Arithmetic Theta Kernel and liftings

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(joint work with J. Bruinier, B. Howard, S. Kulda, and M. Rapoport)

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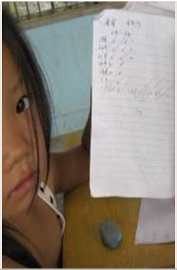

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Goal

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field with odd fund. disc $d < 0$, and let $\mathcal{X}^*$ be the compactified Shimura variety over O_K (Kramer model) ass. to a unimodular Hermitian lattice of signature $(n-1, 1)$.

Goal

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field with odd fund. disc $d < 0$, and let $\mathcal{X}^*$ be the compactified Shimura variety over O_K (Kramer model) ass. to a unimodular Hermitian lattice of signature $(n-1, 1)$.
— Construct an arithmetic theta series

$$\theta^{ar}(\tau) = \sum_{m \geq 0} \hat{\mathcal{Z}}^{tot}(m) q^m \in \mathbb{C}[[q]] \otimes \widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{X}^*)$$

and prove that it is a modular form for $\Gamma_0(|d|)$ of wt n, character χ_d^n, and with values in $\widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{X}^*)$.
—Use this arithmetic theta 'kernel' to study arithmetic theta liftings.

Plan

—Classical Theta Kernel and classical theta liftings
—Kudla-Millson theory
—Regularized theta liftings, Brocherds liftings, and Geometric theta kernel/liftings
—Arithmetic theory of Borcherds Lifttings, Arithmetic theta kernel/liftings.
—Comments/questions (if time permits)

Classical theta kernel and liftings

—$(G, H) = (U(r, r), U(V)$ or $(\mathrm{Sp}_{2r}, O(V))$, ..., reductive dual pair.
— V Hermitian(quadratic) space of dimension m.
Key: Weil representation ω of $G(\mathbb{A})$ on $S(V_{\mathbb{A}}^r)$, $H(\mathbb{A})$ acts on $S(V_{\mathbb{A}}^r)$ linearly.
—theta kernel for any $\phi \in S(V_{\mathbb{A}}^r)$

$$\theta(g, h, \phi) = \sum_{x \in V^n} \omega(g)\phi(h^{-1}x)$$

is a two variable automorphic forms on $[G] \times [H]$, where $[G] = G(\mathbb{Q})\backslash G(\mathbb{A})$.

Ancient Example

Let V be positive definite quadratic space with an lattice L, and $r = 1$. Take $\phi_f = \text{char}(\hat{L})$ and $\phi_\infty(x) = e^{-\pi(x,x)}$. Then $(g_\tau(i) = \tau)$

$$v^{-m/2}\theta(g_\tau, h, \phi_f\phi_\infty) = \sum_{x \in hL} e^{\pi i(x,x)\tau} = \sum_{m \geq 0} r_{hL}(m)q^m$$

is the classical theta function associated to the lattice hL.

Classical Theta liftings:
Given an automorphic form f on $[H]$, we obtain an automorphic form on $[G]$

$$\theta(g, f, \phi) = \int_{[H]} \theta(g, h, \phi) f(h) dh$$

if the integral converges (true if f is cuspidal).
Similarly, we have theta liftings from $[G]$ to $[H]$.
—Produce more automorphic forms from known modular forms (from different groups).
—A lot of applications to automorphic representations and L-functions.

When $f = 1$ on $[H]$, this theta liftings can be realized as Eisenstein Series (Siegel-Weil).

Unitary Shimura Varieties of signature $(n-1,1)$

—Let L be an integral Hermitian O_K-lattice of signature $(n-1,1)$ and $V = L \otimes \mathbb{Q}$. $H = U(V)$
—X the associated Shimura variety over K with

$$X(\mathbb{C}) = H(\mathbb{Q}) \backslash \mathbb{D} \times H(\mathbb{A}_f)/K_L$$

where

$$\mathbb{D} = H(\mathbb{R})/K_\infty = \{z \in V_{\mathbb{C}} : (z,z) < 0\}/\mathbb{C}^\times = \mathcal{L}/C^\times.$$

—$\mathcal{L}$ descends to a line bundle $\mathcal{L}$ of modular forms of wt 1 over X.

—L' the orthogonal dual of L (with respect to $Q(x) = (x, x)$)
—The Weil representation ω induces a Weil rep. ω of $\mathrm{SL}_2(\mathbb{Z})$ on $S_L = \mathbb{C}[L'/L]$.
—Standard basis of S_L: $\{\phi_\mu : \mu \in L'/L\}$
—Special divisors $Z(m, \mu)$ for $m > 0$ and $\mu \in L'/L$ with $Q(\mu) \equiv m$ mod 1: At a connected component, it looks like

$$\Gamma \backslash \{z \in \mathbb{D} : (x, z) = 0 \text{ for some } x \in \mu + L, (x, x) = m\}$$

—$Z(0,\mu) = -\frac{1}{2}[\mathcal{L}]$ or 0 depending on whether $\mu = 0$ or not.
—Geometric theta series in open Shimura variety

$$\theta^{geo}(\tau) = \sum_{\mu \in L'/L} \sum_{m \geq 0} Z(m,\mu) q^m \phi_\mu \in S_L[[q]] \otimes \mathrm{CH}^1_{\mathbb{Q}}(X)$$

via Chern class maps 'cl', we have

$$\theta^{col}(\tau) = \mathrm{cl}(\theta^{geo}) \in S_L[[q]] \otimes H^2(X,\mathbb{Q})$$

Instead of scalar valued Schwartz functions at ∞ in the theta kernel, Kudla and Millson constructed a Schwartz functions $\phi_{KM,\infty}(z,x)$
— with values in closed $(1,1)$-differentials on $z \in \mathbb{D}$ variable
— weight n on variable $\tau \in \mathbb{H}$ via (local) Weil representation
— the associated cohomology class $[\phi_{KM,\infty}]$ is 'holomorphic' on τ, such that

$$\theta^{col}(\tau, \mu)\text{"} = \text{"}\theta(\tau, z, h, \phi_\mu \phi_{KM,\infty})$$

which is modular!!!
—Kudla-Millson theory is much more general.

Regularized theta lifting and Borcherds lifting

—What about θ^{geo}??

$S_k(\omega)$—holomorphic cusp forms with values in S_L. $f : \mathbb{H} \to S_L$

$$f(\gamma\tau) = (c\tau + d)^k \omega(\gamma) f(\tau).$$

$M^!_k(\omega)$—weakly holomorphic forms with values in S_L: meromorphic at the cusp ∞.

$j(\tau) \in M^!_0(trivial)$.

$H_k(\omega)$—Harmonic Maass forms with values in S_L.

$$0 \to M^!_{2-n}(\omega) \to H_{2-n}(\omega) \to S_n(\bar{\omega}) \to 0$$

Where the last map is given by ξ_{2-n}:

$$\xi_k(f) = 2iv^k \overline{\frac{\partial f}{\partial \bar{\tau}}} = -2iv^{k-2}\overline{L_k(f)}.$$

where L_k is Maass weight raising operator.

$f \in H_k$ can be written as

$$f = f^+ + f^- = \sum_{m,\mu} c_f^+(m,\mu) q^m \phi_\mu + \text{non-holomorphic exponentially decay}$$

Finally,

$$Z(f) = \sum_{m>0} \sum_{\mu} c_f^+(-m,\mu) Z(m,\mu) \in \mathrm{CH}^1(X)$$

is the special divisor associated to f.

Regularized theta lifting and automorphic Green function
$f \in H_{2-n}$ gives regularized theta lifting ($(z,h) \in X$)

$$\Phi(z,h,f) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{reg} f(\tau)\theta(\tau,z,h)d\mu(\tau)$$

where

$$\theta(\tau,z,h)\text{''} = \text{''} \sum_{\mu \in L'/L} \theta(g_\tau, h_z h, \phi_\mu \phi_\infty)$$

is the classical theta kernel rewritten 'geometrically'.
—ϕ_∞ is 'Gaussian' function of weight $n-2$.
—$h_z(z_0) = z$, $z_0 \in \mathbb{D}$ prefixed.

Theorem 1

(1) (Bruinier, 02, Bruinier-Funke, 04) $\Phi(z, h, f)$ *is well-defined on* X*, smooth away from* $Z(f)$*, and is a Green function for* $Z(f)$*.*
(2)(Borcherds, 98) When $f \in M^!_{2-n}$ *and* $c_f(m, \mu) = c_f^+(m, \mu) \in \mathbb{Z}$ *for* $m \leq 0$*, there is a memomorphic modular form* $\Psi(z, h, f)$ *(called Borcherds lifting of* f *) of weight* $\frac{1}{2}c_f(0, 0)$ *such that*

$$-\log |\Psi(z, h, f)|^2_{Pet} = \Phi(z, h, f),$$

and

$$Div\, \Psi(z, h, f) = Z(f).$$

(3) (Borcherds 98, Kudla 16) Around a cusp of X*,* $\Psi(z, h, f)$ *has an infinite 'Borcherds' product expansion.*

Examples

Let $12\theta(\tau) = 12 + \sum_{n=1}^{\infty} 24q^{n^2}$ and $L = M_2(\mathbb{Z})^{tr=0}$ with $Q(x) = \det x$. Then

$$\Psi(z, 12\theta) = \Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

On $Y(1)$, $\mathrm{Div}(\Psi) = 0$ by Borcherds.
On $X(1)$, $\mathrm{Div}(\Psi) = \{\infty\}$ by Borherds product expansion.

Take $L = M_2(\mathbb{Z})$ with $Q(x) = detx$, Borcherds obtained the famous

$$\Psi(z_1, z_2, j(\tau) - 744) = j(z_1) - j(z_2) = q_1^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - q_1^m q_2^n)^{c(mn)}$$

where $c(n)$ is the n-th coefficient of $j(\tau) - 744$. $q_k = e^{2\pi i z_k}$.
On $Y(1) \times Y(1)$, Borcherds' theorem asserts

$$\mathrm{Div}(\Psi) = Y(1)^{\mathrm{diag}} = Z(1).$$

Whole story: one $X(1) \times X(1)$, Borcherds product expansion gives

$$\mathrm{Div}(\Psi) = X(1)^{\mathrm{daig}} - X(1) \times \{\infty\} - \{\infty\} \times X(1).$$

— $\Psi(z, h, f)$ is very special in the sense that its divisor is known (like Δ, E_4 and E_6)
—It gives relations about special divisors:

$$\sum_{m \geq 0, \mu} c_f(-m, \mu) Z(m, \mu) = 0 \in \mathrm{CH}^1(X).$$

Theorem 2

(Borcherds, 1999) $\theta^{geo}(\tau)$ *is a modular form of weight* n *valued in* $S_L \otimes \mathrm{CH}^1_{\mathbb{Q}}(X)$.

Reason: Each $f \in M^!_{2-n}$ gives a relation among $Z(m, \mu)$, and thus a lot of relations among them.

Borcherds (Serre duality): Let A be an abelian group and $a(m, \mu) \in A$. Then a power series $\sum_{m \geq 0, \mu} a(m, \mu) q^m \phi_\mu$ is a holomorphic modular form for $\mathrm{SL}_2(\mathbb{Z})$ of wt n valued in $S_L \otimes A$ if and only if

$$\sum_{m \geq 0, \mu} c_f(-m, \mu) a(m, \mu) = 0$$

for every $f \in M^!_{2-n}(\omega_L)$.

—This theorem implies that the subspace of special divisors in $\mathrm{CH}^1_{\mathbb{Q}}(X)$ is finite, and
—we have geometric theta lifting from $S_n(\omega)$ to $\mathrm{CH}^1_{\mathbb{Q}}(X)$.

X^*—smooth (canonical) Toriodal compactification
Boundaries $\partial X = X^* - X = \sum_{P \text{ cusp}} B_P$, the boundary component B_P at each cusp is an Abelian variety of dimension $n-2$, thus a divisor of X^*.

By studying the behavior of the Green functions $\Phi(z, h, f)$ ($f \in H_{2-n}$) around each boundary component, we have

Proposition 1

(Bruinier-Howard-Y, '15) For $f \in H_{2-n}(\omega)$, $\Phi(z, h, f)$ is the Green function on X^* for $Z^{tot}(f) = Z^*(f) + B(f)$, where

$$B(f) = \sum_{P \ Cusp} c_P(f) B_P,$$

and $Z^*(f)$ is the Zariski closure of $Z(f)$ in X^*. $c_P(f) \in \mathbb{Q}$ explicit.

For (m,μ), take $f = f_{m,\mu} = q^{-m}(\phi_\mu + \phi_{-\mu}) + O(1) \in H_{2-n}$ (unique for $n \geq 3$), we obtain $Z^{tot}(m,\mu)$. For example

$$c_P(m) = c_P(f_{m,0}) = \frac{m}{n-2}|\{x \in L_0 : (x,x) = m\}|$$

Where $L_0 = L \cap (J \oplus J^\vee)^\perp$, and J is the isotropic line defining the cusp P.

The line bundle $\mathcal{L}$ of modular forms on X can also be extended to X^* naturally. The same argument as Borcherds now gives

Theorem 3

When $n \geq 3$,

$$\theta^{geo,*}(\tau) = \sum_{m \geq 0, \mu} Z^{tot}(m, \mu) q^m \phi_\mu$$

is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ *of wt* n*, valued in* $S_L \otimes \mathrm{CH}^1_{\mathbb{Q}}(X^*)$.

Geometric Theta Liftings I

$$\theta^{geo}: \mathrm{CH}_{\mathbb{Q}}^{n-2}(X^*) \to S_n(\omega), \quad \theta^{geo}(C) = \langle \theta^{geo,*}, C \rangle$$

It should be interesting to study this map. For example,
— we can take C to be Shimura curves in X^*, understand the decomposition of $\theta^{geo}(C)$ with respect to Hecke Eigenforms.
— classify the image of this map when restricting on all (or split) Shimura curves in X^*

Geometric theta lifting II

$$\theta^{geo} : S_n(\omega) \to \mathrm{CH}^1_{\mathbb{Q}}(X^*), \quad \theta^{geo}(f) = \langle \theta^{geo,*}(\tau), f \rangle_{Pet}.$$

It is interesting to figure out the kernel and image of this map (likely injective for $n \geq 3$).

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Proposition 2

$$\langle \theta^{geo}(C), f \rangle_{Pet} = \langle \theta^{geo}(f), C \rangle.$$

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Proposition 2

$$\langle \theta^{geo}(C), f \rangle_{Pet} = \langle \theta^{geo}(f), C \rangle.$$

This pairing is related to special value of some Rankin-Selberg L-function $\langle E(\tau, s)\theta_{n-2}(\tau), f \rangle_{Pet}$ when C is Shimura curve.

Integral model and Arithmetic theta series

To extend $\theta^{geo,*}$ to arithmetic situation, we need integral model and slightly different setting with more restriction on lattice L.
Let $\mathfrak{a}_0$ and $\mathfrak{a}$ be unimodular Hermitian O_K-modules of signature $(1,0)$ and $(n-1,1)$, and let G be the subgroup of $\mathrm{GU}(\mathfrak{a}_0) \times \mathrm{GU}(\mathfrak{a})$ consisting pairs (g_0, g_1) with equal similitude. Let $L = \mathrm{Hom}_{O_K}(\mathfrak{a}_0, \mathfrak{a})$, then we have exact sequence

$$1 \to \mathbb{G}_{m, O_K/\mathbb{Z}} \to G \to U(L) \to 1.$$

The integral Shimura variety $\mathcal{X}$ parametrizes tuples $(\mathsf{A}_0 = (A_0, \iota_0, \lambda_0), \mathsf{A}_1 = (A_1, \iota_1, \lambda_1, \mathcal{F})) \in \mathcal{X}_{(1,0)} \times \mathcal{X}^{Kra}_{(n-1,1)}$ with an extra condition.

$\mathcal{X}$ is regular over O_K but has bad reduction for $p|d$.

Similarly we extend compactification $\mathcal{X}^*$ with boundaries $\mathcal{B}_P$ integrally.

$\mathcal{X}^{Kra}_{(n-1,1)}$ parametrizes tuples $\mathsf{A} = (A, \iota, \lambda, \mathcal{F})$ where
— (A, ι, λ) are p.p. Abelian scheme of relative dimension n with O_K-action ι,
—$\mathcal{F}$ is a $O_K \otimes O_S$ submodule of Lie(A), and locally a direct summand of rank $n-1$ as O_S-submodule such that
—- O_K acts on $\mathcal{F}$ via structure map $O_K \to O_S$,
— O_K acts on Lie$(A)/\mathcal{F}$ by conjugation of the structure map. (signature $(n-1,1)$ condition)

Associated to $(\mathrm{A}_0, \mathrm{A}_1) \in \mathcal{X}$, is a positive Hermitian form on $\mathrm{Hom}_{O_K}(A_0, A_1)$:

$$(f, g) = \lambda_0^{-1} \circ g^\vee \circ \lambda_1 \circ f \in O_K = \mathrm{End}_{O_K}(A_0).$$

The arithmetic divisor $\mathcal{Z}(m)$ parametrizes $(A_0, A_1, x : A_0 \to A_1)$ with $(x, x) = m$.
The line bundle $\mathcal{L}$ can also extend naturally over $\mathcal{X}^*$, together with a natural metrization to make it a metrized line bundle $\hat{\mathcal{L}} = (\mathcal{L}, \| \, \|)$, viewed as an element in $\widehat{\mathrm{CH}}^1(\mathcal{X}^*)$.

We define similarly,

$$\hat{\mathcal{Z}}^{tot}(m) = (\mathcal{Z}^{tot}(m), \Phi_m) \in \widehat{\mathrm{CH}}^1(\mathcal{X}^*)$$

with

$$\mathcal{Z}^{tot}(m) = \mathcal{Z}^*(m) + \sum_{P \text{ Cusp}} c_P(m)\mathcal{B}_P,$$

and

$$\Phi_m = \Phi_{f_{m,0}}.$$

Define

$$\hat{\mathcal{Z}}(0) = -[\hat{\mathcal{L}}] + (\mathrm{Exc}, -\log|d|)$$

where Exc is sum of exceptional divisors at $p|d$.

Main Theorem (Bruinier-Howard-Kudla-Rapoport-Y, 2020) The arithmetic theta function

$$\theta^{ar}(\tau) = \sum_{m \geq 0} \hat{\mathcal{Z}}^{tot}(m) q^m$$

is a modular form for $\Gamma_0(|d|)$, wt n, character χ_d^n, and with values in $\widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{X}^*)$.

With this kernel function, we can study arithmetic theta liftings.
Arithmetic Theta Lifting I:

$$f \in S_n(\Gamma_0(|d|), \chi_d^n) \mapsto \theta^{ar}(f) = \langle \theta^{ar}(\tau), f \rangle_{Pet} \in \widehat{CH}^1_{\mathbb{Q}}(\mathcal{X}^*).$$

—What can we say the kernel and image of this map?

Arithmetic Theta Lifting II:

$$Z^{n-1}(\mathcal{X}^*) \to S_n(\Gamma_0(|d|), \chi_d^n),$$
$$\theta^{ar}(\mathcal{Z}) = \langle \theta^{ar}(\tau), \mathcal{Z} \rangle_{Fal} = \widehat{deg}\theta^{ar}|_{\mathcal{Z}}$$

and

$$\widehat{CH}_{\mathbb{Q}}^{n-1}(\mathcal{X}^*) \to S_n(\Gamma_0(|d|), \chi_d^n), \quad \theta^{ar}(\hat{\mathcal{Z}}) = \langle \theta^{ar}(\tau), \hat{\mathcal{Z}} \rangle_{GS}$$

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and

$$\widehat{CH}_{\mathbb{Q}}^{n-1}(\mathcal{X}^*) \to S_n(\Gamma_0(|d|), \chi_d^n), \quad \theta^{ar}(\hat{\mathcal{Z}}) = \langle \theta^{ar}(\tau), \hat{\mathcal{Z}} \rangle_{GS}$$

—Can either of the map surjective? It would give an arithmetic construction of classical modular forms.
—Decomposition of $\theta^{ar}(\hat{\mathcal{Z}})$ with respect to Hecke eigenforms.

Adjoint Property: for $f \in S_n(\Gamma_0(|d|), \chi_d^n)$ and $\mathcal{Z} \in Z^{n-1}(\mathcal{X}^*)$ we have

$$\langle \theta^{ar}(f), \mathcal{Z} \rangle_{Fal} = \langle \theta^{ar}(\mathcal{Z}), f \rangle_{Pet}$$

Theorem 4

(BHKRY, 20)(analogue of the Gross-Zagier formula) When $\mathcal{Z}$ is a CM cycle,

$$\langle \theta^{ar}(f), \mathcal{Z} \rangle_{Fal} \text{"} = \text{"} \langle \mathcal{E}'(\tau^{\Delta}, 0)\theta(\tau), f(\tau) \rangle_{Pet}$$

is the central derivative of Rankin-Selberg L-function of f.
—$\mathcal{E}(\vec{\tau}, s)$ *is some 'incoherent' Eisenstein series over the totally real number subfield F^+ if the CM cycle is associated to a CM number field F.*
—$\theta(\tau)$ *is a classical theta function associated to L and the CM cycle.*

—If the central derivative is non-zero, then $\theta^{ar}(f) \neq 0$, f is NOT in the kernel of the arithmetic theta lifting.

Another arithmetic theta series, using Kudla Green functions

$$\theta_K^{ar}(\tau) = \sum_{m \in \mathbb{Z}} \hat{\mathcal{Z}}^{tot}(m, v) q^m$$

which is also a non-holomorphic modular form with values in $\widehat{\mathrm{CH}}^1(\mathcal{X}^*)$ by our result above and Ehlen and Sankaran's result (their difference is a modular form).

Issue with $n = 2$.
Solution: embedding method (in progress with Qiao He and Yousheng Shi). It turns out that the arithmetic theta function using Kudla Green functions has desired property:

$$j^* \theta^{ar}_{K,n}(\tau) = \theta^{ar}_{K,2}(\tau)\theta_{n-2}(\tau).$$

—Use this to prove modularity of $\theta^{ar}_{K,2}(\tau)$
—Use Ehlen-Sankaran result to get modularity of $\theta^{ar}_{2}(\tau)$.

Main Theorem again (Bruinier-Howard-Kudla-Rapoport-Y, 2020) The arithmetic theta function

$$\theta^{ar}(\tau) = \sum_{m \geq 0} \hat{\mathcal{Z}}^{tot}(m) q^m$$

is a modular form for $\Gamma_0(|d|)$, wt n, character χ_d^n, and with values in $\widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{X}^*)$.

Main Idea of Proof: algebraic definition of Borcherds products!
For $f \in M^!_{2-n}$ we have memomorphic form Ψ of weight $k = \frac{1}{2} c_f(0,0)$ with

$$\mathrm{Div}\Psi = Z^{tot}(f).$$

—Ψ is a 'section' of $\mathcal{L}_{\mathbb{C}}^k$.
Basic Question: How to make Ψ a 'section' of the integral and compactified line bundle $\mathcal{L}^k$ with

$$\mathrm{Div}\Psi = \mathcal{Z}^{tot}(f).$$

In reality,

$$\mathrm{Div}\Psi = \mathcal{Z}^{tot}(f) + \text{explicit vertical divisors}$$

and the explicit vertical divisors do not affect the modularity of our arithmetic theta series.

Key: 1. The Bocherds product expansion around cusps of the Borcherds Lifting $\Psi(z, h, f)$ over both unitary Shimura variety (Kudla, Fourier-Jacobi expansion) and the orthogonal Shimura variety (Borcherds)
2. q-principle.
3. Fourier-Jacobi forms.

The important toy examples:
Let $\mathcal{Y}$ be the modular curve over $\mathbb{Z}$ with universal elliptic curve $\mathcal{E}$ over it. Let $\omega_{\mathcal{Y}}$ be the line bundle over $\mathcal{Y}$ of modular forms of wt 1. Then

$$(2\pi i\eta^2)^{12} = (2\pi i)^{12}\Delta = (2\pi i)^{12} q \prod_{n\geq 1}(1-q^n)^{24}$$

is a nowhere vanishing section of $\omega_{\mathcal{Y}}^{12}$.

Let $\mathcal{J}_{0,1} = (\text{diag})^*\mathcal{P}$ is the line bundle of Jacobi forms of wt 0 and index 1 over $\mathcal{E}$. Here $\mathcal{P}$ is the Poincare line bundle over $\mathcal{E} \times_{\mathcal{Y}} \mathcal{E}$.
Let

$$\theta_1(\tau, z) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+1/2)^2\tau + 2\pi i(n+1/2)(z-1/2)}$$

be the classical Jacobi theta function, and

$$\Theta(\tau, z) = i\frac{\theta_1(\tau, z)}{\eta(\tau)} = q_\tau^{\frac{1}{2}}(q_z^{\frac{1}{2}} - q_z^{-\frac{1}{2}})\prod_{n>0}(1 - q_z q_\tau)^n(1 - q_z^{-1}q_\tau^n).$$

Then

$$\Theta^{24} \in H^0(\mathcal{E}, \mathcal{J}_{0,12})$$

is a global section of $\mathcal{J}_{0,12}$ with

$$\mathrm{Div}\Theta^{24} = 24(0)$$

where (0) is the divisor of $\mathcal{E}$ given by the 0-section of $\mathcal{E} \to \mathcal{Y}$.

Comments/Questions

1. The modularity result of arithmetic theta series in $O(n-1,2)$ case should follow from Howard and Madapusi-Pera's work on integral model, and Bruinier and Zemel's work on Green function behavior at Boundary (plus some technical work).
2. Higher Codimension situation for Open Shimura varieties of $O(n-1,2)$ and $U(n-1,2)$.
Kudla conjectured the modularity of

$$\theta_r^{geo} = \sum_{T \in \mathrm{Herm}_r^{\geq 0}} \sum_{\mu \in (L'/L)^r} Z(T,\mu) q^T \phi_\mu \in S_L^{\otimes r}[[q]] \otimes \mathrm{CH}_{\mathbb{Q}}^r(X)$$

Over $O(n-1,2)$
Wei Zhang (thesis): θ_r^{geo} is formally modular.
Bruinier and Raum: Formal Siegel modular forms are modular.

Over $O(n-1,2)$
Wei Zhang (thesis): θ_r^{geo} is formally modular.
Bruinier and Raum: Formal Siegel modular forms are modular.
Over $U(n-1,1)$, Wei Zhang's work extends without problem.
JieCheng Xia extends Bruinier and Raum's work to this case with assumption that O_K is Euclidean.

3. Howard and Madapusi-Pera extends the modularity of θ_r^{geo} to integral moduel in $O(n-1,2)$ case.
4. We still don't know how to extend the geometric theta series to include boundary, unfortunately.
5. Kudla's Green functions was extended to Green currents for general r systematically by Garcia and Sankaran (non-holomorphic). It is still a challenging problem to do the same for Bruinier's Green functions (holomorphic).

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