Diagonal classes and the Birch and Swinnerton-Dyer conjecture

MSRI Workshop celebrating Shou-Wu Zhang’s 60th Birthday

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Setting (triple product)

- $E/\mathbb{Q}$ elliptic curve.

- $$\varrho : G_{\mathbb{Q}} \longrightarrow \mathrm{SL}_4(\mathbb{Q}_\varrho)$$

  is the Artin representation equal to $\varrho_1 \otimes \varrho_2$ for *two-dimensional, odd, irreducible* Artin representations ϱ_1 and ϱ_2 such that $\det(\varrho_1) = \det(\varrho_2)^{-1}$.
  By definition ϱ factors through a number field $K_\varrho = \bar{\mathbb{Q}}^{\ker(\varrho)}$.

- $E(K_\varrho)^\varrho := \mathrm{Hom}_{G_{\mathbb{Q}}}(V(\varrho), E(K_\varrho) \otimes \mathbb{Q}_\varrho)$, the *$\varrho$-component* of the Mordell–Weil group of E.

Let $L(E, \varrho, s)$ be the L-function of E twisted by ϱ, i.e. the L-function attached to the system of 8-dimensional ℓ-adic representations $V_\ell(E) \otimes_{\mathbb{Q}_\ell} V_\ell(\varrho)$. It converges for $\Re(s) > 3/2$ and extends analytically at $s = 1$.

The equivariant Birch and Swinnerton-Dyer conjecture

Conjecture ($BSD(E, \varrho)$)

One has the equality

$$r_{\mathrm{an}}(E, \varrho) := \mathrm{ord}_{s=1} L(E, \varrho, s) \stackrel{?}{=} \dim_{\mathbb{Q}_\varrho} E(K_\varrho)^\varrho =: r(E, \varrho)$$

between the *algebraic rank* $r(E, \varrho)$ and the *analytic rank* $r_{\mathrm{an}}(E, \varrho)$ of E.

The *finiteness* of (the ϱ-component of) the Shafarevich–Tate group $\underline{III}(E/K_\varrho)$ is also conjectured.

Theorem (Darmon–Rotger)

$BSD(E, \varrho)$ holds in analytic rank zero.

I will comment about the proof of this theorem later.

The cases of ϱ_i reducible are also very interesting.

What if $r_{\mathrm{an}}(E, \varrho) > 0$?

Question

Suppose that $r_{\mathrm{an}}(E, \varrho) > 0$, i.e. $L(E, \varrho, 1) = 0$. In light of $BSD(E, \varrho)$ one expects non-trivial points in the Mordell-Weil group $E(K_\varrho)^\varrho$. How can one construct such points? On a less ambitious level, how can one construct non-trivial cohomology classes in a p-adic Selmer group $S_p(E, \varrho) = S_p(E/K_\varrho)^\varrho$ containing $E(K_\varrho)^\varrho$?

This question, as well as the case $r_{\mathrm{an}}(E, \varrho) = 0$ case described above, can be addressed by p-adic methods. The analytic side of these methods involves p-adic L-functions.

Testing ground

Classical setting:

$K = \mathbb{Q}[\sqrt{D}]$, $D < 0$ quadratic imaginary; $\eta_i : G_K \longrightarrow \bar{\mathbb{Q}}^\times$, $i = 1, 2$ ray class characters; $\varrho_i := \mathrm{Ind}_K^{\mathbb{Q}}(\eta_i) : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{Q}_{\varrho_i})$ satisfying our assumptions. One has

$$\varrho = \varrho_1 \otimes \varrho_2 = \mathrm{Ind}_K^{\mathbb{Q}}(\phi) \oplus \mathrm{Ind}_K^{\mathbb{Q}}(\psi), \quad \phi = \eta_1\eta_2, \quad \psi = \eta_1\eta_2^c$$

for ring class characters ϕ and ψ. It follows

$$L(E, \varrho, s) = L(E/K, \phi, s) \cdot L(E/K, \psi, s), \quad E(K_\varrho)^\varrho = E(K_\phi)^\phi \oplus E(K_\psi)^\psi.$$

Testing ground

Consider the above “Question” in the classical setting. Since $L(E, \varrho, 1) = 0$, one has $L(E/K, \phi, 1) = 0$ or $L(E/K, \psi, 1) = 0$. Say $L(E/K, \phi, 1) = 0$. If in addition $L'(E/K, \phi, 1) \neq 0$, the *Gross–Zagier–Zhang* formula implies the existence of a non-trivial *Heegner point* in $E(K_\phi)^\phi$, hence $E(K_\varrho)^\varrho \neq 0$.

In general, if $L(E/K, \phi, 1) = 0$, one can construct a non-trivial Selmer class in $S_p(K_\phi)^\phi$ and hence in $S_p(K_\varrho)^\varrho$, by using *Iwasawa theory* at an ordinary prime p for E.

Garrett p-adic L-functions

We describe the analytic tool used to undertake the above mentioned results and questions.

By the modularity theorem of *Wiles, Taylor–Wiles, et al.*, $L(E, s) = L(f, s)$ for a weight 2 cuspidal eigenform f.

By the solution of the Serre conjecture by *Khare–Wintenberger*, $L(\varrho_1, s) = L(g, s)$ and $L(\varrho_2, s) = L(h, s)$ for weight 1 cuspidal eigenforms g and h.

Choose an *ordinary* prime p for E (i.e. $p \nmid a_p(f)$) and assume that g, h are *p-regular*. By Hida theory, the triple (f, g, h) belongs to a triple of p-adic families of ordinary eigenforms $(\mathbf{f}, \mathbf{g}, \mathbf{h})$.

Garrett p-adic L-functions

This means that

$$\mathbf{f} = \sum_{n \geqslant 1} \mathbf{a}_n(\mathbf{k}) q^n \in \mathcal{O}(U_f)[\![q]\!],$$

where U_f is a p-adic disc centred at 2 and $\mathcal{O}(U_f)$ is a ring of bounded analytic functions on U_f. For a classical weight $k \in U_f \cap \mathbb{Z}_{\geqslant 2}$, the specialisation $\mathbf{f}(k)$ of $\mathbf{f}$ at k is (the p-stabilisation of) a weight k cuspidal eigenform f_k.

Similarly for the definition of $\mathbf{g}$ and $\mathbf{h}$.

Consider the *triple-product Garrett* complex L-function $L(f_k \otimes g_\ell \otimes h_m, s)$ with $(k, \ell, m) \in (U_f \times U_g \times U_h) \cap \mathbb{Z}^3_{\geqslant 1}$. It admits an analytic continuation to $\mathbb{C}$ and a functional equation with sign $\varepsilon(k, \ell, m) = \pm 1$ for $s \longmapsto k + \ell + m - 2 - s$.

Note that $L(E, \varrho, s) = L(f \otimes g \otimes h, s)$, so that $L(E, \varrho, s)$ is defined at $s = 1$.

Garrett p-adic L-functions

Assumption

Assume that $\varepsilon(k,\ell,m) = +1$ either

a) (**unbalanced case**) in the region Σ^f of weights $(k,\ell,m) \in (U_f \times U_g \times U_h) \cap \mathbb{Z}^3_{\geqslant 1}$ such that $k \geqslant \ell + m$, or

b) (**balanced case**) in the region Σ^{bal} of weights $(k,\ell,m) \in (U_f \times U_g \times U_h) \cap \mathbb{Z}^3_{\geqslant 1}$ such that $k < \ell + m$, $\ell < k + m$ and $m < k + \ell$.

Definition

The *Garrett p-adic L-function* $L_p(E,\varrho)$ is defined to be an element of $\mathcal{O}_{fgh} := \mathcal{O}(U_f \times U_g \times U_h)$, such that for $(k,\ell,m) \in \Sigma^f$ or Σ^{bal}

$$L_p(E,\varrho)(k,\ell,m) = C(k,\ell,m) \cdot L(f_k \otimes g_\ell \otimes h_m, (k+\ell+m-2)/2),$$

where $C(k,\ell,m)$ is a generically non-zero explicit constant.

p-adic BSD conjecture

Remark

1) The construction of the above p-adic L-functions in the explicit version necessary here is due to *Hsieh*. It builds on the work of several people, including *Gross–Kudla, Harris–Kudla, Ichino, Hida, Harris–Tilouine, Darmon–Rotger, Greenberg–Seveso*.

2) In the unbalanced case the point $(2, 1, 1)$ belongs to the region of classical interpolation Σ^f. In particular, the sign of the functional equation of $L(E, \varrho, s)$ is $+1$. If $L(E, \varrho, 1) = 0$, then $L(E, \varrho, s)$ vanishes to *even* order at $s = 1$ and $BSD(E, \varrho)$ leads to expect that the rank of $E(K_\varrho)^\varrho$ is even.

In this case the behaviour of $L_p(E, \varrho)$ at $(2, 1, 1)$ should reflect the arithmetic of $E(K_\varrho)^\varrho$ and may lead to a *p-adic analogue* $BSD_p(E, \varrho)$ of $BSD(E, \varrho)$.

p-adic BSD conjecture

Remark

3) In the balanced case $L(E, \varrho, s)$ vanishes to *odd* order at $s = 1$, so that the rank of $E(K_\varrho)^\varrho$ is conjecturally odd. In this setting the point $(2, 1, 1)$ lies outside the region of p-adic interpolation. The value of $L_p(E, \varrho)$ at $(2, 1, 1)$ may be seen as a p-adic avatar of the leading term of $L(E, \varrho, s)$ at $s = 1$.

We first focus on the *unbalanced* case, with the formulation of a p-adic BSD conjecture.

p-adic BSD conjecture (unbalanced case)

Conjecture ($\mathrm{BSD}_p(E, \varrho)$, B.–Seveso–Venerucci)

Assume for simplicity that E has good reduction at p.

1) Then $L_p(E, \varrho)$ belongs to $\mathcal{I}^{r(E,\varrho)}$ where $\mathcal{I}$ is the ideal in $\mathcal{O}_{fgh}$ of functions vanishing at $(2, 1, 1)$, i.e. $L_p(E, \varrho)$ vanishes to order $\geqslant r(E, \varrho)$ at $(2, 1, 1)$.

2) Let $L_p(E, \varrho)^*$ be the image of $L_p(E, \varrho)$ in $\mathcal{I}^{r(E,\varrho)}/\mathcal{I}^{r(E,\varrho)+1}$. Then

$$L_p(E, \varrho)^* = R_p(E, \varrho) \quad (\text{up to } (\mathbb{Q}_\varrho^\times)^2),$$

where $R_p(E, \varrho)$ is the discriminant of a p-adic weight height pairing *à la Nekovàř*

$$\langle\langle\ ,\ \rangle\rangle : E(K_\varrho)^\varrho \otimes E(K_\varrho)^\varrho \longrightarrow \mathcal{I}/\mathcal{I}^2$$

attached to our Hida p-adic deformation.

p-adic BSD conjecture (unbalanced case)

Remark

1) $BSD_p(E, \varrho)$ can be formulated also at *p multiplicative* for E, where $E(K_\varrho)^\varrho$ must be replaced by an *extended Mordell–Weil group*.

2) We have verified $BSD_p(E, \varrho)$ for p multiplicative in low rank cases.

3) Both these cases of $BSD_p(E, \varrho)$, as well as the theorem of Darmon–Rotger mentioned at the beginning of this talk, and partly also the previous “Question” can be addressed via an *Explicit reciprocity law* for $L_p(E, \varrho)$ which I now briefly describe.

The explicit reciprocity law (unbalanced case)

One can define a 3-variable *diagonal class*

$$\kappa(E,\varrho) \in S_p(V(\mathbf{f},\mathbf{g},\mathbf{h})) \subset H^1(G_{\mathbb{Q}}, V(\mathbf{f},\mathbf{g},\mathbf{h})),$$

where $V(\mathbf{f},\mathbf{g},\mathbf{h})$ is the "big" Galois representation attached to the triple $(\mathbf{f},\mathbf{g},\mathbf{h})$ and $S_p(V(\mathbf{f},\mathbf{g},\mathbf{h}))$ is a so-called balanced p-adic Selmer group. The class $\kappa(E,\varrho)$ arises from the diagonal embedding $X_1(N) \longrightarrow X_1(N)^3$ of the modular curve $X_1(N)$ in its triple product.

Moreover, there is a "big" *Perrin-Riou logarithm*

$$\mathrm{Log} : S_p(V(\mathbf{f},\mathbf{g},\mathbf{h})) \longrightarrow \mathcal{O}_{fgh},$$

which interpolates the relevant branch of Bloch–Kato logarithms.

The *explicit reciprocity law (ERL)* states that

$$L_p(E,\varrho) = \mathrm{Log}(\kappa(E,\varrho))^2.$$

The explicit reciprocity law (unbalanced case)

Remark

These diagonal cycles play a prominent role in the work by *S-W. Zhang* with *X. Yuan* and *W. Zhang* on the study of the derivative of the complex triple product L-function for forms in the balanced domain.

Applications of the ERL

I) When $r_{\mathrm{an}}(E, \varrho) = 0$ the specialisation $\kappa(E, \varrho)(2, 1, 1)$ of $\kappa(E, \varrho)$ at $(2, 1, 1)$ gives rise to *p-ramified* classes in $H^1(G_{\mathbb{Q}}, V_p(f) \otimes_{\mathbb{Q}_p} V_p(g) \otimes_{\mathbb{Q}_p} V_p(h))$. This can be used to bound $E(K_\varrho)^\varrho$. (Cf. the result of Darmon–Rotger mentioned above.)

II) When $r_{\mathrm{an}}(E, \varrho) > 0$, $\kappa(E, \varrho)(2, 1, 1)$ is a Selmer class. In the *classical setting* with p multiplicative and inert in K, this class can be related to Heegner points in $E(K_\varrho)^\varrho$. In the analogous setting with K real quadratic one obtains a relation to Stark–Heegner points. (Cf. the recent Astérisque volume by B.–Seveso–Venerucci and Darmon–Rotger.)

The p-adic BSD conjecture in rank 2

Assume that the $\mathbb{Q}(\varrho)$-vector space $E(K_\varrho)^\varrho$ has dimension 2 with basis (P, Q) and that $\underline{III}(E, K_\varrho)^\varrho$ is finite.

Theorem (B.–Seveso–Venerucci)

$BSD_p(E, \varrho)$ implies the identity

$$\kappa(E, \varrho)(2, 1, 1) = \log(P) \cdot Q - \log(Q) \cdot P$$

in $S_p(E, \varrho)$ up to $\mathbb{Q}(\varrho)^\times$.

Remark

The above identity has been conjectured by *Darmon–Lauder–Rotger*, based on experimental evidence and an analysis of some instances of the *classical setting*.

The p-adic BSD conjecture in rank 2

Let $\log : S_p(E, \varrho) \to \mathbb{C}_p$ be a suitable branch of the Bloch–Kato logarithm.

A proof of this identity is obtained from the following formula for the Nekovàř height extended to $S_p(E, \varrho)$ (up to explicit Euler factors):

Theorem (B.–Seveso–Venerucci)

For all $s \in S_p(E, \varrho)$ one has

$$\langle\langle \kappa(E, \varrho)(2, 1, 1), s \; \rangle\rangle \approx \log(s) \cdot L_p(E, \varrho)^{\frac{1}{2}} \pmod{\mathcal{I}^2}.$$

The balanced setting

Assume we are in the balanced setting, so that $L(E, \varrho, s)$ vanishes to *odd* order at $s = 1$ and by the equivariant BSD-conjecture one expects that $E(K_\varrho)^\varrho$ has odd rank. Note that the point $(2, 1, 1)$ does not belong to the region Σ^{bal} of classical interpolation for the definite p-adic L-function $L_p(E, \varrho)$.

Theorem (Andreatta–B.–Seveso–Venerucci)

If $L_p(E, \varrho) \not\equiv 0$, then $S_p(E, \varrho)$ is non-zero.

Remark

1) The regulator $R_p(E, \varrho)$ considered in the above $\mathrm{BSD}_p(E, \varrho)$ must vanish in the definite setting, since the Nekovàř pairing $\langle\langle\ ,\ \rangle\rangle$ on $E(K_\varrho)^\varrho$ is skew-symmetric. We can consider here a regulator of the form $\log(P)^2 \cdot R_p'(E, \varrho)$, where P is a point in the radical of the above pairing and $R_p'(E, \varrho)$ is the discriminant for a complement of P.

The balanced setting

Remark

2) In rank 1 one deduces the formula $L_p(E,\varrho)(2,1,1) \approx \log(P)^2$, which we have verified in the classical setting building on a result of B.–Darmon–Prasanna.

3) The proof of the above theorem is based on an ERL in the balanced setting (see below).

4) Assuming $L(E,\varrho,1)=0$ in the *unbalanced* setting, the ERL implies that the diagonal class $\kappa(E,\varrho)(2,1,1)$ is crystalline. Although one expects that $S_p(E,\varrho)$ is non-zero, the analogue of the above theorem does not seem to be within reach without imposing a non-vanishing condition on $\kappa(E,\varrho)(2,1,1)$.

Strategy of proof

For classical weights (k,ℓ,m) let $V(f_k, g_\ell, h_m)$ be the specialisation of $V(\mathbf{f},\mathbf{g},\mathbf{h})$ at (k,ℓ,m).

Let

$\mathcal{X}^{\mathrm{geom}} := \{(k,k',\ell,m) \in (U_f \times U_f \times U_g \times U_h) \cap \mathbb{Z}^4_{\geqslant 2} : (k,\ell,m) \in \Sigma^f,\ (k',\ell,m) \in \Sigma^{\mathrm{bal}},\ k \geqslant k'\}.$

The theorem is a consequence of the following ERL.

Theorem ((ERL) Andreatta–B.–Seveso–Venerucci)

For $(k,k',\ell,m) \in \mathcal{X}^{\mathrm{geom}}$ *there is a class* $\kappa_{k'}(f_k, g_\ell, h_m) \in S_p(V(f_k, g_\ell, h_m))$ *such that*

$$\log^2\left(\kappa_{k'}(f_k, g_\ell, h_m)\right) = L_p(E,\varrho)(k',\ell,m) \cdot L_p(E,\varrho)(k,\ell,m),$$

where $\log$ *is a branch of the Bloch–Kato logarithm.*

Strategy of proof

To construct $\kappa_{k'}(f_k, g_\ell, h_m)$, one considers the holomorphic genus 2 Siegel eigenform $F_{k,k'} = Y(\ f_k,\ f_{k'})$ defined as a Yoshida lift of the Jacquet–Langlands lift of the pair $(f_k, f_{k'})$ associated with the definite quaternion algebra dictated by $\mathrm{sign}(E, \varrho) = -1$. Recall that $L(F_{k,k'}, s) = L(f_k, s) \cdot L(f_{k'}, s + (k - k')/2)$.

By a theorem of Weissauer one has $H^3_{et}(X_{\bar{\mathbb{Q}}}, \mathcal{S}_{k,k'})[F_{k,k'}] = V(f_k)$, where X is a Siegel 3-fold and $\mathcal{S}_{k,k'}$ is a suitable étale sheaf.

One has a natural embedding $Y^2 \longrightarrow X$, where Y is a modular curve, which gives rise to a diagonal embedding $Y^2 \longrightarrow Y^2 \times X$.

The Abel-Jacobi formalism combined with Weissauer's theorem defines the class $\kappa_{k'}(f_k, g_\ell, h_m)$ in $H^1(\mathbb{Q}, V(\ f_k,\ g_\ell,\ h_m))$, which is Selmer by a theorem of *Nekovàř-Nizioł* (and also follows as a byproduct of our work).

Strategy of proof

Finally, the ERL follows from the p-adic interpolation (and geometric interpretation) of a formula of *Böcherer–Furusawa–Shulze-Pillot* extended by *Gan–Ichino* when (k, ℓ, m) and (k', ℓ, m) both belong to Σ^{bal}:

$$L(f_k \otimes g_\ell \otimes h_m, \frac{k+\ell+m-2}{2}) \cdot L(f_{k'} \otimes g_\ell \otimes h_m, \frac{k'+\ell+m-2}{2}) =$$

$$\langle \delta F_{k,k'}|_{\mathcal{H}\times\mathcal{H}}, g_\ell \otimes h_m \rangle,$$

where δ is a certain Shimura–Maaß differential operator and $\langle\ ,\ \rangle$ denotes the Petersson inner product on the product $\mathcal{H} \times \mathcal{H}$ of two copies of the upper half plane.