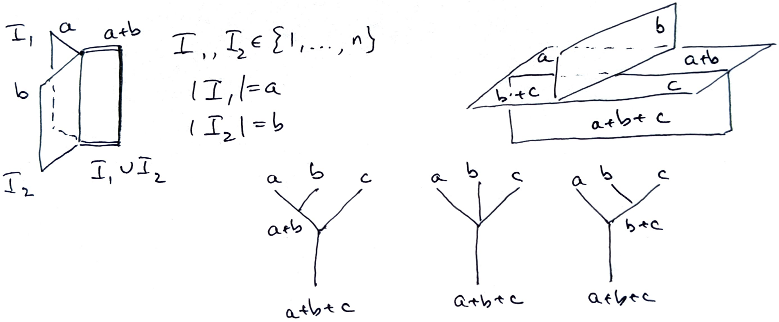
Over $\mathbb{F}_2$: unoriented $SL(3)$ foams.

Over $\mathbb{Z}$: oriented $SL(3)$ foams, no singular vertices, only seams. Closed foams easier to evaluate ($\mathbb{Z}$ M.K. 2004, $\mathbb{Z}[E_1, E_2, E_3]$ M. Mackaay, P. Vaz 2007).

$GL(N)$ and $SL(N)$ foams, $N > 3$. Vertices present. Lots of results (M. Mackaay, P. Vaz, A. Lauda, D. Rose, P. Wedrich, H. Queffelec, A. Sartori, D. Tubbenhauer, M. Stosic, ...)

Explicit evaluation: L. H. Robert and E. Wagner (arxiv 2017).

Evaluation for $GL(N)$ foams: Colors $\{1, \ldots, N\}$. Facets carry thickness from 1 to N.



Color a facet of thickness a by a subset of cardinality a. Disjoint union condition along seams.

$F_{ij}(c)$ all facets which contain exactly one color from $\{i, j\}$. Term $(x_i - x_j)^{\chi(F_{ij}(c))/2}$ in the denominator. Subtle signs that involve $\chi(F_i(c))/2$.

Categorical constructions of $sl(N)$ homology:

1. Explicit description for $N = 2, 3$,
2. Highest weight categories for $sl(k)$ (Bernstein-I.Frenkel-M.K., Sussan, Brundan-Stroppel, Sussan-Stroppel, ...),
3. Coherent sheaves on suitable quiver varieties and convolution varieties for affine Grassmannians (Cautis-Kamnitzer),
4. Fukaya-Floer categories on quiver varieties (Seidel-Smith, Manolescu, Abouzaid-Smith),
5. Webster homology (Koszul dual, works for all $\mathfrak{g}$, all reps),
6. Equivalent categories sit in many rep theory structures,
7. Soergel category (categorification of Hecke algebras).

Equivalences are known in some but not all cases. Foam evaluation is different in that introducing categories can be postponed until the very end and everything is manifestly combinatorial (algebraic). Expect that all these examples hide foams and foam evaluation (or some modification of it).

Problem: Extend to other representations of $sl(N)$, beyond $\Lambda^k V$, and to other $\mathfrak{g}$.

Question: Is there a relation between these surface counting models and 3D statistical mechanics?

Symmetric homology (Cautis; Rose, Queffelec, Sartori; Robert and Wagner); categorifies quantum invariants for $S^k V$ representations.

Categorification of Jones polynomial at roots of unity (Sussan and Y.Qi).

To construct homology, we break the symmetry from $\mathbb{R}^3$ to $\mathbb{R}^2 \times \mathbb{R}$. Construction of state spaces does not use homological algebra. Homology appears later, when extending to links and link cobordisms and forming complexes of state spaces (also for tangles and tangle cobordisms).

Symmetric homology and root of unity require further symmetry breaking and working with braid closures (from $\mathbb{R}^2$ to annulus). Open problem to find a more invariant definition.

Happy anniversary, Chern-Simons theory!