On the modularity of elliptic curves over imaginary quadratic fields

Ana Caraiani (Bonn / Imperial)
joint work w James Newton

Thm A (Wiles, Taylor-Wiles, Breuil-Conrad-Diamond-Taylor)

Let $E/\mathbb{Q}$ be an elliptic curve. Then E is modular, i.e. $\exists$ $f \in S_2(\Gamma_0(N), \mathbb{C})$ a Hecke eigenform s.t. $a_\ell(f) = a_\ell(E)$ $\forall\, \ell \nmid N$ prime

eigenvalue of T_ℓ on f

$\ell + 1 - \# E(\mathbb{F}_\ell)$

Thm A (Wiles, Taylor-Wiles, Breuil-Conrad-Diamond-Taylor)

Let $E/\mathbb{Q}$ be an elliptic curve. Then E is modular, i.e. $\exists$ $f \in S_2(\Gamma_0(N), \mathbb{C})$ a Hecke eigenform s.t. $a_\ell(f) = a_\ell(E)$ $\forall$ $\ell \nmid N$ prime

eigenvalue of T_ℓ on f $\qquad$ $\ell + 1 - \# E(\mathbb{F}_\ell)$

By Eichler-Shimura, $\exists$ $m_f \subset \mathbb{T}$ in support of $H^1(X_{\Gamma_0(N)}(\mathbb{C}), \mathbb{C})$ w prescribed Hecke eigenvalues.

$$\mathbb{T} = \mathcal{H}_{\Gamma_0(N)}$$

Consequences :

1). Analytic continuation of $L(E,s)$ to all of $\mathbb{C}$: Hasse-Weil conjecture

2). Diophantine applications : Fermat's last theorem !

3). Input for results on BSD in ranks 0 & 1 e.g. Gross-Zagier.

Structure of the proof: Let p be a prime.

Want to show $\rho_{f,p} \simeq \rho_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}_p)$.

$\hookrightarrow$ some f

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) =: G_{\mathbb{Q}}$

Structure of the proof: Let p be a prime.

Want to show $\rho_{f,p} \simeq \rho_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}_p)$.

$\hookrightarrow$ some f $\qquad\qquad\qquad\qquad \overset{\shortparallel}{G_{\mathbb{Q}}}$

① Modularity lifting:

if $\bar{\rho}_{E,p} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_p)$ is modular & has large image,

then $\rho_{E,p}$ is modular via Taylor-Wiles method

Structure of the proof: Let p be a prime.

Want to show $\rho_{f,p} \simeq \rho_{E,p} : \underset{\underset{G_{\mathbb{Q}}}{\parallel}}{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \to GL_2(\mathbb{Z}_p)$.

$\hookrightarrow$ some f

① Modularity lifting:

if $\bar{\rho}_{E,p} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_p)$ is modular & has large image,

then $\rho_{E,p}$ is modular via Taylor-Wiles method

② Residual modularity:

Langlands-Tunnell thm implies that $\bar{\rho}_{E,3}$ is modular

subtlety: weight 1 $\underset{\text{congruence}}{\rightsquigarrow}$ weight 2

3 3-5 modularity switch: if $\bar{\rho}_{E,3}$ has small image, find $E'/\mathbb{Q}$ s.t.

$\bar{\rho}_{E,5} \simeq \bar{\rho}_{E',5}$ & $\bar{\rho}_{E',3}$ has large image

$X_E(5) \simeq \mathbb{P}^1 \rightsquigarrow$ use Hilbert irreducibility to produce rat'l points

3 3-5 modularity switch: if $\bar{\rho}_{E,3}$ has small image, find $E'/\mathbb{Q}$ s.t.

$\bar{\rho}_{E,5} \simeq \bar{\rho}_{E',5}$ & $\bar{\rho}_{E',3}$ has large image

$X_E(5) \simeq \mathbb{P}^1 \rightsquigarrow$ use Hilbert irreducibility to produce rat'l points

4 Understand exceptions, e.g. if $\bar{\rho}_{E,3}$ & $\bar{\rho}_{E,5}$ are both reducible $\rightsquigarrow$ E corresponds to a point in $X_0(15)(\mathbb{Q})$

$X_0(15)/\mathbb{Q}$ is an elliptic curve w MW rk 0

Thm B (Freitas - Le Hung - Siksek)

Let $F/\mathbb{Q}$ be real quadratic. Let E/F be an elliptic curve.

Then E is modular, i.e. $\exists$ parallel wt 2 Hilbert modular eigenform w. eigenvalues $a_{\mathfrak{p}}(E) = \# k(\mathfrak{p}) + 1 - \# E(k(\mathfrak{p}))$

$\forall\ \mathfrak{p} \nmid N$ prime of F.

Thm B (Freitas - Le Hung - Siksek)

Let $F/\mathbb{Q}$ be real quadratic. Let E/F be an elliptic curve.

Then E is modular, i.e. $\exists$ parallel wt 2 Hilbert modular eigenform w. eigenvalues $a_\mathfrak{p}(E) = \# k(\mathfrak{p}) + 1 - \# E(k(\mathfrak{p}))$

$\forall\, \mathfrak{p} \nmid N$ prime of F.

Again, by Eichler-Shimura, it's equivalent to find $m \subset \mathbb{T}$ in support of $H^2(X_\Gamma(\mathbb{C}), \mathbb{C})$

Hilbert modular surface

1. MLT proved using Kisin's improvements to the Taylor-Wiles method

3. Modularity switch 3-5 & 3-7

$X_E(7)$ = twist of Klein quartic

$$x^3y + y^3z + z^3x = 0$$

4. Handling exceptional cases is more involved, all classified by curves of genus >1

Now let $F/\mathbb{Q}$ be imaginary quadratic

$\rightsquigarrow X_\Gamma$ Bianchi 3-manifold

Thm (C-Newton) Let E/F be a non-CM elliptic curve s.t. one of the following holds:

1). the action of G_F on $E[5]$ is irreducible

2). the action of G_F on $E[3]$ is irreducible & the image of G_F in $\mathrm{Aut}(E[3])$ is not the normalizer of a split Cartan subgroup.

Then E is modular.

Corollary: If $X_0(15)$ has MW rk 0 over F, then every (non-CM) elliptic curve E/F is modular

Rmk: This applies to $\mathbb{Q}(\sqrt{d})$ for $d \leq 5$. Goldfeld's conjecture predicts this applies to $> 50\%$ of IQFs.

Previous results:

- potential modularity: Boxer-Calegari-Gee-Pilloni
  ten authors
- modularity of positive proportion of elliptic curves: Allen-Khare-Thorne
  Whitmore

1. MLT via Calegari-Geraghty method, needs LGC in torsion setting.

2, 3. residual modularity results mod 3 & mod 5 of AKT

avoid Langlands-Tunnell, use 2-3 switch
3-5 switch

4. exceptional cases lie on six modular curves

$X(b3, b5) = X_0(15)$, $X(s3, b5)$, $X(ns3^\circ, b5)$

$X(b3, ns5)$, $X(s3, ns5)$, $X(ns3^\circ, ns5)$

1. MLT via Calegari-Geraghty method, needs LGC in torsion setting.

2, 3. residual modularity results mod 3 & of AKT
mod 5

avoid Langlands-Tunnell, use 2-3 switch
3-5 switch

4. exceptional cases lie on six modular curves

$X(b3, b5) = X_0(15)$, $X(s3, b5)$, $X(ns3^\circ, b5)$

$X(b3, ns5)$, $X(s3, ns5)$, $X(ns3^\circ, ns5)$

Problem: can't do 3-7 switch. Hard to produce solvable CM points on $X_E(7)$.

Details of the proof:

$E/\mathbb{Q}_p$ finite, $\mathcal{O} \subset E$ integers, $\varpi \in \mathcal{O}$, $\mathbb{F} = \mathcal{O}/(\varpi)$

F - CM field

$G = \mathrm{Res}_{F/\mathbb{Q}} GL_n$ $\Big\}$ $\rightsquigarrow$ X_K loc symm space

$K \subset GL_n(\mathbb{A}_{F,f})$ suff small

λ highest wt for G $\rightsquigarrow$ σ_λ alg rep'n of G over E

$\sigma_\lambda^\circ \subset \sigma_\lambda$ K_p-stable $\mathcal{O}$-lattice $\rightsquigarrow$ local system of $\mathcal{O}$-modules $\mathcal{V}_\lambda$ on X_K

$$\mathbb{T}(K,\lambda) := \mathrm{Im}\left(\mathbb{T} \longrightarrow \mathrm{End}_{\mathcal{O}}(H^*(X_K, \mathcal{V}_\lambda))\right)$$

$\cup$

$\mathfrak{m}$ max'l ideal

Scholze: $\bar{\rho}_{\mathfrak{m}} : G_F \to GL_n(\mathbb{F})$ cts, semi-simple s.t. Satake parameters of $\mathfrak{m}$ match Frobenius eigenvalues of $\bar{\rho}_{\mathfrak{m}}$.

Assume $\bar{\rho}_{\mathfrak{m}}$ is abs irreducible. Scholze also constructed

$$\rho_{\mathfrak{m}} : G_F \longrightarrow GL_n(\mathbb{T}(K,\lambda)^{\wedge}_{\mathfrak{m}}/I)$$

Assume $\forall\, v | p$ prime of F, $K_v = GL_n(\mathcal{O}_{F,v})$

Conj (LGC as formulated by Gee-Newton):

$$\begin{tikzcd}
R^{\square}_{\bar{\rho}_{\mathfrak{m}}|G_{F_v}} \arrow[rr, "\text{restriction}"'] \arrow[dr] & & R_{\bar{\rho}_{\mathfrak{m}}} \arrow[r, "\text{induced by } \rho_{\mathfrak{m}}"'] & \mathbb{T}(K,\lambda)^{\wedge}_{\mathfrak{m}}/I \\
& R^{\square,\, \mathrm{cris}\,(\lambda_v)}_{\bar{\rho}_{\mathfrak{m}}|G_{F_v}} \arrow[urr, dashed] & &
\end{tikzcd}$$

constructed by Kisin, Liu

$$\begin{tikzcd} R^{\square}_{\bar\rho_m|G_{F_v}} \arrow[r] \arrow[dr] & R_{\bar\rho_m} \arrow[r] & \mathbb{T}(K,\lambda)^{\wedge}_{m}/I \\ & R^{\square,\mathrm{cris}}_{\bar\rho_m|G_{F_v}}(\lambda_v) \arrow[ur, dashed] & \end{tikzcd}$$

Thm (C-Newton): Under certain technical assumptions, (e.g. need enough places $v | p$ & $\bar\rho_m$ to have large image) diagram factors as desired.

Rmk:
- A'Campo proved related results in char 0
- Hevesi working on pot ss case, understanding Hecke action at p.

Starting point:

- $\tilde{G}/\mathbb{Q}$ quasi-split unitary gp, $P \subset \tilde{G}$ max'l parabolic

$$P = G \ltimes U$$

- $\tilde{X}_{\tilde{K}}$ Shimura variety for $\tilde{G}$ of some level $\tilde{K}$, $\dim_{\mathbb{C}} \tilde{X}_{\tilde{K}} = d$
- $\tilde{m} \subset \tilde{\mathbb{T}}(\tilde{K}, \tilde{\lambda})$ pullback of $m \subset \mathbb{T}(K, \lambda)$

Thm (Scholze, Koshikawa): There exists a $\tilde{\mathbb{T}}$-equivariant diagram:

$$H^d(\tilde{X}_{\tilde{K}}, \mathcal{V}_{\tilde{\lambda}})_{\tilde{m}}[1/p] \hookleftarrow H^d(\tilde{X}_{\tilde{K}}, \mathcal{V}_{\tilde{\lambda}})_{\tilde{m}} \twoheadrightarrow H^d(\partial \tilde{X}_{\tilde{K}}^{BS}, \mathcal{V}_{\tilde{\lambda}})_{\tilde{m}}$$

understand this from essentially self-dual case

show that $H^i(X_K, \mathcal{V}_\lambda)_m$ contributes here for $0 \leq i \leq d-1$

New ideas (roughly):

1). at $\overline{v}$, work w P-ordinary parts:

- $\widetilde{K}_{\overline{v}}$ = parahoric corr to Siegel parabolic $P \subset GL_{2n}(\mathcal{O}_{F,v})$
- we apply ord parts w.r.t. $(\underbrace{\varpi_v, \dots, \varpi_v}_{n \text{ times}}, 1, \dots, 1)$

Upshot: produce a global lift $\rho_{\tilde{m}}$ of $\overline{\rho}_m \oplus \overline{\rho}_m^\vee(1-2n)$

$$\rho_{\tilde{m}}|_{G_{F_v}} \simeq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

char 0 lift of $\rho_m|_{G_{F_v}}$

2). at auxiliary primes $\bar{v}' \neq \bar{v}$, $\bar{v}' | p$, we want degree-shifting when p is small, e.g. 3 or 5, & highly ramified in F

Set $\mathcal{U}^{\circ}_{\bar{v}'} := \mathcal{U}(\mathcal{O}_{F^+_{\bar{v}'}})$

Want $R\Gamma(\mathcal{U}^{\circ}_{\bar{v}'}, \mathcal{O}/\varpi^m) \simeq \bigoplus_{i=0}^{d} H^i(\mathcal{U}^{\circ}_{\bar{v}'}, \mathcal{O}/\varpi^m)[-i]$

in $D^+_{sm}(G(\mathcal{O}_{F^+_{\bar{v}'}}), \mathcal{O}/\varpi^m)$

replace by deeper level

$K_{\bar{v}'}(M)$, $M >> m$.

Happy birthday,

Shou-Xu!