Zeta morphisms for rank two universal deformations

Kentaro Nakamura

Saga University

March 22, 2023

Notation

- p: a prime number.
- $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\quad \iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$: fixed embeddings.
- $L/\mathbb{Q}_p$: a (sufficiently large) finite extension in $\overline{\mathbb{Q}}_p$, $\quad \mathcal{O} = \mathcal{O}_L$,
  $\varpi \in \mathcal{O}$: a uniformizer, $\quad \mathbb{F} = \mathcal{O}/(\varpi)$.
- $\Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$, $\quad \Lambda = \mathcal{O}[[\Gamma]]$: the Iwasawa algebra of Γ.
- For a field F, we set $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$.

Kato's zeta morphisms

- $f = \sum_{n=1}^{\infty} a_n q^n \in S_k^{\mathrm{new}}(\Gamma_1(N))$: a normalized Hecke eigen cusp newform of level $N \geqq 1$, weight $k \geqq 2$ with a neben type character $\chi_f : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$.
- $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$: a Galois representation associated to f, i.e. odd and unramified outside $\Sigma_f = \mathrm{prime}(N) \cup \{p\}$ satisfying

$$\mathrm{tr}(\rho_f(\mathrm{Frob}_\ell)) = a_l$$

  for all $\ell \notin \Sigma_f$.
- $\mathrm{H}^i_{\mathrm{Iw}}(\mathbb{Z}[1/Np], \rho_f^*(1)) := \varprojlim_{m \geqq 0} H^i(\mathbb{Z}[1/Np, \zeta_{p^m}], \rho_f^*(1))$.
- $\mathbf{H}^1(\rho_f^*(1)) := \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/Np], \rho_f^*(1))$,

$$\mathbf{H}^2(\rho_f^*(1)) := \mathrm{Ker}(\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/Np], \rho_f^*(1)) \to \oplus_{\ell \in \Sigma_f \setminus \{p\}} \mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Q}_\ell, \rho_f^*(1))).$$

  These are Λ-modules.

Kato defined a non zero Euler system, i.e.

$$\{z_{np^m} \in H^1(\mathbb{Q}(\zeta_{np^m}), \rho_f^*(1))\}_{m\geq 0, n\geq 1, (n,Np)=1}$$

satisfying the norm relation.

Theorem (12.4 of Kato (04))

- $\mathbf{H}^2(\rho_f^*(1))$ is a torsion Λ-module.
- $\mathbf{H}^1(\rho_f^*(1)) \otimes \mathbb{Q}$ (resp. $\mathbf{H}^1(\rho_f^*(1))$) is free of rank one over $\Lambda \otimes \mathbb{Q}$ (resp. free over Λ if $\overline{\rho}_f$ is absolutely irreducible).

We can define

$$\{z_{p^m}\}_{m \geqq 1} \in \mathbf{H}^1(\rho_f^*(1)),$$

but it is **not canonical**, $\{z_{np^m}\}_{n,m}$ depends on many choices $c, d \geqq 2$ s.t. $(cd, 6pN) = 1$, $1 \leqq j \leqq k-1$ and $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, etc (cf. Kato(04)).

Dividing its dependent factors (and the L-factors at the bad primes $\ell \in \Sigma_f \setminus \{p\}$), Kato obtained the following :

Theorem (12.5 of Kato (04))

(1) $^{\exists}$ a **canonical** $\mathcal{O}$-linear map (**zeta morphism** for f)

$$\mathbf{z}(f)\colon \rho_f^* \to \mathbf{H}^1(\rho_f^*(1)) \otimes \mathbb{Q}$$

interpolating, via Bloch-Kato's dual exponentials and period maps, all the critical values of

$$L_{\{p\}}(f, \chi, s) = \sum_{n=1,(n,p)=1}^{\infty} \frac{a_n \chi(n)}{n^s}$$

for all the finite characters $\chi\colon \Gamma (\xrightarrow{\sim} \mathbb{Z}_p^{\times}) \to \mathbb{C}^{\times}$.

(2) If p is odd and $\overline{\rho}_f = \rho_f \pmod{\varpi}$ is absolutely irreducible,

$$\mathrm{Char}_{\Lambda}(\mathbf{H}^1(\rho_f^*(1))/\Lambda \cdot \mathrm{Im}(\mathbf{z}(f))) \subseteq \mathrm{Char}_{\Lambda}(\mathbf{H}^2(\rho_f^*(1)))$$

Conjecture (12.10 of Kato (04), Kato (93))

(1) (Kato main conjecture, KMC)

$$\mathrm{Char}_\Lambda(\mathbf{H}^1(\rho_f^*(1))/\Lambda\cdot\mathrm{Im}(\mathbf{z}(f))) = \mathrm{Char}_\Lambda(\mathbf{H}^2(\rho_f^*(1)))$$

(2) Such **zeta morphisms exist for all the families of p-adic representations of $G_{\mathbb{Q}}$** which are unramified outside a finite set of primes.

- When $\pi_p(f)$ is non supercuspidal, KMC is equivalent to the usual Iwasawa main conjecture (IMC), i.e. the equality

$$(p\text{-adic } L\text{-function}) = \mathrm{Char}_\Lambda((\text{cyclotomic Selmer group})^\vee),$$

  formulated by Mazur (72), Greenberg (89), Pollack (03)-Kobayashi (03), Lei-Loeffler-Zerbes (10), etc.
- **KMC is formulated for arbitrary** f, e.g. for f s.t. $\pi_p(f)$ is supercuspidal.

Zeta morphisms for rank two universal deformations

- Σ: a finite set of primes containing p.
- $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$: odd, absolutely irreducible, unramified outside Σ.
- $\mathrm{Comp}(\mathcal{O})$: the category of commutative local Noetherian complete $\mathcal{O}$-algebras with finite residue field.
- $\rho_\Sigma\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(R_\Sigma)$: the universal deformation for the deformations $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ ($A \in \mathrm{Comp}(\mathcal{O})$) s.t. $\rho(\mathrm{mod}\,\mathfrak{m}_A) \xrightarrow{\sim} \overline{\rho} \otimes_{\mathbb{F}} A/\mathfrak{m}_A$, unramified outside Σ (no condition at the primes in Σ).
- $X_\Sigma(\overline{\rho}) = \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}-\mathrm{alg}}(R_\Sigma, \overline{\mathbb{Z}}_p)$.
- $X_\Sigma^{\mathrm{mod}}(\overline{\rho})$: the subset of modular points.
- For $\ell \notin \Sigma$, we set

$$P_\ell(T) = \det(1 - \mathrm{Frob}_\ell \cdot T \mid \rho_\Sigma) \in R_\Sigma[T].$$

- For $f \in S_k^{\mathrm{new}}(\Gamma_1(N))$ and $\ell \neq p$, we set

$$P_{f,\ell}(T) = \det(1 - \mathrm{Frob}_\ell \cdot T \mid \rho_f^{I_\ell}) \in \mathcal{O}[T].$$

Theorem (Main Theorem, Nakamura (20))

Assume the following:

(i) $\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible, (ii) $p \geqq 5$, (iii) $\mathrm{End}_{\mathbb{F}[G_{\mathbb{Q}_p}]}(\overline{\rho}) = \mathbb{F}$,

(iv) $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is not of the form $\begin{pmatrix} \overline{\chi}_p^{\pm 1} & * \\ 0 & 1 \end{pmatrix} \otimes \eta \quad (\eta : G_{\mathbb{Q}_p} \to \mathbb{F}^{\times})$.

Then, $^{\exists}$ R_Σ-linear map

$$Z_{\Sigma,n} \colon \rho_\Sigma^* \to \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma_n, \zeta_n], \rho_\Sigma^*(1))$$

for each $n \geqq 1$ s.t. $(n, \Sigma) = 1$ ($\Sigma_n = \Sigma \cup \mathrm{prime}(n)$), satisfying :

(1) $\mathrm{Cor} \circ Z_{\Sigma,n\ell} = \begin{cases} Z_{\Sigma,n} & \text{if } \ell | n \\ P_\ell(\mathrm{Frob}_\ell) \cdot Z_{\Sigma,n} & \text{otherwise} \end{cases}$

(2) $x_f^*(Z_{\Sigma,1}) = \prod_{\ell \in \Sigma \setminus \{p\}} P_{f,\ell}(\mathrm{Frob}_\ell) \cdot \mathbf{z}(f)$ for arbitrary $x_f \in X_\Sigma^{\mathrm{mod}}(\overline{\rho})$ (this is an equality as a map $\rho_f^* \to \mathbf{H}^1(\rho_f^*(1))$).

Namely, $Z_{\Sigma,1}$ interpolates zeta elements which are related with $L_\Sigma(f, \chi, s) = \sum_{n=1,(n,\Sigma)=1}^{\infty} \frac{a_n \chi(n)}{n^s}$.

Some works on zeta morphisms (or Euler systems) for families.

- Ochiai (06), **Fukaya-Kato** (12): Hida families (of ordinary p-adic modular forms).
- Hansen (15), Ochiai (17), Wang (13), Benois-Buyukboduk (21) : Coleman-Mazur eigencurves (families of overconvergent of p-finite slope modular forms).
- Fouquet, Wang: for universal deformations.
- **Colmez-Wang** (21) : similar results (essentially same (?), but different proof).

Application to KMC

For $f_i = \sum_{n=1}^{\infty} a_n(f_i)q^n \in S_{k_i}^{\mathrm{new}}(N_i)$ $(i = 1, 2)$, we say that f_1 and f_2 are **congruent** if

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\varpi}$$

for all but finitely many primes ℓ ($\iff$ $\overline{\rho}_{f_1} \xrightarrow{\sim} \overline{\rho}_{f_2}$ if $\overline{\rho}_{f_1}$ is abs irr).

Corollary

Assume that $\overline{\rho}_{f_1}$ satisfies all the assumptions in Main Theorem. For $\Sigma := \mathrm{prime}(N_1) \cup \mathrm{prime}(N_2) \cup \{p\}$, one has

$$\prod_{\ell \in \Sigma \setminus \{p\}} P_{f_1,\ell}(\mathrm{Frob}_\ell) \cdot \mathbf{z}(f_1) \equiv \prod_{\ell \in \Sigma \setminus \{p\}} P_{f_2,\ell}(\mathrm{Frob}_\ell) \cdot \mathbf{z}(f_2) \pmod{\varpi}.$$

(equality as a map $\overline{\rho}_{f_1}^* \to \mathbf{H}^1(\overline{\rho}_{f_1}^*(1))$)

Remark Kim-Lee-Ponsinet (19) (essentially) proved that such a congruence between zeta morphisms implies the equivalence of KMC.

Corollary

Assume f_1 *and* f_2 *are congruent,* $\overline{\rho}_{f_1}$ *satisfies all the assumptions in Main theorem, and*

$$\mathbf{z}(f_1) \pmod{\varpi} \neq 0.$$

Then one also has

$$\mathbf{z}(f_2) \pmod{\varpi} \neq 0,$$

and one has the following equivalence

$$\text{KMC for } f_1 \text{ holds} \iff \text{KMC for } f_2 \text{ holds.}$$

- I expect that the assumption $\mathbf{z}(f_1) \pmod{\varpi} \neq 0$ always holds.

Known results (Assume that $\overline{\rho}_{f_1}$ is absolutely irreducible and $\mu(f_1) = 0$)

- Greenberg-Vatsal (00): congruent elliptic curves E_1 and E_2 with good ordinary reduction at p (i.e. of weight two).
- Emerton-Pollack-Weston (06): congruent eigenforms which are ordinary at p (of arbitrary weights).
- many related results in many related settings $\cdots$
- Kim-Lee-Ponsinet (19): congruent eigenforms which are of finite slope (not ordinary in general) but with a fixed weight $2 \leqq k \leqq p-1$.
- (Na): all the congruent eigenforms with **arbitrary levels and weights**.

Therefore, we can compare (under the assumption that $\mathbf{z}(f_1) \bmod \varpi \neq 0$)

known IMC (=KMC) for ordinary case (Kato, Skinner-Urban),

or of finite slope case (Kato, X.Wan,,,,)

with

unknown KMC, e.g. for supercuspidal case.

The proof of the main theorem

We mainly explain how to construct our zeta morphism (for $n = 1$)

$$Z_\Sigma := Z_{\Sigma,1} \colon \rho_\Sigma^* \to \mathbf{H}^1(\rho_\Sigma^*(1)).$$

Idea of the proof
Combine

Fukaya-Kato's method

with

the p-adic Langlands correspondence for $\mathrm{GL}_{2/F}$ for $F = \mathbb{Q}, \mathbb{Q}_p, \mathbb{Q}_\ell$

(Colmez (10), Emerton (11), Paskunas (13), (16), Emerton-Helm (14)).

Fukaya-Kato's construction for Hida families

Set $H^1(Y_1(N)) = H^1(Y_1(N)(\mathbb{C}), \mathcal{O})$, etc.

Theorem (Fukaya-Kato (12))

There exists a canonical Hecke equivariant $\mathcal{O}$-linear map

$$\mathbf{z}_{1,N} \colon H^1(Y_1(N)) \to \mathbf{H}^1(H^1(Y_1(N))(1)) \otimes_\Lambda \mathrm{Frac}(\Lambda)$$

interpolating the operator valued L-functions $\sum_{n \geqq 1, (n,p)=1} \frac{T_n \cdot \chi(n)}{n^s}$ acting on $H^1(Y_1(N)(\mathbb{C}), \mathbb{C})$.

We can take, for $N \geqq 1$ such that $p \nmid N$, the limit

$$\mathbf{z}_{1,Np^\infty} : \varprojlim_{m \geqq 1} H^1(Y_1(Np^m)) \to \mathbf{H}^1(\varprojlim_{m \geqq 1} H^1(Y_1(Np^m))(1)) \otimes_\Lambda \Lambda[1/\lambda]$$

for some $\lambda \in \Lambda$. Applying Hida's ordinary projection defined using U_p-operator, we can obtain **the zeta morphisms for Hida families**.

A refined local-global compatibility (Emerton)

For each $N_0 \geqq 1$ such that $\mathrm{prime}(N_0) = \Sigma \setminus \{p\} =: \Sigma_0$, we set

$$\widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0)) := \varprojlim_{m \geqq 0} H^1(Y(N_0 p^m))(1)$$

w.r.t. the corestrictions $H^1(Y(N_0 p^{m+1}))(1) \to H^1(Y(N_0 p^m))(1)$ $(k \geqq 1)$, and

$$\widetilde{H}_{1,\Sigma}^{BM} := \varinjlim_{N_0} \widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0))$$

w.r.t. the restrictions $\widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0)) \to \widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0'))$ for $N_0 | N_0'$.
We set $G_\ell = \mathrm{GL}_2(\mathbb{Q}_\ell)$, $G_\Sigma = \prod_{\ell \in \Sigma} G_\ell$, $G_{\Sigma_0} = \prod_{\ell \in \Sigma_0} G_\ell$.
$\widetilde{H}_{1,\Sigma}^{BM}$ is equipped with actions of $G_{\mathbb{Q}}$ and G_Σ, and (homological) Hecke actions at the primes $\ell \notin \Sigma$. Using its Hecke actions, one can define its $\overline{\rho}$-part

$$\widetilde{H}_{1,\Sigma,\overline{\rho}}^{BM}$$

which is a topological $R_\Sigma[G_{\mathbb{Q}} \times G_\Sigma]$-module.

The following is the dual version of Emerton's theorem.

Theorem (A refined local-global compatibility, Emerton (11))

There exists a topological $R_\Sigma[G_{\mathbb{Q}} \times G_\Sigma]$-linear isomorphism

$$\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}} \stackrel{\sim}{\to} \Pi_p^* \otimes_{R_\Sigma} \rho_\Sigma^* \otimes_{R_\Sigma} \widetilde{\pi}_{\Sigma_0},$$

where

- Π_p is the representation of G_p corresponding to $\rho_\Sigma|_{G_{\mathbb{Q}_p}}$,
- π_{Σ_0} is the representation of G_{Σ_0} corresponding to $\{\rho_\Sigma|_{G_{\mathbb{Q}_\ell}}\}_{\ell\in\Sigma_0}$

by the family version of p-adic local Langlands correspondence defined by Colmez (10) (+many people) for Π_p and Emerton-Helm (14) for π_{Σ_0}.

G_Σ-equivariant zeta morphisms

Proposition (Na)

For each $N_0 \geqq 1$ and $m \geqq 1$ as before, there exists a canonical Hecke equivariant $\mathcal{O}$-linear map

$$\mathbf{z}_{N_0p^m,\overline{\rho}} \colon H^1(Y(N_0p^m))_{\overline{\rho}}(1) \to \mathbf{H}^1(H^1(Y(N_0p^m))_{\overline{\rho}}(2))$$

characterized by a similar interpolation property using the L-**functions removing its Euler factors at all** $\ell \in \Sigma$, which is compatible with corestrictions for $m \geqq 1$ and restrictions for N_0.

- (A subtle point) We can define the map $\mathbf{z}_{N_0p^m,\overline{\rho}}$ over Λ (not over $\mathrm{Frac}(\Lambda)$) after taking the $\overline{\rho}$-part.

By this integrality and the compatibilities, we can define the following maps.

We set

$$\mathbf{z}_{N_0p^\infty,\overline{\rho}} := \varprojlim_{m \geqq 1} \mathbf{z}_{N_0p^m,\overline{\rho}} \colon \widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0))_{\overline{\rho}} \to \mathbf{H}^1(\widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0))_{\overline{\rho}}(1))$$

and

$$\mathbf{z}_{\Sigma,\overline{\rho}} := \varinjlim_{N_0} \mathbf{z}_{N_0p^\infty,\overline{\rho}} \colon \widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}} \to \mathbf{H}^1(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}(1)).$$

Proposition (Na)

The map $\mathbf{z}_{\Sigma,\overline{\rho}}$ is continuous and $R_\Sigma[G_\Sigma]$-linear.

- All the equivariances for Fukaya-Kato's and our maps follow from the interpolation property, which follows from Kato's very deep result, i.e. **the explicit reciprocity law**.

Factoring out the ρ_Σ^*-part from $\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}$

Since one has an isomorphism

$$\psi_1 \colon \widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}} \xrightarrow{\sim} \Pi_p^* \otimes_{R_\Sigma} \rho_\Sigma^* \otimes_{R_\Sigma} \widetilde{\pi}_{\Sigma_0},$$

it suffices to remove Π_p^*-and $\widetilde{\pi}_{\Sigma_0}$-parts.

Removing $\widetilde{\pi}_{\Sigma_0}$-part: For π a smooth admissible representation of G_ℓ $(\ell \neq p)$ defined over $\overline{\mathbb{Q}}_p$, we set $\Psi_\ell(\pi)$ the largest quotient on which $U_\ell = \begin{pmatrix} 1 & \mathbb{Q}_\ell \\ 0 & 1 \end{pmatrix}$ acts by a fixed non-trivial additive character $U_\ell \to \overline{\mathbb{Q}}_p$.

Emerton-Helm extended this exact functor for smooth admissible representations of G_ℓ defined over more general $\mathbb{Z}_p$-algebras, e.g. for $\widetilde{\pi}_{\Sigma_0}$. We set

$$\Psi_{\Sigma_0}(\widetilde{\pi}_{\Sigma_0}) := \Psi_{\ell_1} \circ \cdots \circ \Psi_{\ell_d}(\widetilde{\pi}_{\Sigma_0})$$

for $\Sigma_0 = \{\ell_1, \cdots, \ell_d\}$. By the characterization property of their correspondence, one has a R_Σ-linear map

$$\psi_2 \colon \Psi_{\Sigma_0}(\widetilde{\pi}_{\Sigma_0}) \xrightarrow{\sim} R_\Sigma \quad (\textbf{genericity} \text{ of } \widetilde{\pi}_{\Sigma_0}),$$

which follows from mod p multiplicity one and Ihara's lemma.

Removing Π_p^*-part:

- $\mathfrak{C}(\mathcal{O})$: the category which is the Pontryagin dual of the category of locally admissible G_p-representations on torsion $\mathcal{O}$-modules (Emerton).
- $\rho_p\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(R_p)$: the universal deformation of $\overline{\rho}|_{G_{\mathbb{Q}_p}}$.
- Π_p^{univ}: the representation of G_p over R_p corresponding to ρ_p.

Theorem (Paskunas (13), a very rough form)

- $P := (\Pi_p^{\mathrm{univ}})^*$ is a projective object in $\mathfrak{C}(\mathcal{O})$.
- $R_p = \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P)$.

By the universality for ρ_p, one has $R_p \to R_\Sigma$ and

$$\Pi_p^* \xrightarrow{\sim} P\widehat{\otimes}_{R_p} R_\Sigma.$$

Hence, one also has

$$\psi_1\colon \widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}} \xrightarrow{\sim} P\widehat{\otimes}_{R_p}\rho_\Sigma^* \otimes_{R_\Sigma} \widetilde{\pi}_{\Sigma_0}.$$

Definition of Z_Σ

The isomorphisms ψ_1 and ψ_2 induce the following isomorphisms.

Corollary

- *One has* $\Psi_{\Sigma_0}(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}) \in \mathfrak{C}(\mathcal{O})$, *and*

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \Psi_{\Sigma_0}(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}})) \xrightarrow{\sim} \rho^*_\Sigma.$$

- *One has* $\Psi_{\Sigma_0}(\mathbf{H}^1(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}(1))) \in \mathfrak{C}(\mathcal{O})$, *and*

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \Psi_{\Sigma_0}(\mathbf{H}^1(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}(1)))) \xrightarrow{\sim} \mathbf{H}^1(\rho^*_\Sigma(1)).$$

Applying $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \Psi_{\Sigma_0}(-))$ to the continuous $R_\Sigma[G_\Sigma]$-linear map

$$\mathbf{z}_{\Sigma,\overline{\rho}} \colon \widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}} \to \mathbf{H}^1(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}(1)),$$

we can finally define

$$Z_\Sigma := \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \Psi_{\Sigma_0}(\mathbf{z}_{\Sigma,\overline{\rho}})) \colon \rho^*_\Sigma \to \mathbf{H}^1(\rho^*_\Sigma(1)).$$